PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 94 (108) (2013), 91–98 DOI: 10.2298/PIM1308091G

4-DIMENSIONAL (PARA)-KÄHLER–WEYL STRUCTURES

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Abstract. We give an elementary proof of the fact that any 4-dimensional para-Hermitian manifold admits a unique para-Kähler–Weyl structure. We then use analytic continuation to pass from the para-complex to the complex setting and thereby show that any 4-dimensional pseudo-Hermitian manifold also admits a unique Kähler–Weyl structure.

1. Introduction

1.1. Weyl manifolds. Let (M, g) be a pseudo-Riemannian manifold of dimension m. A triple (M, g, ∇) is said to be a *Weyl manifold* and ∇ is said to be a *Weyl connection* if ∇ is a torsion free connection with $\nabla g = -2\phi \otimes g$ for some smooth 1-form ϕ. This is a conformal theory; if $\tilde{g} = e^{2f} g$ is a conformally equivalent metric, then $(M, \tilde{g}, \nabla)$ is a Weyl manifold with associated 1-form $\tilde{\phi} = \phi - df$. If ∇^g is the Levi-Civita connection, we may then express $\nabla = \nabla^\phi$ in the form:

$$\nabla^\phi_x y = \nabla^g_x y + \phi(x)y + \phi(y)x - g(x, y)\phi^\# \tag{1.1}$$

where $\phi^\#$ is the dual vector field. Thus ϕ determines ∇. Conversely, if ϕ is given and if we use Equation (1.1) to define ∇, then ∇ is a Weyl connection with associated 1-form ϕ. We refer to [**5**] for further details concerning Weyl geometry.

1.2. Para-Hermitian manifolds. Let $m = 2\bar{m}$. A triple (M, g, J_+) is said to be an *almost para-Hermitian manifold* with an *almost para-complex structure* J_+ if g is a pseudo-Riemannian metric on M of neutral signature $(\bar{m}, \bar{m})$ and if J_+ is an endomorphism of the tangent bundle TM so that $J_+^2 = \mathrm{Id}$ and so that $J_+^* g = -g$; (M, g, J_+) is said to be *para-Hermitian* with an *integrable complex structure* J_+ if the *para-Nijenhuis* tensor

$$N_{J_+}(x, y) := [x, y] - J_+[J_+x, y] - J_+[x, J_+y] + [J_+x, J_+y]$$

2010 *Mathematics Subject Classification:* Primary 53B05; Secondary 15A72, 53A15, 53C07.
Key words and phrases: Weyl geometry, Hermitian manifold, para-Hermitian manifold, pseudo-Hermitian manifold, Kähler manifold, para-Kähler manifold, 4-dimensional geometry.
Research of the authors partially supported by project MTM2009-07756 (Spain), and by project 174012 (Serbia).

vanishes or, equivalently, if there are local coordinates $(u^1, \dots, u^{\bar m}, v^1, \dots, v^{\bar m})$ centered on an arbitrary point of M so that $J_+\partial_{u_i} = \partial_{v_i}$ and $J_+\partial_{v_i} = \partial_{u_i}$.

1.3. Pseudo-Hermitian manifolds. Let $m = 2\bar m$. A triple (M, g, J_-) is said to be an *almost pseudo-Hermitian manifold* with an *almost complex structure* J_- if (M, g) is a pseudo-Riemannian manifold, if J_- is an endomorphism of the tangent bundle so that $J_-^2 = -\operatorname{id}$ and so that $J_-^* g = g$; (M, g, J_-) is said to be a *pseudo-Hermitian* manifold with an *integrable complex structure* J_- if the *Nijenhuis* tensor

$$N_{J_-}(x, y) := [x, y] + J_-[J_-x, y] + J_-[x, J_-y] - [J_-x, J_-y]$$

vanishes or, equivalently, if there are local coordinates $(u^1, \dots, u^{\bar m}, v^1, \dots, v^{\bar m})$ centered on an arbitrary point of M so that $J_-\partial_{u_i} = \partial_{v_i}$ and $J_-\partial_{v_i} = -\partial_{u_i}$.

1.4. Para-Kähler and Kähler manifolds. One says that a Weyl connection ∇ on a para-Hermitian manifold (M, g, J_+) is a *para-Kähler–Weyl* connection if $\nabla J_+ = 0$. Similarly, one says that a Weyl connection ∇ on a pseudo-Hermitian manifold (M, g, J_-) is a *Kähler–Weyl* connection if $\nabla J_- = 0$. Since $\nabla J_\pm = 0$ implies $J_\pm$ to be integrable, we assume this condition henceforth. If $\nabla = \nabla^g$ is the Levi-Civita connection, then $(M, g, J_\pm)$ is said to be *(para)-Kähler*.

Let $\star$ be the Hodge operator and let $\Omega_\pm(x, y) := g(x, J_\pm y)$ be the (para)-Kähler form. The co-derivative $\delta\Omega_\pm$ is given, see [**2**] for example, by $\delta\Omega_\pm = -\star d \star \Omega_\pm$.

The following is well known – see, for example, the discussion in [**9**] of the Riemannian setting (which uses results of [**10, 11**]) and the generalization given in [**3**] to the more general context.

Theorem 1.1. *Let $m \geqslant 6$. If $(M, g, J_\pm, \nabla)$ is a (para)-Kähler–Weyl structure, then the associated Weyl structure is trivial, i.e., there is always locally a conformally equivalent metric $\tilde g = e^{2f} g$ so that $(M, \tilde g, J_\pm)$ is (para)-Kähler and so that $\nabla = \nabla^{\tilde g}$.*

By Theorem 1.1, only the 4-dimensional setting is relevant. The following is the main result of this short note; it plays a central role in the discussion of [**1**].

Theorem 1.2. (1) *If $\mathcal{M} = (M, g, J_+)$ is a para-Hermitian manifold of signature* $(2, 2)$*, then there is a unique para-Kähler–Weyl structure on $\mathcal{M}$ with $\phi = \frac{1}{2} J_+ \delta\Omega_+$.*

(2) *If $\mathcal{M} = (M, g, J_-)$ is a pseudo-Hermitian manifold of signature* $(2, 2)$*, then there is a unique Kähler–Weyl structure on $\mathcal{M}$ with $\phi = -\frac{1}{2} J_- \delta\Omega_-$.*

(3) *If $\mathcal{M} = (M, g, J_-)$ is a Hermitian manifold of signature* $(0, 4)$*, then there is a unique Kähler–Weyl structure on $\mathcal{M}$ with $\phi = -\frac{1}{2} J_- \delta\Omega_-$.*

Assertion (3) of Theorem 1.2, which deals with the Hermitian setting, is well known – see, for example, the discussion in [**8**]. Subsequently, Theorem 1.2 was established in full generality (see [**3, 4**]) by extending the Higa curvature decomposition [**6, 7**] from the real to the Kähler–Weyl and to the para-Kähler Weyl contexts.

Here is a brief outline to this paper. In Section 2, we show that if a (para)-Kähler–Weyl structure exists, then it is unique. In Section 3, we give a direct proof of Assertion (1) of Theorem 1.2 in the para-Hermitian setting. In Section 4, we use analytic continuation to derive Assertions (2) and (3), which deal with the complex setting, from Assertion (1). This reverses the usual procedure of viewing para-complex geometry setting as an adjunct to complex geometry and is a novel feature of this paper.

2. Uniqueness of the (para)-Kähler–Weyl structure

This section is devoted to the proof of the following uniqueness result.

LEMMA 2.1. (1) *If* ∇^{ϕ_1} *and* ∇^{ϕ_2} *are two para-Kähler–Weyl connections on a* 4*-dimensional para-Hermitian manifold* (M, g, J_+)*, then* $\phi_1 = \phi_2$*.*

(2) *If* ∇^{ϕ_1} *and* ∇^{ϕ_2} *are two Kähler–Weyl connections on a* 4*-dimensional pseudo-Hermitian manifold* (M, g, J_-)*, then* $\phi_1 = \phi_2$*.*

PROOF. Let $\phi = \phi_1 - \phi_2$ and let $\Theta_X(Y) = \phi(X)Y + \phi(Y)X - g(X, Y)\phi^{\#}$. By Equation (1.1), $\nabla_X^{\phi_1} - \nabla_X^{\phi_2} = \Theta_X \in \mathrm{End}(TM)$. Consequently, $\{\nabla^{\phi_1} - \nabla^{\phi_2}\}J_\pm = 0$ implies $[\Theta_X, J_\pm] = 0$ for all X.

We first deal with the para-Hermitian case. This is a purely algebraic computation. Let $\{e_1, e_2, e_3, e_4\}$ be a local frame for TM so that

$$\begin{gathered} J_+e_1 = e_1, \quad J_+e_2 = e_2, \quad J_+e_3 = -e_3, \quad J_+e_4 = -e_4, \\ g(e_1, e_3) = g(e_2, e_4) = 1. \end{gathered} \tag{2.1}$$

We expand $\phi = a_1e^1 + a_2e^2 + a_3e^3 + a_4e^4$ and compute:

$$\begin{aligned} &\Theta_{e_1}e_4 = a_1e_4 + a_4e_1, && J_+\Theta_{e_1}e_4 = -a_1e_4 + a_4e_1, && \Theta_{e_1}J_+e_4 = -a_1e_4 - a_4e_1, \\ &\Theta_{e_2}e_3 = a_2e_3 + a_3e_2, && J_+\Theta_{e_2}e_3 = -a_2e_3 + a_3e_2, && \Theta_{e_2}J_+e_3 = -a_2e_3 - a_3e_2, \\ &\Theta_{e_4}e_1 = a_4e_1 + a_1e_4, && J_+\Theta_{e_4}e_1 = a_4e_1 - a_1e_4, && \Theta_{e_4}J_+e_1 = a_4e_1 + a_1e_4, \\ &\Theta_{e_3}e_2 = a_3e_2 + a_2e_3, && J_+\Theta_{e_3}e_2 = a_3e_2 - a_2e_3, && \Theta_{e_3}J_+e_2 = a_3e_2 + a_2e_3. \end{aligned}$$

Equating $\Theta_{e_i}J_+e_j$ with $J_+\Theta_{e_i}e_j$ then implies $a_1 = a_2 = a_3 = a_4 = 0$ so $\phi = 0$ and $\phi_1 = \phi_2$. This establishes Assertion (1).

Next assume we are in the pseudo-Hermitian setting. Complexify and extend g to be complex bilinear. Choose a local frame $\{Z_1, Z_2, \bar{Z}_1, \bar{Z}_2\}$ for $TM \otimes_{\mathbb{R}} \mathbb{C}$ so

$$\begin{aligned} &J_-Z_1 = \sqrt{-1}Z_1, && J_-Z_2 = \sqrt{-1}Z_2, \\ &J_-\bar{Z}_1 = -\sqrt{-1}\bar{Z}_1, && J_-\bar{Z}_2 = -\sqrt{-1}\bar{Z}_2, \\ &g(Z_1, \bar{Z}_1) = 1, && g(Z_2, \bar{Z}_2) = \varepsilon_2 \end{aligned}$$

where we take $\varepsilon_2 = +1$ in signature $(0, 4)$ and $\varepsilon_2 = -1$ in signature $(2, 2)$. We set $J_+ := -\sqrt{-1}J_-$, $e_1 := Z_1$, $e_2 := Z_2$, $e_3 := \bar{Z}_1$, and $e_4 := \varepsilon_2\bar{Z}_2$ and apply the argument given to prove Assertion (1) (where the coefficients a_i are now complex) to derive Assertion (2). ∎

3. Para-Hermitian geometry

3.1. The algebraic context. Let $(V, \langle\cdot,\cdot\rangle, J_+)$ be a para-Hermitian vector space of dimension 4. Here $\langle\cdot,\cdot\rangle$ is an inner product on V of signature $(2,2)$ and J_+ is an endomorphism of V satisfying $J_+^2 = \mathrm{Id}$ and $J_+^*\langle\cdot,\cdot\rangle = -\langle\cdot,\cdot\rangle$. We may then choose a basis $\{e_1, e_2, e_3, e_4\}$ for $V = \mathbb{R}^4$ so that the relations of Equation (2.1) are satisfied. The Kähler form and orientation μ are then given by

$$\Omega_+ = -e^1\wedge e^3 - e^2\wedge e^4 \quad\text{and}\quad \mu = \tfrac{1}{2}\Omega_+\wedge\Omega_+ = e^1\wedge e^3\wedge e^2\wedge e^4.$$

Let $\star$ be the *Hodge operator*, characterized by

$$\omega_1\wedge\star\omega_2 = \langle\omega_1,\omega_2\rangle e^1\wedge e^3\wedge e^2\wedge e^4 \text{ for all } \omega_i.$$

Consequently:

$$\begin{aligned}
&\star\, e^1\wedge e^3 = -e^2\wedge e^4, &&\star\, e^2\wedge e^4 = -e^1\wedge e^3,\\
&\star\, e^1\wedge e^2\wedge e^3 = -e^2, &&\star\, e^1\wedge e^2\wedge e^4 = \ \ e^1,\\
&\star\, e^1\wedge e^3\wedge e^4 = -e^4, &&\star\, e^2\wedge e^3\wedge e^4 = \ \ e^3.
\end{aligned}\tag{3.1}$$

3.2. Example. We begin the proof of Theorem 1.2 by considering a very specific example. Let $(x^1, x^2, x^3, x^4\}$ be the usual coordinates on $\mathbb{R}^4$, let $\partial_i := \partial_{x_i}$, and let J_+ be the standard para-complex structure:

$$J_+\partial_1 = \partial_1,\quad J_+\partial_2 = \partial_2,\quad J_+\partial_3 = -\partial_3,\quad J_+\partial_4 = -\partial_4.$$

Let $f(0) = 0$. We take the metric to have non-zero components determined by $g(\partial_1,\partial_3) = 1$ and $g(\partial_2,\partial_4) = e^{2f}$. Let $f_i := \{\partial_i f\}(0)$. The (possibly) non-zero Christoffel symbols of ∇^g at the origin are given by:

$$\begin{gathered}
g(\nabla^g_{\partial_1}\partial_2,\partial_4) = g(\nabla^g_{\partial_2}\partial_1,\partial_4) = g(\nabla^g_{\partial_1}\partial_4,\partial_2) = g(\nabla^g_{\partial_4}\partial_1,\partial_2) = f_1,\\
g(\nabla^g_{\partial_3}\partial_2,\partial_4) = g(\nabla^g_{\partial_2}\partial_3,\partial_4) = g(\nabla^g_{\partial_3}\partial_4,\partial_2) = g(\nabla^g_{\partial_4}\partial_3,\partial_2) = f_3,\\
g(\nabla^g_{\partial_4}\partial_4,\partial_2) = 2f_4,\quad g(\nabla^g_{\partial_2}\partial_2,\partial_4) = 2f_2,\\
g(\nabla^g_{\partial_2}\partial_4,\partial_1) = g(\nabla^g_{\partial_4}\partial_2,\partial_1) = -f_1,\quad g(\nabla^g_{\partial_2}\partial_4,\partial_3) = g(\nabla^g_{\partial_4}\partial_2,\partial_3) = -f_3.
\end{gathered}$$

Consequently the (possibly) non-zero covariant derivatives at the origin are:

$$\begin{gathered}
\nabla^g_{\partial_1}\partial_2 = \nabla^g_{\partial_2}\partial_1 = f_1\partial_2,\quad \nabla^g_{\partial_1}\partial_4 = \nabla^g_{\partial_4}\partial_1 = f_1\partial_4,\\
\nabla^g_{\partial_3}\partial_2 = \nabla^g_{\partial_2}\partial_3 = f_3\partial_2,\quad \nabla^g_{\partial_3}\partial_4 = \nabla^g_{\partial_4}\partial_3 = f_3\partial_4,\\
\nabla^g_{\partial_4}\partial_4 = 2f_4\partial_4,\quad \nabla^g_{\partial_2}\partial_2 = 2f_2\partial_2,\\
\nabla^g_{\partial_2}\partial_4 = \nabla^g_{\partial_4}\partial_2 = -f_1\partial_3 - f_3\partial_1.
\end{gathered}$$

Since $\nabla^g_{\partial_1}$ and $\nabla^g_{\partial_3}$ are diagonal, they commute with J_+ so $\nabla^g_{\partial_1}(J_+) = \nabla^g_{\partial_3}(J_+) = 0$. We compute

$$\begin{aligned}
&(\nabla^g_{\partial_2}J_+)\partial_1 = (1-J_+)\nabla^g_{\partial_2}\partial_1 = (1-J_+)f_1\partial_2 = 0,\\
&(\nabla^g_{\partial_2}J_+)\partial_2 = (1-J_+)\nabla^g_{\partial_2}\partial_2 = (1-J_+)2f_2\partial_2 = 0,\\
&(\nabla^g_{\partial_2}J_+)\partial_3 = (-1-J_+)\nabla^g_{\partial_2}\partial_3 = (-1-J_+)f_3\partial_2 = -2f_3\partial_2,\\
&(\nabla^g_{\partial_2}J_+)\partial_4 = (-1-J_+)\nabla^g_{\partial_2}\partial_4 = (-1-J_+)(-f_1\partial_3 - f_3\partial_1) = 2f_3\partial_1,
\end{aligned}$$

$$\begin{aligned}
&(\nabla^g_{\partial_4}J_+)\partial_1=(1-J_+)\nabla^g_{\partial_4}\partial_1=(1-J_+)f_1\partial_4=2f_1\partial_4,\\
&(\nabla^g_{\partial_4}J_+)\partial_2=(1-J_+)\nabla^g_{\partial_4}\partial_2=(1-J_+)(-f_1\partial_3-f_3\partial_1)=-2f_1\partial_3,\\
&(\nabla^g_{\partial_4}J_+)\partial_3=(-1-J_+)\nabla^g_{\partial_4}\partial_3=(-1-J_+)f_3\partial_4=0,\\
&(\nabla^g_{\partial_4}J_+)\partial_4=(-1-J_+)\nabla^g_{\partial_4}\partial_4=(-1-J_+)2f_4\partial_4=0.
\end{aligned}$$

We apply Equation (3.1). We have $\star\Omega_+=-\Omega_+$. Setting $e^1=dx^1$, $e^2=e^f dx^2$, $e^3=dx^3$, and $e^4=e^f dx^4$ and recalling $f(0)=0$ yields

$$\begin{gathered}
\star\Omega_+=-\Omega_+=dx^1\wedge dx^3+e^{2f}dx^2\wedge dx^4,\\
d\star\Omega_+=2f_1dx^1\wedge dx^2\wedge dx^4-2f_3dx^2\wedge dx^3\wedge dx^4,\\
\delta_g\Omega_+(0)=-\star d\star\Omega_+(0)=-2f_1dx^1+2f_3dx^3,\\
\phi(0)=\tfrac{1}{2}J\delta_g\Omega_+=-f_1dx^1-f_3dx^3,\ \text{and}\ \phi^\#=-f_1\partial_3-f_3\partial_1.
\end{gathered}$$

Let $\Theta_{ij}:=\phi(\partial_i)\partial_j+\phi(\partial_j)\partial_i-g(\partial_i,\partial_j)\phi^\#=(\nabla^\phi-\nabla^g)_{\partial_i}\partial_j$ at 0. Then:

$$\begin{gathered}
\Theta_{11}=-2f_1\partial_1,\ \ \Theta_{12}=-f_1\partial_2,\ \ \Theta_{13}=(-f_1\partial_3-f_3\partial_1)+(f_1\partial_3+f_3\partial_1)=0,\\
\Theta_{14}=-f_1\partial_4,\quad \Theta_{22}=0,\ \ \Theta_{23}=-f_3\partial_2,\ \ \Theta_{24}=(f_1\partial_3+f_3\partial_1),\\
\Theta_{33}=-2f_3\partial_3,\ \ \Theta_{34}=-f_3\partial_4,\quad \Theta_{44}=0.
\end{gathered}$$

Since $\Theta(\partial_1)$ and $\Theta(\partial_3)$ are diagonal, $[\Theta(\partial_1),J_+]=[\Theta(\partial_3),J_+]=0$. We compute:

$$\begin{aligned}
&[\Theta(\partial_2),J_+]\partial_1=(1-J_+)\Theta_{12}=0,\\
&[\Theta(\partial_2),J_+]\partial_2=(1-J_+)\Theta_{22}=0,\\
&[\Theta(\partial_2),J_+]\partial_3=(-1-J_+)\Theta_{23}=2f_3\partial_2,\\
&[\Theta(\partial_2),J_+]\partial_4=(-1-J_+)\Theta_{24}=-2f_3\partial_1,\\
&[\Theta(\partial_4),J_+]\partial_1=(1-J_+)\Theta_{14}=-2f_1\partial_4,\\
&[\Theta(\partial_4),J_+]\partial_2=(1-J_+)\Theta_{24}=2f_1\partial_3,\\
&[\Theta(\partial_4),J_+]\partial_3=(-1-J_+)\Theta_{34}=0,\\
&[\Theta(\partial_4),J_+]\partial_4=(-1-J)\Theta_{44}=0.
\end{aligned}$$

We now observe that $[\nabla^g,J_+]+[\Theta,J_+]=0$. Consequently $\nabla^\phi J_+=0$ for this metric and Assertion (1) of Theorem 1.2 holds in this special case.

Proof of Theorem 1.2 (1). Let $V=\mathbb{R}^4$, let S^2_- be the vector space of symmetric 2-cotensors ω so that $J_+^*\omega=-\omega$, and let $\varepsilon\in C^\infty(S^2)$ satisfy $\varepsilon(0)=0$. We use ε to define a perturbation of the flat metric by setting:

$$g=dx^1\circ dx^3+dx^2\circ dx^4+\varepsilon.$$

This is non-degenerate near the origin. Since only the 1-jets of ε are relevant in examining $\nabla^\phi(J_+)(0)$, this is a linear problem and we may take $\varepsilon\in S^2_-\otimes V^\star$ so:

$$g=g_0+\sum_i x^i\varepsilon(e_i).$$

Then $\varepsilon \to (\nabla^\phi J_+)(0)$ defines a linear map

$$\mathcal{E}: S_-(V)\otimes V^* \to \mathrm{End}(V)\otimes V^* \quad \text{or equivalently}$$
$$\mathcal{E}: S_-(V) \to \mathrm{Hom}(V^*, \mathrm{End}(V)\otimes V^*).$$

The analysis of Section 3.2 shows that $\mathcal{E}(dx^2 \circ dx^4) = 0$. Permuting the indices $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ then yields $\mathcal{E}(dx^1 \circ dx^3) = 0$. The question is invariant under the action of the para-unitary group; we must preserve J_+ and we must preserve the inner product at the origin. Define a unitary transformation T by setting:

$$T(e^1) = e^1 + ae^2, \quad T(e^2) = e^2, \quad T(e^3) = e^3, \quad T(e^4) = e^4 - ae^3.$$

Then

$$T(e^1 \circ e^3) = e^1 \circ e^3 + ae^2 \circ e^3.$$

Consequently, $\mathcal{E}(e^2 \circ e^3) = 0$. Permuting the indices $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ then yields $\mathcal{E}(e^1 \circ e^4) = 0$. Since $S_- = \mathrm{Span}\{e^1 \circ e^3, e^1 \circ e^4, e^2 \circ e^3, e^2 \circ e^4\}$, we see that $\mathcal{E} = 0$ in general; this completes the proof of Assertion (1) of Theorem 1.2. □

4. Hermitian and pseudo-Hermitian manifolds

In this section, we will use analytic continuation to derive Theorem 1.2 in the complex setting from Theorem 1.2 in the para-complex setting. Let $V = \mathbb{R}^4$ with the usual basis $\{e_1, e_2, e_3, e_4\}$ and coordinates $\{x^1, x^2, x^3, x^4\}$, where we expand $v = x^1e_1 + x^2e_2 + x^3e_3 + x^4e_4$. Let S^2 denote the space of symmetric 2-tensors. We complexify and consider

$$\mathcal{S} := \left\{S^2 \otimes_{\mathbb{R}} \mathbb{C}\right\} \oplus \left\{(V^* \otimes_{\mathbb{R}} S^2) \otimes_{\mathbb{R}} \mathbb{C}\right\}.$$

Let $J_+ \in M_2(\mathbb{C})$ be a complex 2×2 matrix with $J_+^2 = \mathrm{Id}$ and $\mathrm{Tr}(J_+) = 0$. Let

$$\mathcal{S}(J_+) := \{(g_0, g_1) \in \mathcal{S} : \det(g_0 - J_+^* g_0) \neq 0\}. \tag{4.1}$$

For $(g_0, g_1) \in \mathcal{S}(J_+)$, define:

$$\begin{aligned} g(x)(X,Y) := {}& \tfrac12\big\{g_0(X,Y) - g_0(J_+X, J_+Y)\big\} \\ & + \sum_{i=1}^4 x^i \cdot \tfrac12\big\{g_1(e_i, X, Y) - g_1(e_i, J_+X, J_+Y)\big\}. \end{aligned}$$

By Equation (4.1), this is non-degenerate at 0 and defines a complex metric on some neighborhood of 0 so $J_+^* g = -g$. Let ∇^g be the complex Levi-Civita connection:

$$\nabla^g_{\partial_i}\partial_j = \tfrac12 g^{kl}\{\partial_i g_{jl} + \partial_j g_{il} - \partial_{x_l} g_{ij}\}\partial_k.$$

Then ∇^g is a torsion free connection on $T_{\mathbb{C}}M := T_M \otimes_{\mathbb{R}} \mathbb{C}$. The para-Kähler form is defined by setting $\Omega_+(x,y) = g(x, J_+y)$ and we have

$$\delta\Omega_+ = \star d\Omega_+ \text{ and } \phi := \tfrac12 J_+ \delta_g \Omega.$$

We then use ϕ to define a complex Weyl connection ∇^ϕ on $T_{\mathbb{C}}M$ and define a holomorphic map from $\mathcal{S}(J_+)$ to $\mathfrak{V} := V^* \otimes M_4(\mathbb{C})$ by setting

$$\mathcal{E}(g_0, g_1; J_+) := \nabla^\phi(J_+)|_{x=0}.$$

LEMMA 4.1. *Let* $J_+ \in M_4(\mathbb{C})$ *with* $J_+^2 = \mathrm{id}$ *and* $\mathrm{Tr}(J_+) = 0$. *Suppose that* $(g_0, g_1) \in \mathcal{S}(J_+)$.

(1) *If* J_+ *is real and if* (g_0, g_1) *is real, then* $\mathcal{E}(g_0, g_1; J_+) = 0$.
(2) *If* J_+ *is real and if* (g_0, g_1) *is complex, then* $\mathcal{E}(g_0, g_1; J_+) = 0$.
(3) *If* J_+ *is complex and if* (g_0, g_1) *is complex, then* $\mathcal{E}(g_0, g_1; J_+) = 0$.

PROOF. Assertion (1) follows from Theorem 1.2 (1). We argue as follows to prove Assertion (2). $\mathcal{S}(J_+)$ is an open dense subset of $\mathcal{S}$ and inherits a natural holomorphic structure thereby. Assume that J_+ is real. The map $\mathcal{E}$ is a holomorphic map from $\mathcal{S}(J_+)$ to $\mathfrak{V}$. By Assertion (1), $\mathcal{E}(g_0, g_1; J_+)$ vanishes if (g_0, g_1) is real. Thus, by the identity theorem, $\mathcal{E}(g_0, g_1; J_+)$ vanishes for all $(g_0, g_1) \in \mathcal{S}_{J_+}$. This establishes Assertion (2) by removing the assumption that (g_0, g_1) is real.

We complete the proof by removing the assumption that J_+ is real. The general linear group $\mathrm{GL}_4(\mathbb{C})$ acts on the structures involved by change of basis (i.e., conjugation). Let $(g_0, g_1) \in \mathcal{S}(J_+)$ where J_+ is real and $\mathrm{Tr}(J_+) = 0$. We consider the real and complex orbits

$$\mathcal{O}_{\mathbb{R}}(g_0, g_1; J_+) := \mathrm{GL}_4(\mathbb{R}) \cdot (g_0, g_1; J_+),$$
$$\mathcal{O}_{\mathbb{C}}(g_0, g_1; J_+) := \mathrm{GL}_4(\mathbb{C}) \cdot (g_0, g_1; J_+).$$

Let $\mathcal{F}(A) := \mathcal{E}(A \cdot (g_0, g_1; J_+))$ define a holomorphic map from $\mathrm{GL}_4(\mathbb{C})$ to $\mathfrak{V}$. By Assertion (2), $\mathcal{F}$ vanishes on $\mathrm{GL}_4(\mathbb{R})$. Thus by the identity theorem, $\mathcal{F}$ vanishes on $\mathrm{GL}_4(\mathbb{C})$ or, equivalently, $\mathcal{E}$ vanishes on the orbit space $\mathcal{O}_{\mathbb{C}}(g_0, g_1; J_+)$. Given any $J_+ \in M_4(\mathbb{C})$ with $J_+^2 = \mathrm{Id}$ and $\mathrm{Tr}(J_+) = 0$, we can choose $A \in \mathrm{GL}_4(\mathbb{C})$ so that $A \cdot J_+$ is real. The general case now follows from Assertion (2). □

Proof of Theorem 1.2 (2,3). Let (M, g, J_-) be a 4-dimensional pseudo-Hermitian manifold of dimension 4. Fix a point P of M. Since J_- is integrable, we may choose local coordinates (x^1, x^2, x^3, x^4) so the matrix of J_- relative to the coordinate frame $\{\partial_i\}$ is constant. Define a Weyl connection with associated 1-form given by $\phi = -\frac{1}{2} J_- \delta \Omega_-$. Only the 0 and the 1-jets of the metric play a role in the computation of $(\nabla^\phi J_-)(P)$. So we may assume $g = g(g_0, g_1)$. We set $J_+ = \sqrt{-1}\, J_-$. We have that

$$J_+^2 = \sqrt{-1}\, J_- \sqrt{-1}\, J_- = -J_-^2 = \mathrm{id}, \quad \mathrm{Tr}(J_+) = \sqrt{-1}\ \mathrm{Tr}(J_-) = 0,$$
$$J_+^\star(g)(X, Y) = g\big(\sqrt{-1}\, J_- X, \sqrt{-1}\, J_- Y\big) = -g(J_- X, J_- Y) = -g(X, Y)$$

so $J_+^\star(g) = -g$ and $(g_0, g_1) \in \mathcal{S}_{J_+}$. Finally, since $J_- = -\sqrt{-1}\, J_+$, we have

$$\Omega_- = -\sqrt{-1}\, \Omega_+,$$
$$\phi_{J_-} = -\tfrac{1}{2} J_- \delta_g \Omega_- = -\tfrac{1}{2}\big(-\sqrt{-1}\, J_+\big)\delta_g\big(-\sqrt{-1}\, \Omega_+\big) = \tfrac{1}{2} J_+ \delta_g \Omega_+ = \phi_{J_+}.$$

We apply Lemma 4.1 to complete the proof. □

References

1. M. Brozos-Vázquez, E. García-Río, P. Gilkey, R. Vázquez-Lorenzo, *Homogeneous 4-dimensional Kähler–Weyl Structures*, Results Math. (to appear); DOI: 10.1007/s00025-013-0319-5

2. P. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem*, CRC Press, 1994.
3. P. Gilkey, S. Nikčević, *Kähler-Weyl manifolds of dimension* 4, Rendiconti del Seminario Matematico (to appear) (see arXiv:1109.4532).
4. ______, *(para)-Kähler Weyl structures*; in: M. Sanchez, M. Ortega, A. Romero (eds.), *Recent Trends in Lorentzian Geometry*, Springer-Verlag, 2013, 335–353.
5. P. Gilkey, S. Nikčević, U. Simon, *Geometric realizations, curvature decompositions, and Weyl manifolds*, J. Geom. Phys. **61** (2011), 270–275.
6. T. Higa, *Weyl manifolds and Einstein-Weyl manifolds*, Comm. Math. Univ. St. Pauli **42** (1993), 143–160.
7. ______, *Curvature tensors and curvature conditions in Weyl geometry*, Comm. Math. Univ. St. Pauli **43** (1994), 139–153.
8. G. Kokarev, D. Kotschick, *Fibrations and fundamental groups of Kähler-Weyl manifolds*, Proc. Am. Math. Soc. **138** (2010), 997–1010.
9. H. Pedersen, Y. Poon, A. Swann, *The Einstein-Weyl equations in complex and quaternionic geometry*, Diff. Geom. Appl. **3** (1993), 309–321.
10. I. Vaisman, *Generalized Hopf manifolds*, Geom. Dedicata **13** (1982), 231–255.
11. ______, *A survey of generalized Hopf manifolds* in: *Differential Geometry on Homogeneous Spaces*, Proc. Conf. Torino Italy (1983), Rend. Semin. Mat. Torino, Fasc. Spec. 205–221.

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