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SPACES WITH σ-LOCALLY FINITE LINDELÖF sn-NETWORKS

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Abstract. We prove that a space X has a σ-locally finite Lindelöf sn-network if and only if X is a compact-covering compact and mssc-image of a locally separable metric space, if and only if X is a sequentially-quotient π and mssc-image of a locally separable metric space, where "compact-covering" (or "sequentially-quotient") can not be replaced by "sequence-covering". As an application, we give a new characterization of spaces with locally countable weak bases.

1. Introduction

In [**17**] Lin introduced the concept of mssc-maps to characterize spaces with certain σ-locally finite networks by mssc-images of metric spaces. After that, some characterizations for certain mssc-images of metric (or semi-metric) spaces are obtained by many authors ([**11, 12, 14**], for example). Recently, Dung gave some characterizations for certain mssc-images of locally separable metric spaces (see in [**3**]).

We prove that a space X has a σ-locally finite Lindelöf sn-network if and only if X is a compact-covering compact and mssc-image of a locally separable metric space, if and only if X is a sequentially-quotient π and mssc-image of a locally separable metric space, where "compact-covering" (or "sequentially-quotient") can not be replaced by "sequence-covering". As an application, we give a new characterization of spaces with locally countable weak bases.

Throughout this paper, all spaces are assumed to be T_1 and regular, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers. Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of subsets of X and $x \in X$, we denote $(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$, $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$, $\bigcap \mathcal{P} = \bigcap\{P : P \in \mathcal{P}\}$, $\mathrm{st}(x, \mathcal{P}) = \bigcup(\mathcal{P})_x$ and $\mathcal{P} \bigwedge \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$. For a sequence $\{x_n\}$ converging to x and $P \subset X$, we say

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that $\{x_n\}$ is *eventually* in P if $\{x\} \bigcup \{x_n : n \geqslant m\} \subset P$ for some $m \in \mathbb{N}$, and $\{x_n\}$ is *frequently* in P if some subsequence of $\{x_n\}$ is eventually in P.

Definition 1.1. Let X be a space, $P \subset X$ and let $\mathcal{P}$ be a cover of X.

(1) P is a *sequential neighborhood* of x in X [**5**], if each sequence S converging to x is eventually in P.
(2) P is a *sequentially open* subset of X [**5**], if P is a sequential neighborhood of x in X for every $x \in P$.
(3) $\mathcal{P}$ is an *so-cover* for X [**19**], if each element of $\mathcal{P}$ is sequentially open in X.
(4) $\mathcal{P}$ is a *cfp-cover* for X [**29**], if whenever K is compact subset of X, there exist a finite family $\{K_i : i \leqslant n\}$ of closed subsets of K and $\{P_i : i \leqslant n\} \subset \mathcal{P}$ such that $K = \bigcup \{K_i : i \leqslant n\}$ and each $K_i \subset P_i$.
(5) $\mathcal{P}$ is an *cs***-cover* for X [**28**], if every convergent sequence is frequently in some $P \in \mathcal{P}$.

Definition 1.2. Let $\mathcal{P}$ be a family of subsets of a space X.

(1) For each $x \in X$, $\mathcal{P}$ is a *network* at x in X [**18**], if $x \in \bigcap \mathcal{P}$, and if $x \in U$ with U open in X, then there exists $P \in \mathcal{P}$ such that $x \in P \in U$.
(2) $\mathcal{P}$ is a *cs-network* for X [**28**], if each sequence S converging to a point $x \in U$ with U open in X, S is eventually in $P \subset U$ for some $P \in \mathcal{P}$.
(3) $\mathcal{P}$ is a *cs***-network* for X [**28**], if for each sequence S converging to a point $x \in U$ with U open in X, S is frequently in $P \subset U$ for some $P \in \mathcal{P}$.
(4) $\mathcal{P}$ is *Lindelöf*, if each element of $\mathcal{P}$ is a Lindelöf subset of X.
(5) $\mathcal{P}$ is *point-countable* [**4**], if each point $x \in X$ belongs to only countably many members of $\mathcal{P}$.
(6) $\mathcal{P}$ is *locally countable* [**4**], if for each $x \in X$, there exists a neighborhood V of x such that V meets only countably many members of $\mathcal{P}$.
(7) $\mathcal{P}$ is *locally finite* [**4**], if for each $x \in X$, there exists a neighborhood V of x such that V meets only finite many members of $\mathcal{P}$.
(8) $\mathcal{P}$ is *star-countable* [**23**], if each $P \in \mathcal{P}$ meets only countably many members of $\mathcal{P}$.

Definition 1.3. Let $\mathcal{P} = \bigcup \{\mathcal{P}_x : x \in X\}$ be a family of subsets of a space X satisfying that, for every $x \in X$, $\mathcal{P}_x$ is a network at x in X, and if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

(1) $\mathcal{P}$ is a *weak base* for X [**1**], if $G \subset X$ such that for every $x \in G$, there exists $P \in \mathcal{P}_x$ satisfying $P \subset G$, then G is open in X. Here, $\mathcal{P}_x$ is a *weak base* at x in X.
(2) $\mathcal{P}$ is an *sn-network* for X [**16**], if each member of $\mathcal{P}_x$ is a sequential neighborhood of x for all $x \in X$. Here, $\mathcal{P}_x$ is an *sn-network* at x in X.

Definition 1.4. Let X be a space.

(1) X is an *sn-first countable space* [**8**], if there is a countable sn-network at x in X for all $x \in X$.
(2) X is an *sn-metrizable space* [**7**] (resp., a *g-metrizable space* [**25**]), if X has a σ-locally ŕfinite sn-network (resp., weak base).

(3) X is a *cosmic space* [**21**], if X has a countable network.
(4) X is an $\aleph_0$-*space* [**21**], if X has a countable cs-network.
(5) X is an $\aleph$-*space* [**22**], if X has a σ-locally finite cs-network.
(6) X is a *sequential space* [**5**], if each sequentially open subset of X is open.
(7) X is a *Fréchet space* [**4**], if for each $x \in \overline{A}$, there exists a sequence in A converging to x.

Definition 1.5. Let $f : X \to Y$ be a map.

(1) f is *sequence-covering* [**24**], if for each convergent sequence S of Y, there exists a convergent sequence L of X such that $f(L) = S$. Note that a sequence-covering map is a *strong sequence-covering* map in the sense of [**14**].
(2) f is *compact-covering* [**21**], if for each compact subset K of Y, there exists a compact subset L of X such that $f(L) = K$.
(3) f is *pseudo-sequence-covering* [**13**], if for each convergent sequence S of Y, there exists a compact subset K of X such that $f(K) = S$.
(4) f is *sequentially-quotient* [**2**], if for each convergent sequence S of Y, there exists a convergent sequence L of X such that $f(L)$ is a subsequence of S.
(5) f is a *quotient map* [**4**], if whenever $U \subset Y$, U open in Y if and only if $f^{-1}(U)$ open in X.
(6) f is an *mssc-map* [**17**], if X is a subspace of the product space $\prod_{i\in\mathbb{N}} X_i$ of a family $\{X_i : i \in \mathbb{N}\}$ of metric spaces and for each $y \in Y$, there is a sequence $\{V_i : i \in \mathbb{N}\}$ of open neighborhoods of y such that each $\overline{p_i f^{-1}(V_i)}$ is compact in X_i.
(7) f is *compact* [**4**], if each $f^{-1}(y)$ is compact in X.
(8) f is a π-*map* [**13**], if for each $y \in Y$ and for each neighborhood U of y in Y, $d\big(f^{-1}(y), X - f^{-1}(U)\big) > 0$, where X is a metric space with a metric d.

Definition 1.6. [**18**] Let $\{\mathcal{P}_i\}$ be a cover sequence of a space X. $\{\mathcal{P}_i\}$ is called a *point-star network*, if $\{\mathrm{st}(x, \mathcal{P}_i) : i \in \mathbb{N}\}$ is a network of x for each $x \in X$.

For some undefined or related concepts, we refer the reader to [**4, 13, 18**].

2. Main Results

Lemma 2.1. *Let $f : M \to X$ be a sequentially-quotient mssc-map, and M be a locally separable metric space. Then, X has a σ-locally finite Lindelöf cs-network.*

Proof. By using the proof of (3) $\Rightarrow$ (1) in [**14**, Theorem 4], there exists a base $\mathcal{B}$ of M such that $f(\mathcal{B})$ is a σ-locally finite network for X. Since M is locally separable, for each $a \in M$, there exists a separable open neighborhood U_a. Denote

$$\mathcal{C} = \{B \in \mathcal{B} : B \subset U_a \text{ for some } a \in M\}.$$

Then, $\mathcal{C} \subset \mathcal{B}$ and $\mathcal{C}$ is a separable base for M. If put $\mathcal{P} = f(\mathcal{C})$, then $\mathcal{P} \subset f(\mathcal{B})$, and $\mathcal{P}$ is a σ-locally finite Lindelöf network. Since f is sequentially-quotient and $\mathcal{C}$ is a base for M, $\mathcal{P}$ is a cs*-network. Therefore, $\mathcal{P}$ is a σ-locally finite Lindelöf cs*-network.

Let $\mathcal{P} = \bigcup\{\mathcal{P}_i : i \in \mathbb{N}\}$, we can assume that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$. Since each element of $\mathcal{P}_i$ is Lindelöf, each $\mathcal{P}_i$ is star-countable. It follows from [**23**, Lemma 2.1] that for each $i \in \mathbb{N}$, $\mathcal{P}_i = \bigcup\{\mathcal{Q}_{i,\alpha} : \alpha \in \Lambda_i\}$, where $\mathcal{Q}_{i,\alpha}$ is a countable subfamily of $\mathcal{P}_i$ for all $\alpha \in \Lambda_i$ and $\left(\bigcup \mathcal{Q}_{i,\alpha}\right) \cap \left(\bigcup \mathcal{Q}_{i,\beta}\right) = \emptyset$ for all $\alpha \neq \beta$. For each $i \in \mathbb{N}$ and $\alpha \in \Lambda_i$, we put $\mathcal{R}_{i,\alpha} = \left\{\bigcup \mathcal{F} : \mathcal{F} \text{ is a finite subfamily of } \mathcal{Q}_{i,\alpha}\right\}$. Since each $\mathcal{R}_{i,\alpha}$ is countable, we can write $\mathcal{R}_{i,\alpha} = \{R_{i,\alpha,j} : j \in \mathbb{N}\}$. Now, for each $i, j \in \mathbb{N}$, put $\mathcal{F}_{i,j} = \{R_{i,\alpha,j} : \alpha \in \Lambda_i\}$, and denote $\mathcal{G} = \bigcup\{\mathcal{F}_{i,j} : i, j \in \mathbb{N}\}$. Then, each $R_{i,\alpha,j}$ is Lindelöf and each family $\mathcal{F}_{i,j}$ is locally finite. Now, we shall show that $\mathcal{G}$ is a cs-network. In fact, let $\{x_n\}$ be a sequence converging to $x \in U$ with U open in X. Since $\mathcal{P}$ is a point-countable cs*-network, it follows from [**27**, Lemma 3] that there exists a finite family $\mathcal{A} \subset (\mathcal{P})_x$ such that $\{x_n\}$ is eventually in $\bigcup \mathcal{A} \subset U$. Furthermore, since $\mathcal{A}$ is finite and $\mathcal{P}_i \subset \mathcal{P}_{i+1}$ for all $i \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that $\mathcal{A} \subset \mathcal{P}_i$. So, there exists unique $\alpha \in \Lambda_i$ such that $\mathcal{A} \subset \mathcal{Q}_{i,\alpha}$, and $\bigcup \mathcal{A} \in \mathcal{R}_{i,\alpha}$. Thus, $\bigcup \mathcal{A} = R_{i,\alpha,j}$ for some $j \in \mathbb{N}$. Hence, $\bigcup \mathcal{A} \in \mathcal{G}$, and $\mathcal{G}$ is a cs-network. Therefore, $\mathcal{G}$ is a σ-locally finite Lindelöf cs-network. □

THEOREM 2.1. *The following are equivalent for a space X.*

(1) *X is an sn-metrizable space and has an so-cover consisting of $\aleph_0$-subspaces;*
(2) *X has a σ-locally finite Lindelöf sn-network;*
(3) *X is a compact-covering compact and mssc-image of a locally separable metric space;*
(4) *X is a pseudo-sequence-covering compact and mssc-image of a locally separable metric space;*
(5) *X is a subsequence-covering compact and mssc-image of a locally separable metric space;*
(6) *X is a sequentially-quotient π and mssc-image of a locally separable metric space.*

PROOF. (1) → (2). Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a σ-locally finite sn-network and $\mathcal{O}$ be an so-cover consisting of $\aleph_0$-subspaces for X. For each $x \in X$, pick $O_x \in \mathcal{O}$ such that $x \in O_x$ and put $\mathcal{G}_x = \{P \in \mathcal{P}_x : P \subset O_x\}$ and $G = \bigcup\{\mathcal{G}_x : x \in X\}$. Then, $\mathcal{G}$ is a σ-locally finite Lindelöf sn-network for X.

(2) → (3). Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\} = \{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-locally finite Lindelöf sn-network for X, where each $\mathcal{P}_n$ is locally finite and each $\mathcal{P}_x$ is an sn-network at x. Since X is a regular space, we can assume that each element of $\mathcal{P}$ is closed. On the other hand, since each element of $\mathcal{P}_i$ is Lindelöf, each $\mathcal{P}_i$ is star-countable. It follows from [**23**, Lemma 2.1] that for each $i \in \mathbb{N}$, $\mathcal{P}_i = \bigcup\{\mathcal{Q}_{i,\alpha} : \alpha \in \Phi_i\}$, where $\mathcal{Q}_{i,\alpha}$ is a countable subfamily of $\mathcal{P}_i$ for all $\alpha \in \Phi_i$ and $\left(\bigcup \mathcal{Q}_{i,\alpha}\right) \cap \left(\bigcup \mathcal{Q}_{i,\beta}\right) = \emptyset$ for all $\alpha \neq \beta$. Since each $\mathcal{Q}_{i,\alpha}$ is countable, we can write $\mathcal{Q}_{i,\alpha} = \{P_{i,\alpha,j} : j \in \mathbb{N}\}$. Now, for each $i, j \in \mathbb{N}$, put $\mathcal{F}_{i,j} = \{P_{i,\alpha,j} : \alpha \in \Phi_i\}$, and $A_{i,j} = \{x \in X : \mathcal{P}_x \cap \mathcal{F}_{i,j} = \emptyset\}$ and $\mathcal{H}_{i,j} = \mathcal{F}_{i,j} \cup \{A_{i,j}\}$. Then, $\mathcal{P} = \bigcup\{\mathcal{F}_{i,j} : i, j \in \mathbb{N}\}$, and

(a) Each $\mathcal{H}_{i,j}$ is locally finite. It is obvious.

(b) Each $\mathcal{H}_{i,j}$ is a cfp-cover. Let K be a non-empty compact subset of X. We shall show that there exists a finite subset of $\mathcal{H}_{i,j}$ which forms a cfp-cover of K.

In fact, since X has a σ-locally finite sn-network, K is metrizable. On the other hand, since $\mathcal{P}_i$ is locally finite, K meets only finitely many members of $\mathcal{P}_i$. Thus, K meets only finitely many members of $\mathcal{H}_{i,j}$. Let

$$\Gamma_{i,j} = \big\{\alpha \in \Phi_i : P_{i,\alpha,j} \in \mathcal{H}_{i,j}, P_{i,\alpha,j} \cap K \neq \emptyset\big\}.$$

For each $\alpha \in \Gamma_{i,j}$, put $K_{i,\alpha,j} = P_{i,\alpha,j} \cap K$, $F_{i,j} = \overline{K - \bigcup_{\alpha\in\Gamma_{i,j}} K_{i,\alpha,j}}$. It is obvious that all $K_{i,\alpha,j}$ and $F_{i,j}$ are closed subset of K, and $K = F_{i,j} \cup \big(\bigcup_{\alpha\in\Gamma_{i,j}} K_{i,\alpha,j}\big)$. Now, we only need to show $F_{i,j} \subset A_{i,j}$. Let $x \in F_{i,j}$; then there exists a sequence $\{x_n\}$ of $K - \bigcup_{\alpha\in\Gamma_{i,j}} K_{i,\alpha,j}$ converging to x. If $P \in \mathcal{P}_x \cap \mathcal{H}_{i,j}$, then P is a sequential neighborhood of x and $P = P_{i,\alpha,j}$ for some $\alpha \in \Gamma_{i,j}$. Thus, $x_n \in P$ whenever $n \geqslant m$ for some $m \in \mathbb{N}$. Hence, $x_n \in K_{i,\alpha,j}$ for some $\alpha \in \Gamma_{i,j}$, a contradiction. So, $\mathcal{P}_x \cap \mathcal{H}_{i,j} = \emptyset$, and $x \in A_{i,j}$. This implies that $F_{i,j} \subset A_{i,j}$ and $\{A_{i,j}\} \cup \{P_{i,\alpha,j} : \alpha \in \Gamma_{i,j}\}$ is a cfp-cover of K.

(c) $\{\mathcal{H}_{i,j} : i, j \in \mathbb{N}\}$ is a point-star network for X. Let $x \in U$ with U open in X. Then, $x \in P \subset U$ for some $P \in \mathcal{P}_x$. Thus, there exists $i \in \mathbb{N}$ such that $P \in \mathcal{P}_i$. Hence, there exists a unique $\alpha \in \Phi_i$ such that $P \in \mathcal{Q}_{i,\alpha}$. So, $P = P_{i,\alpha,j} \in \mathcal{H}_{i,j}$ for some $j \in \mathbb{N}$. Since $P \in \mathcal{P}_x \cap \mathcal{H}_{i,j}$, $x \notin A_{i,j}$. Noting that $P \cap P_{i,\alpha,j} = \emptyset$ for all $j \neq i$. Therefore, $\mathrm{st}\big(x, \mathcal{H}_{i,j}\big) = P \subset U$.

Next, we write $\{\mathcal{H}_{m,n} : m, n \in \mathbb{N}\} = \{\mathcal{G}_i : i \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, put $\mathcal{G}_n = \{P_\alpha : \alpha \in \Lambda_n\}$ and endow Λ_n with the discrete topology. Then,

$$M = \Big\{\alpha = (\alpha_n) \in \prod\nolimits_{n\in\mathbb{N}} \Lambda_n : \{P_{\alpha_n}\} \text{ forms a network at some point } x_\alpha \in X\Big\}$$

is a metric space and the point x_α is unique in X for every $\alpha \in M$. Define $f : M \to X$ by $f(\alpha) = x_\alpha$. It follows [**20**, Lemma 13] that f is a compact-covering and compact map. On the other hand, we have

Claim 1. M is locally separable.

Let $a = (\alpha_i) \in M$. Then, $\{P_{\alpha_i}\}$ is a network at some point $x_a \in X$, and $x_a \in P$ for some $P \in \mathcal{P}_{x_a}$. Thus, there exists $m \in \mathbb{N}$ such that $P \in \mathcal{P}_m$. Hence, there exists a unique $\alpha \in \Phi_m$ such that $P \in \mathcal{Q}_{m,\alpha}$. Therefore, $P = P_{m,\alpha,n} \in \mathcal{H}_{m,n}$ for some $n \in \mathbb{N}$. Since $P \in \mathcal{P}_{x_a} \cap \mathcal{H}_{m,n}$, $x_a \notin A_{m,n}$. Noting that $P \cap P_{m,\alpha,n} = \emptyset$ for all $n \neq m$. This implies that $\mathrm{st}(x, \mathcal{H}_{m,n}) = P$. Then, $\mathcal{H}_{m,n} = \mathcal{G}_{i_0}$ for some $i_0 \in \mathbb{N}$ and $P = P_{\alpha_{i_0}}$. Thus, $P_{\alpha_{i_0}}$ is Lindelöf. Put

$$U_a = M \cap \Big\{(\beta_i) \in \prod\nolimits_{i\in\mathbb{N}} \Lambda_i : \beta_i = \alpha_i, i \leqslant i_0\Big\}.$$

Then, U_a is an open neighborhood of a in M. Now, for each $i \leqslant i_0$, put $\Delta_i = \{\alpha_i\}$, and for each $i > i_0$ we put $\Delta_i = \{\alpha \in \Lambda_i : P_\alpha \cap P_{\alpha_{i_0}} \neq \emptyset\}$. Then, $U_a \subset \prod_{i\in\mathbb{N}} \Delta_i$. Furthermore, since each $\mathcal{P}_i$ is locally finite and $P_{\alpha_{i_0}}$ is Lindelöf, Δ_i is countable for every $i > i_0$. Thus, U_a is separable, and M is locally separable.

Claim 2. f is an mssc-map.

Let $x \in X$. For each $n \in \mathbb{N}$, since $\mathcal{G}_n$ is locally finite, there is an open neighborhood V_n of x such that V_n intersects at most finite members of $\mathcal{G}_n$. Put

$$\Theta_n = \{\alpha \in \Lambda_n : P_\alpha \cap V_n \neq \emptyset\}.$$

Then, Θ_n is finite and $p_n f^{-1}(V_n) \subset \Theta_n$. Hence, $\overline{p_n f^{-1}(V_n)}$ is a compact subset of Λ_n, so f is an mssc-map.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6). It is obvious.

(6) $\Rightarrow$ (1). Let $f : M \to X$ be a sequentially-quotient π and mssc-map, where M be a locally separable metric space. By [**9**, Corollary 2.9], X has a point-star network $\{\mathcal{U}_n\}$, where each $\mathcal{U}_n$ is a cs*-cover. For each $n \in \mathbb{N}$, put $\mathcal{G}_n = \bigwedge_{i \leqslant n} \mathcal{U}_i$. Now, for each $x \in X$, let $\mathcal{G}_x = \{\mathrm{st}(x, \mathcal{G}_n) : n \in \mathbb{N}\}$. Since each $\mathcal{U}_n$ is a cs*-cover, it implies that $\bigcup\{\mathcal{G}_x : x \in X\}$ is an sn-network for X. Hence, X is an sn-first countable space. On the other hand, since f is a sequentially-quotient mssc-map, it follows from Lemma 2.1 that X has a σ-locally finite Lindelöf cs-network $\mathcal{P}$. We can assume that each $\mathcal{P}$ is closed under finite intersections. Then, each element of $\mathcal{P}$ is a cosmic subspace. By [**19**, Theorem 3.4], X has an so-cover consisting of $\aleph_0$-subspaces. Now, we only need to prove that X is an sn-metrizable space. In fact, since X is sn-first countable, X has an sn-network $\mathcal{Q} = \bigcup\{\mathcal{Q}_x : x \in X\}$ with each $\mathcal{Q}_x = \{Q_n(x) : n \in \mathbb{N}\}$ is a countable sn-network at x. For each $x \in X$, put $\mathcal{P}_x = \big\{P \in \mathcal{P} : Q_n(x) \subset P \text{ for some } n \in \mathbb{N}\big\}$. By using proof of [**26**, Lemma 7], we obtain $\mathcal{P}_x$ is an sn-network at x. Then, $\mathcal{G} = \bigcup\{\mathcal{P}_x : x \in X\}$ is an sn-network for X. Since $\mathcal{G} \subset \mathcal{P}$, it implies that $\mathcal{G}$ is σ-locally finite. Thus, X is an sn-metrizable space. □

By Theorem 2.1 and [**28**, Lemma 2.7(2)], we have

Corollary 2.1. *The following are equivalent for a space X.*

1. *X has a locally countable weak base;*
2. *X is a local $\aleph_0$-subspace and g-metrizable space;*
3. *X has a σ-locally finite Lindelöf weak base;*
4. *X is a compact-covering quotient compact and mssc-image of a locally separable metric space;*
5. *X is a pseudo-sequence-covering quotient compact and mssc-image of a locally separable metric space;*
6. *X is a subsequence-covering quotient compact and mssc-image of a locally separable metric space;*
7. *X is a quotient π and mssc-image of a locally separable metric space.*

Example 2.1. Let C_n be a convergent sequence containing its limit point p_n for each $n \in \mathbb{N}$, where $C_m \cap C_n = \emptyset$ if $m \neq n$. Let $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$ be the set of all rational numbers of the real line $\mathbb{R}$. Put $M = \big(\bigoplus\{C_n : n \in \mathbb{N}\}\big) \oplus \mathbb{R}$ and let X be the quotient space obtained from M by identifying each p_n in C_n with q_n in $\mathbb{R}$. Then, by the proof of [**12**, Example 3.1], X has a countable weak base and X is not a sequence-covering quotient π-image of a metric space. Hence,

1. A space with a σ-locally finite Lindelöf sn-network $\not\Rightarrow$ a sequence-covering π and mssc-image of a locally separable metric space.
2. A space with a σ-locally finite Lindelöf weak base $\not\Rightarrow$ a sequence-covering quotient π and mssc-image of a locally separable metric space.

Example 2.2. Using [**10**, Example 3.1], it is easy to see that X is Hausdorff, non-regular and X has a countable base, but it is not a sequentially-quotient

π-image of a metric space. This shows that regular properties of X can not be omitted in Theorem 2.1 and Corollary 2.1.

EXAMPLE 2.3. S_ω is a Fréchet and $\aleph_0$-space, but it is not first countable. Thus, S_ω has a σ-locally finite Lindelöf cs-network. It follows from [**3**, Theorem 2.1] that X is a sequence-covering mssc-image of a locally separable metric space. Furthermore, since S_ω is not first countable, it doesn't have a point-countable sn-network. Hence,

(1) A space with a σ-locally finite Lindelöf cs-network $\not\Rightarrow$ a sequentially-quotient π and mssc-image of a locally separable metric space.
(2) A sequence-covering quotient mssc-image of a locally separable metric space $\not\Rightarrow$ X has a σ-locally finite Lindelöf sn-network.

EXAMPLE 2.4. Using [**15**, Example 2.7], it is easy to see that X is a compact-covering quotient and compact image of a locally compact metric space, but it does not have a point-countable cs-network. Thus, a compact-covering quotient and compact image of a locally separable metric space $\not\Rightarrow$ X has a σ-locally finite Lindelöf sn-network.

EXAMPLE 2.5. There exists a space X having a locally countable sn-network, which is not an $\aleph$-space (see [**6**, Example 2.19]). Then, X has a σ-locally countable Lindelöf sn-network. Therefore,

(1) A space with a locally countable sn-network $\not\Rightarrow$ X has a σ-locally finite Lindelöf sn-network.
(2) A space with a σ-locally countable Lindelöf sn-network $\not\Rightarrow$ X has a σ-locally finite Lindelöf sn-network.

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References

1. A. V. Arhangel'skii, *Mappings and spaces*, Russian Math. Surveys **21**(4) (1966), 115–162.
2. J. R. Boone and F. Siwiec, *Sequentially quotient mappings*, Czech. Math. J. **26** (1976), 174–182.
3. N. V. Dung, *On sequence-covering mssc-images of locally separable metric spaces*, APN Acta Math. **87**(101) (2010), 143–153.
4. R. Engelking, *General Topology*, Sigma Ser. Pure Math. 6, Heldermann, Berlin, 1988
5. S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math. **57** (1965), 107–115.
6. X. Ge, *Spaces with a locally countable sn-network*, Lobachevskiĭ J. Math. **26** (2007), 33–49.
7. Y. Ge, *On sn-metrizable spaces*, Acta Math. Sinica **45** (2002), 355–360.
8. Y. Ge, *Characterizations of sn-metrizable spaces*, Publ. Inst. Math., Nouv. Ser. **74**(88) (2003), 121–128.
9. Y. Ge, *On pseudo-sequence-covering π-images of metric spaces*, Mat. Vesnik **57** (2005), 113–120.
10. Y. Ge, J. S. Gu, *On π-images of separable metric spaces*, Math. Sci. **10** (2004), 65–71.
11. Y. Ge, *$\aleph$-spaces and mssc-images of metric sapces*, J. Math. Res. Exposition **24** (2004), 198–202.
12. Y. Ge, S. Lin, *g-metrizable spaces and the images of semi-metric spaces*, Czech. Math. J. **57**(132) (2007), 1141–1149.

13. Y. Ikeda, C. Liu and Y. Tanaka, *Quotient compact images of metric spaces, and related matters*, Topology Appl. **122**(1–2) (2002), 237–252.
14. Z. Li, *A note on* $\aleph$*-spaces and* g*-metrizable spaces*, Czech. Math. J. **55** (2005), 803–808.
15. Z. Li, *On* π*-*s*-images of metric spaces*, Int. J. Math. Sci. **7** (2005), 1101–1107.
16. S. Lin, *On sequence-covering* s*-mappings*, Adv. Math. **25**(6) (1996), 548–551. (China)
17. S. Lin, *Locally countable collections, locally finite collections and Alexandroff's problems*, Acta Math. Sinica **37** (1994), 491–496 (in Chinese).
18. S. Lin, *Point-Countable Covers and Sequence-Covering Mappings*, Chinese Science Press, Beijing, 2002.
19. S. Lin and P. Yan, *Sequence-covering maps of metric spaces*, Topology Appl. **109** (2001) 301–314.
20. S. Lin, P. Yan, *Notes on cfp-covers*, Comment. Math. Univ. Carolin. **44** (2003), 295–306.
21. E. Michael, $\aleph_0$*-spaces*, J. Math. Mech. **15** (1966), 983–1002.
22. P. O'Meara, *On paracompactness in function spaces with the compact-open topology*, Proc. Amer. Math. Soc. **29** (1971), 183–189
23. M. Sakai, *On spaces with a star-countable* k*-networks*, Houston J. Math. **23**(1) (1997), 45–56.
24. F. Siwiec, *Sequence-covering and countably bi-quotient mappings*, General Topology Appl. **1** (1971), 143–153.
25. F. Siwiec, *On defining a space by a weak base*, Pacific J. Math. **52** (1974), 233–245.
26. S. Lin, Y. Tanaka, *Point-countable* k*-networks, closed maps, and related results*, Topology Appl. **59** (1994), 79–86.
27. Y. Tanaka and Z. Li, *Certain covering-maps and* k*-networks, and related matters*, Topology Proc. **27**(1) (2003), 317–334.
28. Y. Tanaka, Y. Ge, *Around quotient compact images of metric spaces, and symmetric spaces*, Houston J. Math. **32** (1) (2006), 99–117.
29. P. Yan, *On the compact images of metric spaces*, J. Math. Study **30** (1997), 185–187 (in Chinese).

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