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C_∞-STRUCTURE ON THE COHOMOLOGY OF THE FREE 2-NILPOTENT LIE ALGEBRA

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Abstract. We consider the free 2-step nilpotent Lie algebra and its cohomology ring. The homotopy transfer induces a homotopy commutative algebra on its cohomology ring which we describe. We show that this cohomology is generated in degree 1 as C_∞-algebra only by the induced binary and ternary operations.

1. Homotopy algebras

The homotopy associative algebras, or A_∞-algebras were introduced by Jim Stasheff in the 1960's as a tool in algebraic topology for studying 'group-like' spaces. Homotopy algebras received a new attention and further development in the 1990's after the discovery of their relevance into a multitude of topics in algebraic geometry, symplectic and contact geometry, knot theory, moduli spaces and deformation theory.

Definition 1.1. (A_∞-algebra) A homotopy associative algebra, or A_∞-algebra, over a field $\mathbb{K}$ is a $\mathbb{Z}$-graded vector space $A = \bigoplus_{i\in\mathbb{Z}} A^i$ endowed with a family of graded mappings (operations) $m_n : A^{\otimes n} \to A$, $\deg(m_n) = 2 - n$, $n \geqslant 1$ satisfying the Stasheff identities $\mathbf{SI(n)}$ for $n \geqslant 1$

$$\mathbf{SI(n)}: \quad \sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(\mathrm{Id}^{\otimes r} \otimes m_s \otimes \mathrm{Id}^{\otimes t}) = 0 \quad r \geqslant 0,\ t \geqslant 0,\ s \geqslant 1,$$

where the sum runs over all decompositions $n = r + s + t$. Throughout the text we assume the Koszul sign convention $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$.

A morphism of two A_∞-algebras A and B is a family of graded maps $f_n : A^{\otimes n} \to B$ for $n \geqslant 1$ with $\deg f_n = 1 - n$ such that the following conditions hold

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(\mathrm{Id}^{\otimes r} \otimes m_s \otimes \mathrm{Id}^{\otimes t}) = \sum_{1\leqslant r\leqslant n} (-1)^S m_r(f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_r})$$

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where the sum is over all decompositions $i_1 + \cdots + i_r = n$ and the sign $(-1)^S$ on the right-hand side is determined by

$$S = (r-1)(i_1 - 1) + (r-2)(i_2 - 1) + \cdots + 2(i_{r-2} - 1) + (i_{r-1} - 1).$$

The morphism f is a *quasi-isomorphism of A_∞-algebras* if f_1 is a quasi-isomorphism. It is strict if $f_i = 0$ for all $i \neq 1$. The identity morphism of A is the strict morphism f such that f_1 is the identity of A.

We define the shuffle product $\mathrm{Sh}_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A^{\otimes p+q}$ by

$$(a_1 \otimes \cdots \otimes a_p) \sqcup (a_{p+1} \otimes \cdots \otimes a_{p+q}) = \sum_{\sigma \in \mathrm{Sh}_{p,q}} \pm \operatorname{sgn}(\sigma)\, a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(p+q)}$$

where the sum runs over all (p,q)-shuffles $\mathrm{Sh}_{p,q}$, i.e., over all permutations $\sigma \in S_{p+q}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q)$ and the signs $\pm$ on the right-hand side are fixed from the cohomological degrees $\hat{a}_i$ of the elements a_i according to the place permutation action in the tensor powers of graded spaces.

Definition 1.2. (C_∞-algebra [**10**]) A homotopy commutative algebra, or C_∞-algebra, is an A_∞-algebra $\{A, m_n\}$ such that each operation m_n vanishes on non-trivial shuffles $m_n((a_1 \otimes \cdots \otimes a_p) \sqcup (a_{p+1} \otimes \cdots \otimes a_n)) = 0,\ \ 1 \leqslant p \leqslant n-1$.

In particular for m_2 we have $m_2(a \otimes b - (-1)^{\hat{a}\hat{b}} b \otimes a) = 0$, so a C_∞-algebra such that $m_n = 0$ for $n \geqslant 3$ is a (super-)commutative DGA.

A morphism of C_∞-algebras is a morphism of A_∞-algebras vanishing on non-trivial shuffles $f_n((a_1 \otimes \cdots \otimes a_p) \sqcup (a_{p+1} \otimes \cdots \otimes a_n)) = 0,\ \ 1 \leqslant p \leqslant n-1$.

2. Homotopy transfer theorem

Lemma 2.1. *Every cochain complex (A,d) of vector spaces over a field $\mathbb{K}$ has its cohomology $H^\bullet(A)$ as a deformation retract.*

One can always choose a vector space decomposition of the cochain complex (A,d) such that $A^n \cong B^n \oplus H^n \oplus B^{n+1}$ where H^n is the cohomology and B^n is the space of coboundaries, $B^n = dA^{n-1}$. We choose a homotopy $h : A^n \to A^{n-1}$ which identifies B^n with its copy in A^{n-1} and is 0 on $H^n \oplus B^{n+1}$. The projection p to the cohomology and the cocycle-choosing inclusion i given by $A^n \underset{i}{\overset{p}{\rightleftarrows}} H^n$ are chain homomorphisms, satisfying the additional *side conditions*: $hh = 0$, $hi = 0$, $ph = 0$. With these choices done the complex $(H^\bullet(A), 0)$ is a deformation retract of (A,d)

$$h \circlearrowright (A,d) \underset{i}{\overset{p}{\rightleftarrows}} (H^\bullet(A), 0)\ , \quad pi = \mathrm{Id}_{H^\bullet(A)}, \quad ip - \mathrm{Id}_A = dh + hd.$$

Let now (A, d, μ) be a DGA, i.e., A is endowed with an associative product μ compatible with d. The cochain complexes (A,d) and its contraction $H^\bullet(A)$ are homotopy equivalent, but the associative structure is not stable under homotopy equivalence. However the associative structure on A can be transferred to an A_∞-structure on a homotopy equivalent complex, a particular interesting complex being the deformation retract $H^\bullet(A)$. For a friendly introduction to homotopy transfer theorems in much broader context we refer the reader to the textbook [**14**, Chapter 9].

THEOREM 2.1 (Kadeishvili [**10**]). *Let (A, d, μ) be a (commutative) DGA over a field $\mathbb{K}$. There exists an A_∞-algebra (C_∞-algebra) structure on the cohomology $H^\bullet(A)$ and an $A_\infty(C_\infty)$-quasi-isomorphism*

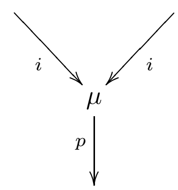
$$f_k : (\otimes^k H^\bullet(A), \{m_j\}) \to (A, \{d, \mu, 0, 0, \dots\})$$

such that the inclusion $f_1 = i : H^\bullet(A) \to A$ is a cocycle-choosing homomorphism of cochain complexes. The differential m_1 on $H^\bullet(A)$ is zero ($m_1 = 0$) and m_2 is the strictly associative operation induced by the multiplication on A. The resulting structure is unique up to quasi-isomorphism.

Kontsevich and Soibelman [**12**] gave explicit expressions for the higher operations of the induced A_∞-structure as sums over decorated planar binary trees with one root where all leaves are decorated by the inclusion i, the root by the projection p, the vertices by the product μ of the (commutative) DGA (A, d, μ) and the internal edges by the homotopy h. The C_∞-structure implies additional symmetries on trees.

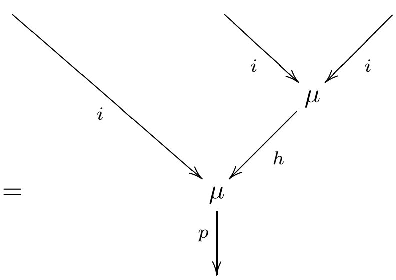
For instance the operation m_2 of the induced A_∞-structure on $H^\bullet(A)$ looks like

$m_2(x, y) := p\mu(i(x), i(y))$ or $m_2 =$



and the ternary one $m_3(x, y, z) = p\mu\big(i(x), h\mu(i(y), i(z))\big) - p\mu\big(h\mu(i(x), i(y)), i(z)\big)$ is the sum of two planar binary trees with three leaves

$m_3 =$



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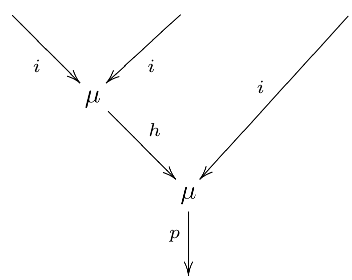

3. Homology and cohomology of a Lie algebra $\mathfrak{g}$

A non-minimal projective (in fact free) resolution of the trivial $U\mathfrak{g}$-module $\mathbb{K}$, $C(\mathfrak{g}) \xrightarrow{\epsilon} \mathbb{K}$ is given by the standard Chevalley–Eilenberg chain complex $C_\bullet(\mathfrak{g}) =$

$(U\mathfrak{g} \otimes_{\mathbb{K}} \wedge^p \mathfrak{g}, d_p)$ with differential maps

$$d_p(u \otimes x_1 \wedge \cdots \wedge x_p) = \sum_i (-1)^{i+1} u x_i \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_p + \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_p$$

The homologies $H_n(\mathfrak{g}, \mathbb{K})$ of the Lie algebra $\mathfrak{g}$ with trivial coefficients are given by the homologies of the derived complex $\mathbb{K} \otimes_{U\mathfrak{g}} C_\bullet(\mathfrak{g})$

$$\mathrm{Tor}_n^{U\mathfrak{g}}(\mathbb{K}, \mathbb{K}) \cong H_n(\mathbb{K} \otimes_{U\mathfrak{g}} C_\bullet(\mathfrak{g})) = H_n(\mathfrak{g}, \mathbb{K}).$$

The complex $\mathbb{K} \otimes_{U\mathfrak{g}} C_\bullet(\mathfrak{g})$ is the chain complex with degrees $\bigwedge^\bullet \mathfrak{g} = \mathbb{K} \otimes_{U\mathfrak{g}} U\mathfrak{g} \otimes \bigwedge^\bullet \mathfrak{g}$ and differentials $\partial_p := id \otimes_{U\mathfrak{g}} d_p : \bigwedge^p \mathfrak{g} \to \bigwedge^{p-1} \mathfrak{g}$ induced by the extension as coderivation of the Lie bracket $\partial_2 := -[\,\cdot\,,\cdot\,] : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$.

The dual cochain complex $\mathrm{Hom}_{U\mathfrak{g}}(C(\mathfrak{g}), \mathbb{K}) = (\bigwedge^\bullet \mathfrak{g}^*, \delta)$ has coboundary map $\delta^p : \bigwedge^p \mathfrak{g}^* \to \bigwedge^{p+1} \mathfrak{g}^*$ (being transposed to the differential ∂_{p+1}) which is the extension as derivation of the dualization of the Lie bracket $\delta^1 := [\,\cdot\,,\cdot\,]^* : \mathfrak{g}^* \to \bigwedge^2 \mathfrak{g}^*$. One calculates the cohomologies[1] of the Lie algebra $\mathfrak{g}$ as

$$\mathrm{Ext}^n_{U\mathfrak{g}}(\mathbb{K}, \mathbb{K}) \cong H^n(\mathrm{Hom}_{U\mathfrak{g}}(C(\mathfrak{g}), \mathbb{K})) = H^n(\mathfrak{g}, \mathbb{K}).$$

Hence the algebra $(\bigwedge^\bullet \mathfrak{g}^*, \delta)$ equipped with δ is a *(super)commutative* DGA and the Yoneda algebra $\mathrm{Ext}^\bullet_{U\mathfrak{g}}(\mathbb{K}, \mathbb{K}) = \bigoplus_n \mathrm{Ext}^n_{U\mathfrak{g}}(\mathbb{K}, \mathbb{K})$ has the structure of commutative associative algebra. Moreover due to the Kadeishvili theorem the Yoneda algebra $\mathrm{Ext}^\bullet_{U\mathfrak{g}}(\mathbb{K}, \mathbb{K}) = H^\bullet(\mathfrak{g}, \mathbb{K})$ is a C_∞-algebra which stems from the homotopy transfer of the wedge product $\wedge$ on cohomology classes $H^i(\mathfrak{g}, \mathbb{K}) \wedge H^j(\mathfrak{g}, \mathbb{K}) \to H^{i+j}(\mathfrak{g}, \mathbb{K})$.

4. Abelian Lie algebra $\mathfrak{h} = V$

Let us take as a basic example the abelian Lie algebra $\mathfrak{h}$, that is, the free nilpotent Lie algebra of rank 1 generated by a finite dimensional vector space V. The Lie bracket of $\mathfrak{h}$ is trivial $[V, V] = 0$. The universal enveloping algebra of the abelian Lie algebra $\mathfrak{h} = V$ is the symmetric algebra $U(\mathfrak{h}) \cong S(V)$. The Chevalley–Eilenberg complex $C_\bullet(\mathfrak{h}) = S(V) \otimes_{\mathbb{K}} \Lambda^\bullet V$ yields the resolution of the trivial $U(\mathfrak{h})$-module $\mathbb{K}$

$$(4.1) \quad 0 \to S(V) \otimes \Lambda^{\dim V} V \to S(V) \otimes \Lambda^{\dim V - 1} V \to \dots \cdots \to S(V) \otimes \Lambda^2 V \to S(V) \otimes V \to S(V) \to \mathbb{K} \to 0.$$

The derived complex $\mathbb{K} \otimes_{U\mathfrak{h}} C(\mathfrak{h})$ has zero differential and the Chevalley–Eilenberg resolution turns out to be minimal (which is not the case in general)

$$H_n(\mathfrak{h}, \mathbb{K}) \cong H_n(\mathbb{K} \otimes_{U\mathfrak{h}} C(\mathfrak{h})) \cong \Lambda^n V.$$

The Chevalley–Eilenberg resolution coincides with the Koszul complex $K(A) = A \otimes (A^!)^*$ of the symmetric algebra $A = S(V)$. The Koszul dual algebra of the symmetric algebra is the exterior algebra $S(V)^! = \Lambda V^*$. A quadratic algebra is

[1] In the presence of any metric on a nilpotent Lie algebra $\mathfrak{g}$ one has $\delta := \partial^*$(see below).

said to be a Koszul algebra when its Koszul complex $K_\bullet(A) = A \otimes (A^!_\bullet)^*$ is acyclic everywhere except in degree 0 (where its homology is $\mathbb{K}$). Then the Koszul complex yields a minimal projective (in fact free) resolution by (left) A-modules of the trivial A-module $\mathbb{K}$

$$K(A) \xrightarrow{\epsilon} \mathbb{K} \to 0.$$

In particular the resolution (4.1) is the same as the the resolution by the Koszul complex $K_n(S(V)) = S(V) \otimes \Lambda^n V^*$ thus the algebra $S(V)$ is a *Koszul algebra.* One has an equivalent definition of Koszul algebra based on the following proposition.

PROPOSITION 4.1. *A finitely generated quadratic algebra A is Koszul iff its Yoneda algebra* $\mathrm{Ext}_A(\mathbb{K},\mathbb{K})$ *is generated in degree* 1*. One has then* $\mathrm{Ext}_A(\mathbb{K},\mathbb{K}) \cong A^!$.

Indeed the Yoneda algebra $\mathrm{Ext}_{S(V)}(\mathbb{K},\mathbb{K})$ of the symmetric algebra $S(V)$ is just the exterior algebra

$$\mathrm{Ext}^n_{S(V)}(\mathbb{K},\mathbb{K}) = (\mathrm{Tor}^{S(V)}_n(\mathbb{K},\mathbb{K}))^* = \Lambda^n V^*$$

which is obviously generated by V^*, *i.e.,* in degree 1, by the wedge product. Through the homotopy transfer the Yoneda algebra $\mathrm{Ext}_{S(V)}(\mathbb{K},\mathbb{K})$ inherits a C_∞-structure but it is easy to show (by a degree preserving argument) that the latter C_∞-algebra is formal, i.e., all higher multiplications are trivial, $m_n = 0$ for $n \neq 2$.

5. Homology of the free 2-nilpotent algebra $\mathfrak{g} = V \oplus \Lambda^2 V$

Let $\mathfrak{g}$ be the free 2-step nilpotent Lie algebra generated by a vector space V in degree 1, $\mathfrak{g} = V \oplus [V,V]$. In other words the Lie bracket of the graded Lie algebra $\mathfrak{g} = V \oplus \Lambda^2 V$ is given by

$$[u,v] = \begin{cases} u \wedge v & u,v \in V \\ 0 & \text{otherwise} \end{cases}.$$

We denote the Universal Enveloping Algebra (UEA) $U\mathfrak{g}$ by PS and refer to it as *parastatistics algebra.*[2] Throughout this note we will consider the generators space V to be an ordinary vector space V which corresponds to a parafermionic algebra $PS(V) = U\mathfrak{g}$. The case of a $\mathbb{Z}_2$-space of generators $V = V_0 \oplus V_1$, that is, $PS(V)$ is the Universal Enveloping Algebra of a Lie super-algebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ (which would include the parabosonic algebras) will be treated elsewhere. More on parastatistics algebras and their application to combinatorics could be found in the articles [**5, 13**].

The parastatistics algebra $PS(V)$ generated by a finite dimensional vector space V is the positively graded algebra with degree induced by the tensor degree

$$PS(V) := U\mathfrak{g} = U\Big(V \oplus \bigwedge^2 V\Big) = T(V)/([[V,V],V]).$$

We shall write simply PS when the space of generators V is clear from the context.

[2]Such cubic algebras arise through the exchange relations between the operators in a quantization procedure introduced by Green [**8**] for particles obeying more general statistics than Bose–Einstein or Fermi–Dirac, coined parabosons and parafermions.

The homologies $H_n(\mathfrak{g},\mathbb{K})$ of the free 2-nilpotent Lie algebra $\mathfrak{g}$ are the homologies of the chain complex

$$\bigwedge^n \mathfrak{g} = \bigwedge^n \left(V \oplus \bigwedge^2 V \right) = \bigoplus_{s+r=n} \bigwedge^s \left(\bigwedge^2 V \right) \otimes \bigwedge^r (V)$$

with differentials $\partial_n : \bigwedge^s \left(\bigwedge^2 V\right) \otimes \bigwedge^r (V) \to \bigwedge^{s+1} \left(\bigwedge^2 V\right) \otimes \bigwedge^{r-2}(V)$ given by

$$\begin{aligned}\partial_n : & e_{i_1 j_1} \wedge \cdots \wedge e_{i_s j_s} \otimes e_{l_1} \wedge \cdots \wedge e_{l_r} \mapsto \\ & \sum_{i<j} (-1)^{i+j} e_{l_i l_j} \wedge e_{i_1 j_1} \wedge \cdots \wedge e_{i_s j_s} \otimes e_{l_1} \wedge \cdots \wedge \hat{e}_{l_i} \wedge \cdots \wedge \hat{e}_{l_j} \wedge \cdots \wedge e_{l_r}.\end{aligned}$$

The differential ∂ identifies a pair of degree 1 generators $e_i, e_j \in V$ with one degree 2 generator $e_{ij} := (e_i \wedge e_j) = [e_i, e_j] \in \Lambda^2 V$.

The cohomologies $H^n(\mathfrak{g},\mathbb{K})$ arise from the dualized complex with coboundary map $\delta^n : \bigwedge^n \mathfrak{g}^* \to \bigwedge^{n+1} \mathfrak{g}^*$ which is transposed to the differential ∂_{n+1}

$$\begin{aligned}&\delta^n : e^*_{i_1 j_1} \wedge \cdots \wedge e^*_{i_s j_s} \otimes e^*_{l_1} \wedge \cdots \wedge e^*_{l_r} \mapsto \\ &\sum_{k=1}^{s} \sum_{i_k < j_k} (-1)^{i+j} e^*_{i_1 j_1} \wedge \cdots \wedge \hat{e}^*_{i_k j_k} \wedge \cdots \wedge e^*_{i_s j_s} \otimes e^*_{i_k} \wedge e^*_{j_k} \wedge e^*_{l_1} \wedge \cdots \wedge \cdots \wedge e^*_{l_r}.\end{aligned}$$

In the presence of a metric g one has identifications $V \overset{g}{\cong} V^*$ and $\bigwedge^\bullet \mathfrak{g} \overset{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$. The adjoint operator $\partial^*_n : \bigwedge^n \mathfrak{g} \to \bigwedge^{n+1} \mathfrak{g}$ is defined by $g(\partial^*_n v, w) = g(v, \partial_{n+1} w)$. One can show that independently of the metric g chosen the action of ∂^*_n takes the form

$$\begin{aligned}&\partial^*_n : e_{i_1 j_1} \wedge \cdots \wedge e_{i_s j_s} \otimes e_{l_1} \wedge \cdots \wedge e_{l_r} \mapsto \\ &\sum_{k=1}^{s} \sum_{i_k < j_k} (-1)^{i+j} e_{i_1 j_1} \wedge \cdots \wedge \hat{e}_{i_k j_k} \wedge \cdots \wedge e_{i_s j_s} \otimes e_{i_k} \wedge e_{j_k} \wedge e_{l_1} \wedge \cdots \wedge \cdots \wedge e_{l_r}.\end{aligned}$$

We will see in the following that after the identification $\bigwedge^\bullet \mathfrak{g} \overset{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$ the map $\partial^* \overset{g}{=} \delta$ will play the role of homotopy for the chain complex $(\bigwedge^\bullet \mathfrak{g}, \partial_\bullet)$, and vice versa: the boundary map $\partial \overset{g}{=} \delta^*$ is a homotopy for the cochain complex $(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$.

The complexes $(\bigwedge^n \mathfrak{g}, \partial_n)$ and $(\bigwedge^n \mathfrak{g}^*, \delta^n)$ are bigraded by two different degrees; the homological degree $n := r + s$ counting the number of Lie algebra generators and the tensor degree $t := 2s+r$ also called weight. The cohomologies $H^n(\mathfrak{g},\mathbb{K})$ can have components of different weight t, $H^n(\mathfrak{g},\mathbb{K}) = \bigoplus_t H^n(\mathfrak{g},\mathbb{K})_t$ and the weight t is in fact the *Adams grading* on the Yoneda algebra $\mathrm{Ext}^n_{U\mathfrak{g}}(\mathbb{K},\mathbb{K})_t$ [**15**]. The differential and the homotopy, $\delta = \partial^*$ and $\partial = \delta^*$ do not alter the weight t, but raise and lower the homological degree n.

The operations m_k in the homotopy algebra are bigraded by homological and Adams gradings of bidegree $(k,t) = (2-k, 0)$. The bi-grading imposes the vanishing of many higher products.

5.1. Homology of $\mathfrak{g}$ as a $GL(V)$-module. A Schur module V_λ is an irreducible polynomial $GL(V)$-module labelled by a Young diagram λ. The basis of a Schur module V_λ is in bijection with semistandard Young tableaux with entries in the set $\{1,\dots,\dim V\}$. The action of the linear group $GL(V)$ on the space V of the generators of the Lie algebra $\mathfrak{g}$ induces a $GL(V)$-action on the universal enveloping algebra $PS = U\mathfrak{g} \cong S(V \oplus \Lambda^2 V)$ and on the space $\bigwedge^\bullet \mathfrak{g} \cong \bigwedge^\bullet(V \oplus \bigwedge^2 V)$.

The maps ∂ and ∂^* both commute with the $GL(V)$-action. It follows that the homology and cohomology carry structure of $GL(V)$-modules and hence can be decomposed into irreducibles.

The *Laplacian* $\Delta = \oplus_{n\geqslant 0}\Delta_n$ is defined to be the self-adjoint operator

$$\Delta_n = \partial_{n+1}\partial^*_{n+1} + \partial^*_n\partial_n \in \operatorname{End}\Big(\bigwedge^n \mathfrak{g}\Big).$$

Its kernel is a complete set of representatives for the homology classes in $H_n(\mathfrak{g},\mathbb{K})$

$$\ker \Delta_n \cong H_n(\mathfrak{g},\mathbb{K}).$$

The decomposition of the $GL(V)$-module $H_n(\mathfrak{g},\mathbb{K})$ into irreducible polynomial representations V_λ is given by the following theorem.

THEOREM 5.1 (Józefiak and Weyman [**9**], Sigg [**16**]). *The homology $H_\bullet(\mathfrak{g},\mathbb{K})$ of the free 2-nilpotent Lie algebra $\mathfrak{g} = V \oplus \bigwedge^2 V$ decomposes into a sum of irreducible $GL(V)$-modules*

$$H_n(\mathfrak{g},\mathbb{K}) \cong \operatorname{Tor}^{PS}_n(\mathbb{K},\mathbb{K})(V) \cong \bigoplus_{\lambda:\lambda=\lambda'} V_\lambda \quad \textit{such that} \quad n = \frac{1}{2}(|\lambda| + r(\lambda)),$$

where the sum is over the self-conjugate Young diagrams λ, $|\lambda|$ stands for the number of boxes in λ and $r(\lambda)$ for the rank of λ (the number of diagonal boxes in λ).

REMARK 5.1. The free 2-step nilpotent Lie algebra $\mathfrak{g}$ is the nilradical of a parabolic subalgebra of a simple Lie algebra of type C and its cohomology can be described by a general result of Bertram Kostant [**11**, Theorem 5.14]. A derivation of the cohomology $H^\bullet(\mathfrak{g},\mathbb{K})$ in these lines has been worked out by Grassberger, King and Tirao [**7**] thus providing one more proof of Theorem 5.1 via the isomorphism $H_n(\mathfrak{g},\mathbb{K}) \cong \operatorname{Tor}^{PS}_n(\mathbb{K},\mathbb{K})(V) \cong \operatorname{Ext}^n_{PS}(\mathbb{K},\mathbb{K})^* \cong H^n(\mathfrak{g},\mathbb{K})^*$.

5.2. Homological interpretation of the Littlewood formula. We recall the beautiful result of Józefiak and Weyman [**9**] giving a representation-theoretic interpretation of the Littlewood formula

$$\prod_i (1-x_i) \prod_{i<j}(1-x_ix_j) = \sum_{\lambda:\lambda=\lambda'} (-1)^{\frac{1}{2}(|\lambda|+r(\lambda))} s_\lambda(x).$$

Here the sum is over all self-conjugate Young diagrams λ and $s_\lambda(x)$ stands for the Schur function with diagram λ.

One knows that for the graded algebra PS there exists a *minimal* resolution[3] by projective modules in the graded category

$$(5.1) \qquad P_\bullet : \quad 0 \to P_d \to \cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} \mathbb{K} \to 0.$$

Here the length d of the resolution is the projective dimension of the algebra PS which is $d = \frac{1}{2}\dim V(\dim V + 1)$. Since PS is positively graded and, in the category of positively graded modules over connected locally finite graded algebras, projective module is the same as free module [**4**], we have $P_n \cong PS \otimes E_n$, where E_n are finite dimensional vector spaces. Thus we deal with a minimal resolution of $\mathbb{K}$ by free PS-modules and the minimality implies that the derived complex $\mathbb{K} \otimes_{PS} P_\bullet$ has vanishing differentials, i.e., $\mathrm{Tor}_\bullet^{PS}(\mathbb{K},\mathbb{K}) = H_\bullet(\mathbb{K} \otimes_{PS} P_\bullet) = \mathbb{K} \otimes_{PS} P_\bullet$. Then the multiplicity spaces $E_n = \mathrm{Tor}_n^{PS}(\mathbb{K},\mathbb{K})$ are fixed by Theorem 5.1 and thus the data $H_n(\mathfrak{g},\mathbb{K}) = \mathrm{Tor}_n^{PS}(\mathbb{K},\mathbb{K})$ encodes the minimal free resolution $P_\bullet$ (cf. 5.1) which is unique (up to isomorphism).

The Euler characteristics of $P_\bullet$ implies an identity about the $GL(V)$-characters

$$\mathrm{ch}\, PS(V).\mathrm{ch}\Bigg(\bigoplus_{\lambda:\lambda=\lambda'} (-1)^{\frac{1}{2}(|\lambda| + r(\lambda))} V_\lambda \Bigg) = 1.$$

The character of a Schur module V_λ is the Schur function, $\mathrm{ch} V_\lambda = s_\lambda(x)$. Due to the Poincaré–Birkhoff–Witt theorem $PS(V) \cong S(V \oplus \bigwedge^2 V)$ thus the identity reads

$$\prod_i \frac{1}{(1-x_i)} \prod_{i<j} \frac{1}{(1-x_i x_j)} \sum_{\lambda:\lambda=\lambda'} (-1)^{\frac{1}{2}(|\lambda| + r(\lambda))} s_\lambda(x) = 1.$$

But the latter identity is nothing but a rewriting of the Littlewood identity (5.1). The moral is that the Littlewood identity reflects a homological property of the algebra PS, namely the above particular structure of the minimal projective (free) resolution of $\mathbb{K}$ by PS-modules.

5.3. $\mathrm{Ext}_{PS}^\bullet(\mathbb{K},\mathbb{K})$ as a C_∞-algebra.

THEOREM 5.2. *The cohomology $H^\bullet(\mathfrak{g},\mathbb{K}) \cong \mathrm{Ext}_{PS}^\bullet(\mathbb{K},\mathbb{K})$ of the free 2-nilpotent Lie algebra $\mathfrak{g} = V \oplus \bigwedge^2 V$ is a homotopy commutative algebra which is generated in degree* 1 *(i.e., in $H^1(\mathfrak{g},\mathbb{K})$) by the operations m_2 and m_3.*

PROOF. We start by choosing a metric g on the vector space V and an orthonormal basis $g(e_i, e_j) = \delta_{ij}$. The choice induces a metric on $\bigwedge^\bullet \mathfrak{g} \overset{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$.

The isomorphisms $V \cong V^*$ and $\mathrm{Tor}_n^{PS}(\mathbb{K},\mathbb{K}) \cong \mathrm{Ext}_{PS}^n(\mathbb{K},\mathbb{K})$ and the Theorem 5.1 imply the decomposition of $H^\bullet(\mathfrak{g},\mathbb{K})$ into irreducible $GL(V)$-modules

$$H^n(\mathfrak{g},\mathbb{K}) \cong H^n(\textstyle\bigwedge \mathfrak{g}^*, \delta) \cong \mathrm{Ext}_{PS}^n(\mathbb{K},\mathbb{K}) \cong \displaystyle\bigoplus_{\lambda:\lambda=\lambda'} V_\lambda,$$

where the sum is over all self-conjugate diagrams λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

[3]The Chevalley–Eilenberg complex does not provide a minimal resolution of the module $\mathbb{K}$, in general.

The adjoint of the boundary map ∂, $\delta \overset{g}{:=} \partial^*$ is the differential in the DGA $\left(\bigwedge \mathfrak{g}^*, \delta\right)$ while $\delta^* \overset{g}{:=} \partial$ plays the role of a homotopy. In view of Lemma 2.1 we have the cohomology $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$ as deformation retract of the complex $\left(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet\right)$,

$$pi = \mathrm{Id}_{H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)}, \quad ip - \mathrm{Id}_{\bigwedge^\bullet \mathfrak{g}^*} = \delta\delta^* + \delta^*\delta, \quad \delta^* \overset{g}{=} \partial.$$

Here the projection p identifies the subspace $\ker\delta \cap \ker\delta^*$ with $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$, which is the orthogonal complement of the space of the coboundaries $\mathrm{im}\delta$. The cocycle-choosing homomorphism i is Id on $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$ and zero on coboundaries.

We apply the Kadeishvili homotopy transfer theorem 2.1 for the commutative DGA $(\bigwedge^\bullet \mathfrak{g}^*, \mu, \delta^\bullet)$ and its deformation retract $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*) \cong H^\bullet(\mathfrak{g}, \mathbb{K})$ and conclude that the cohomology $H^\bullet(\mathfrak{g}, \mathbb{K})$ is a C_∞-algebra.

The Kontsevich and Soibelman tree representations of the operations m_n provide explicit expressions. Let us take μ to be the super-commutative product $\wedge$ on the DGA $(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$. The projection p maps onto the Schur modules V_λ with self-conjugated Young diagram $\lambda = \lambda'$.

The binary operation on the generators $e_i \in H^1(\mathfrak{g}, \mathbb{K})$ is trivial, one gets

$$m_2(e_i, e_j) = p(e_i \wedge e_j) = 0 \quad p(V_{(1^2)}) = 0.$$

Hence $H^\bullet(\mathfrak{g}, \mathbb{K})$ could not be generated in $H^1(\mathfrak{g}, \mathbb{K})$ as an algebra with the binary product m_2.

The ternary operation m_3 restricted to $H^1(\mathfrak{g}, \mathbb{K})$ is nontrivial, indeed taking into account the Koszul sign rule we get the following representative cocycles

$$\begin{aligned} m_3(e_i, e_j, e_k) &= p\left\{-e_i \wedge \partial(e_j \wedge e_k) - \partial(e_i \wedge e_j) \wedge e_k\right\} \\ &= p\left\{e_{ij} \wedge e_k + e_i \wedge e_{jk}\right\} = e_{ij} \wedge e_k - e_{jk} \wedge e_i \in H^2(\mathfrak{g}, \mathbb{K}). \end{aligned}$$

The complete antisymmetrization of the monomial $e_{ik} \wedge e_j$ spans the Schur module $V_{(1^3)}$ and thus it is projected out, $p(e_{ij} \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j) = 0$. Therefore the monomials $e_{ij} \wedge e_k$ modulo $V_{(1^3)}$ span a Schur module $V_{(2,1)} \cong H^2(\mathfrak{g}, \mathbb{K})$ having the representative cocycles in bijection with the semistandard Young tableaux with diagram $(2,1)$,

$$\begin{aligned} e_{ij} \wedge e_k - e_{jk} \wedge e_i \quad &\leftrightarrow \quad \begin{array}{|c|c|}\hline i & k \\ \hline j & \\ \cline{1-1}\end{array} \quad \text{for } i < j,\ i \leqslant k, \\ e_{jk} \wedge e_i - e_{ki} \wedge e_j \quad &\leftrightarrow \quad \begin{array}{|c|c|}\hline i & j \\ \hline k & \\ \cline{1-1}\end{array} \quad \text{for } pi < k,\ i \leqslant j. \end{aligned}$$

We check the symmetry condition on the ternary operation m_3 in the C_∞-algebra; indeed m_3 vanishes on the (signed) shuffles $\mathrm{Sh}_{1,2}$

$$m_3(e_i \sqcup\!\sqcup\, e_j \otimes e_k) = m_3(e_i, e_j, e_k) - m_3(e_j, e_i, e_k) + m_3(e_j, e_k, e_i) = 0.$$

Similarly one gets $m_3(e_i \otimes e_j \sqcup\!\sqcup\, e_k) = 0$ on shuffles $\mathrm{Sh}_{2,1}$.

On the level of Schur modules the ternary operation glues three fundamental $GL(V)$-modules $V_\square$ into a Schur module $V_{(2,1)}$. By iteration of the process of gluing boxes we generate all elementary hooks $V_k := V_{(k+1,1^k)}$,

$$m_3(V_\square, V_\square, V_\square) = V_{\yng(2,1)}$$
$$m_3\left(V_\square, V_{\yng(2,1)}, V_\square\right) = V_{\yng(3,1,1)}$$
$$\cdots$$
$$m_3(V_0, V_k, V_0) = V_{k+1}.$$

In our context the more convenient notation for Young diagrams is due to Frobenius: $\lambda := (a_1, \ldots, a_r | b_1, \ldots b_r)$ stands for a diagram λ with a_i boxes in the i-th row on the right of the diagonal, and with b_i boxes in the i-th column below the diagonal and the rank $r = r(\lambda)$ is the number of boxes on the diagonal.

For self-dual diagrams $\lambda = \lambda'$, i.e., $a_i = b_i$ we set $V_{a_1,\ldots,a_r} := V_{(a_1,\ldots,a_r|a_1,\ldots a_r)}$ when $a_1 > a_2 > \cdots > a_r \geqslant 0$ (and set the convention $V_{a_1,\ldots,a_r} := 0$ otherwise). Any two elementary hooks V_{a_1} and V_{a_2} can be glued together by the binary operation m_2, the decomposition of $m_2(V_{a_1}, V_{a_2}) \cong m_2(V_{a_2}, V_{a_1})$ is given by

$$m_2(V_{a_1}, V_{a_2}) = V_{a_1,a_2} \oplus \left(\bigoplus_{i=1}^{a_2} V_{a_1+i, a_2-i} \right), \quad a_1 \geqslant a_2$$

where the "leading" term V_{a_1,a_2} has the diagram with minimal height. Hence any m_2-bracketing of the hooks $V_{a_1}, V_{a_2}, \ldots, V_{a_r}$ yields[4] a sum of $GL(V)$-modules

$$m_2(\ldots m_2(m_2(V_{a_1}, V_{a_2}), V_{a_3}), \ldots, V_{a_r}) = V_{a_1,\ldots,a_r} \oplus \cdots$$

whose module with minimal height is precisely $V_{a_1,\ldots,a_r}$. We conclude that all elements in the C_∞-algebra $H^\bullet(\mathfrak{g}, \mathbb{K})$ can be generated in $H^1(\mathfrak{g}, \mathbb{K})$ by m_2 and m_3. $\square$

One could draw a parallel between the theorem for the cubic algebra PS and the Proposition 4.1 for the Koszul algebra; in both cases the Yoneda algebra $\mathrm{Ext}^\bullet_{PS}(\mathbb{K}, \mathbb{K})$ is generated only in $\mathrm{Ext}^1_{PS}(\mathbb{K}, \mathbb{K})$. Although we have the notion of N-Koszul algebras for the N-homogeneous algebras [**2, 3**], it turns out that the cubic algebra PS is not 3-Koszul, beside the exceptional case when $\dim V = 2$. Instead the algebra $PS = U\mathfrak{g}$ falls in the class of *Artin–Schelter-regular algebras* [**1**], being an UEA of positively graded Lie algebra (for a proof see [**6**]). The parallel between the quadratic Koszul algebra $S(V)$ and the cubic AS-regular regular algebra $PS(V)$ suggests that the C_∞-algebra $\mathrm{Ext}^\bullet_{PS}(\mathbb{K}, \mathbb{K})$ is a generalization of a Koszul dual algebra of PS in the realm of the homotopy algebras, an idea that has been put forward in [**15**].

The analogy would be complete if we had the following conjectural proposition.

PROPOSITION 5.1. *The cohomology $H^\bullet(\mathfrak{g}, \mathbb{K}) \cong \mathrm{Ext}^\bullet_{PS}(\mathbb{K}, \mathbb{K})$ of the free 2-nilpotent Lie algebra $\mathfrak{g} = V \oplus \bigwedge^2 V$ can be endowed with a structure of C_∞-algebra having trivial higher multiplications $m_k = 0$, $k \geqslant 4$.*

[4]The operation m_2 is associative thus the result does not depend on the choice of the bracketing.

So far we have been able to prove this conjecture only in dimensions $\dim V \leqslant 3$. Our proof rests entirely on the bigrading $(2-k,0)$ of the multiplication m_k by homological and tensor degree in the C_∞-algebra $\mathrm{Ext}^{\bullet}_{PS}(\mathbb{K},\mathbb{K})$. The bigrading arguments work only for $\dim V = 2$ and $\dim V = 3$ thus for a complete proof the conjecture would need more refined methods.

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