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TWISTED PRODUCT CR-SUBMANIFOLDS IN A LOCALLY CONFORMAL KAEHLER MANIFOLD

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Dedicated to Professor Mileva Prvanović on the occasion of her 83rd birthday

Abstract. Recently, we have researched certain twisted product CR-submanifolds in a Kaehler manifold and some inequalities of the second fundamental form of these submanifolds [11].

We consider here two kinds of twisted product CR-submanifolds (the first and the second kind) in a locally conformal Kaehler manifold. In these submanifolds, we give inequalities of the second fundamental form (see Theorems 5.1 and 5.2) and consider the equality case of these.

1. Twisted product manifolds

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds and M be a (topological) product manifold of M_1 and M_2. We define a Riemannian metric g of M as

$$g(U,V) = e^{f^2} g_1(\pi_{1*}U, \pi_{1*}V) + g_2(\pi_{2*}U, \pi_{2*}V)$$

for any $U, V \in TM$, where f denotes a positive differentiable function on M, TM is the tangent bundle of M, π_1 (resp. π_2) is a projection operator of M to M_1 (resp. M_2) and π_{1*} (resp. π_{2*}) is the differential of π_1 (resp. π_2). Then the manifold M is called a *twisted product manifold with an associated* (*or a warping*) *function* f and we write it $M = M_1 \times_f M_2$ [8]. In particular, if the associated function f is in M_2, then the manifold $\tilde{M}$ is a warped product [12].

Let $M = M_1 \times_f M_2$ be a twisted product manifold with the associated function f and let $\dim M_1 = n_1$, $\dim M_2 = n_2$ and $\dim M = n = n_1 + n_2$. Moreover, let $(x^1, x^2, \dots, x^{n_1})$, $(x^{n_1+1}, \dots, x^{n_1+n_2})$ be local coordinate systems of M_1 and M_2, respectively. Then $(x^1, x^2, \dots, x^n)$ is a local coordinate system of M.

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Using the above local coordinate systems, we can write

$$(g_{\mu\lambda}) = \begin{pmatrix} g_{ji} & 0 \\ 0 & g_{ba} \end{pmatrix} = \begin{pmatrix} e^{f^2} g_{1ji} & 0 \\ 0 & g_{2ba} \end{pmatrix}, \tag{1.1}$$

where the indices $(j, i, \ldots, h)$, $(d, c, \ldots, a)$ and $(\nu, \mu, \ldots, \lambda)$ run over the ranges $(1, 2, \ldots, n_1)$, $(n_1 + 1, n_1 + 2, \ldots, n_1 + n_2)$ and $(1, 2, \ldots, n_1 + n_2 = n)$, respectively.

From (1.1), we have

$$\begin{aligned} &\partial_k g_{ji} = e^{f^2}\{2f^2(\partial_k \log f_1) g_{1ji} + \partial_k g_{1ji}\}, \\ &\partial_a g_{ji} = 2f^2 e^{f^2}(\partial_a \log f) g_{1ji}, \\ &\partial_i g_{ba} = 0, \quad \partial_c g_{ba} = \partial_c g_{2ba}, \end{aligned} \tag{1.2}$$

where $\partial_k = \partial/\partial x^k$ and $\partial_a = \partial/\partial x^a$.

Next, using (1.1) and (1.2), we calculate the Christoffel symbols $\{{}_\nu{}^\lambda{}_\mu\}$ with respect to $g_{\mu\lambda}$ which are given by

$$\{{}_\nu{}^\lambda{}_\mu\} = \tfrac{1}{2}\, g^{\lambda\varepsilon}(\partial_\nu g_{\varepsilon\mu} + \partial_\mu g_{\nu\varepsilon} - \partial_\varepsilon g_{\nu\mu}). \tag{1.3}$$

By virtue of (1.2) and (1.3), we obtain

$$\begin{aligned} &\{{}_j{}^h{}_i\} = \{{}_j{}^h{}_i\}_1 + f^2\{(\partial_j \log f)\delta_i{}^h + (\partial_i \log f)\delta_j{}^h - (\partial_1{}^h \log f) g_{1ji}\}, \\ &\{{}_b{}^h{}_i\} = f^2(\partial_b \log f)\delta_i{}^h, \quad \{{}_b{}^h{}_a\} = 0, \\ &\{{}_j{}^a{}_i\} = -f^2 e^{f^2}(\partial_2{}^a \log f) g_{1ji}, \\ &\{{}_j{}^a{}_b\} = 0, \quad \{{}_c{}^a{}_b\} = \{{}_c{}^a{}_b\}_2, \end{aligned} \tag{1.4}$$

where $\partial_1{}^h = g_1{}^{lh}\partial_l$ (resp. $\partial_2^a = g_2{}^{ea}\partial_e$) and $\{{}_j{}^h{}_i\}_1$ (resp. $\{{}_c{}^a{}_b\}_2$) denotes the Christoffel symbol of g_1 (resp. g_2).

By virtue of (1.4), we get

$$\begin{aligned} \nabla_Y X &= \nabla_{1Y} X + f^2\{(Y \log f)X + (X \log f)Y\} \\ &\quad - f^2 g_1(Y, X)\{(\partial_1{}^l \log f)\partial_l + e^{f^2}(\partial_2{}^e \log f)\partial_e\}, \\ \nabla_X Z &= \nabla_Z X = f^2(Z \log f)X, \quad \nabla_Z W = \nabla_{2Z} W \end{aligned}$$

for any $Y, X \in TM_1$ and $Z, W \in TM_2$, where ∇_1 (resp. ∇_2) denotes the covariant differentiation with respect to g_1 (resp. g_2).

2. Locally conformal Kaehler manifolds

A Hermitian manifold $\tilde{M}$ with structure $(J, \tilde{g})$ is called a locally conformal Kaehler (l.c.K.) manifold if each point $x \in \tilde{M}$ has an open neighbourhood U with differentiable function $\rho : U \to \mathcal{R}$ such that $\tilde{g}^* = e^{-2\rho}\tilde{g}_{|U}$ is a Kaehlerian metric on U, that is, $\nabla^* J = 0$, where J is the almost complex structure, $\tilde{g}$ is the Hermitian metric, ∇^* is the covariant differentiation with respect to $\tilde{g}^*$ and $\mathcal{R}$ is a real number space [13]. Then we know [9]

Proposition 2.1. *A Hermitian manifold $\tilde{M}$ with structure $(J, \tilde{g})$ is l.c.K. if and only if there exists a global 1-form α which is called Lee form satisfying*

$$d\alpha = 0 \quad (\alpha : closed), \tag{2.1}$$

$$(\tilde{\nabla}_V J)U = -\tilde{g}(\alpha^\sharp, U)JV + \tilde{g}(V, U)\beta^\sharp + \tilde{g}(JV, U)\alpha^\sharp - \tilde{g}(\beta^\sharp, U)V \tag{2.2}$$

for any $V, U \in \tilde{M}$, where $\tilde{\nabla}$ denotes the covariant differentiation with respect to $\tilde{g}$, $\alpha^\sharp$ is the dual vector field of α, the 1 form β is defined by $\beta(X) = -\alpha(JX)$, $\beta^\sharp$ is the dual vector field of β and $T\tilde{M}$ means the tangent bundle of $\tilde{M}$.

3. CR-submanifolds in an l.c.K.-manifold

In general, between a Riemannian manifold $(\tilde{M}, \tilde{g})$ and its Riemannian submanifold, we know the Gauss and Weingarten formulas

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \qquad \tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$$

for any $X, Y \in TM$ and $\xi \in T^\perp M$, where σ is the second fundamental form and A_ξ is the shape operator with respect to ξ [7]. The second fundamental form σ and the shape operator A are related by $\tilde{g}(A_\xi Y, X) = \tilde{g}(\sigma(Y, X), \xi)$ for any $Y, X \in TM$ and $\xi \in T^\perp M$.

A submanifold M in an l.c.K.-manifold $\tilde{M}$ is called a *CR-submanifold* if there exists a differentiable distribution $\mathcal{D} : x \to \mathcal{D}_x \subset T_x M$ on M satisfying the following conditions

(i) $\mathcal{D}$ is holomorphic, i.e., $J\mathcal{D}_x = D_x$ for each $x \in M$ and

(ii) the complementary orthogonal distribution $\mathcal{D}^\perp : x \to \mathcal{D}^\perp_x \subset T_x M$ is totally real, i.e., $J\mathcal{D}^\perp_x \subset T^\perp_x M$ for each $x \in M$, where $T_x M$ (resp. $T^\perp_x M$) denotes the tangent (resp. normal) vector space at x of M [1, 2, 6, etc.].

If $\dim \mathcal{D}^\perp_x = 0$ (resp. $\dim \mathcal{D}_x = 0$) for each $x \in M$, then the CR-submanifold is *holomorphic* (resp. *totally real*). A CR-submanifold M is said to be *anti-holomorphic* if $J\mathcal{D}^\perp_x = T^\perp_x M$ for any $x \in M$.

In [10], we proved the following

Proposition 3.1. *In a CR-submanifold M in an l.c.K.-manifold $\tilde{M}$, we have*

(i) the distribution $\mathcal{D}^\perp$ is integrable,

(ii) the distribution $\mathcal{D}$ is integrable if and only if

$$\tilde{g}(\sigma(X, JY) - \sigma(Y, JX) + 2\tilde{g}(JX, Y)\alpha^\sharp, JZ) = 0 \tag{3.1}$$

for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$.

A CR-submanifold is said to be *proper* if it is neither holomorphic nor totally real.

In a CR-submanifold M in an l.c.K.-manifold $\tilde{M}$, we know the following formulas [10]

$$\begin{aligned} \tilde{g}(\nabla_U Z, X) = {} & \tilde{g}(JA_{JZ}U, X) + \tilde{g}(\alpha^\sharp, Z)\tilde{g}(U, X) \\ & + \tilde{g}(U, Z)\tilde{g}(\alpha^\sharp, X) - \tilde{g}(\beta^\sharp, Z)\tilde{g}(JU, X), \\ A_{JZ}W = {} & A_{JW}Z + \tilde{g}(\beta^\sharp, Z)W - \tilde{g}(\beta^\sharp, W)Z \end{aligned}$$

for any $U \in TM$, $X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^{\perp}$.

A CR-submanifold is said to be *mixed geodesic* if the second fundamental form σ satisfies $\sigma(\mathcal{D}, \mathcal{D}^{\perp}) = \{0\}$ and to be $\mathcal{D}$ (resp. $\mathcal{D}^{\perp}$)-*geodesic* if the second fundamental form σ satisfies $\sigma(\mathcal{D}, \mathcal{D}) = \{0\}$ (resp. $\sigma(\mathcal{D}^{\perp}, \mathcal{D}^{\perp}) = \{0\}$).

In a CR-submanifold M of an almost Hermitian manifold $\tilde{M}$, we denote by ν the complementary orthogonal subbundle of $J\mathcal{D}^{\perp}$ in the normal bundle $T^{\perp}M$. Then we have the following direct sum decomposition $T^{\perp}M = J\mathcal{D}^{\perp} \oplus \nu$, $J\mathcal{D}^{\perp} \perp \nu$.

Remark 3.1. By the definition of ν, a CR-submanifold is anti-holomorphic if $\nu_x = \{0\}$ for any $x \in M$.

Definition 3.1. Let $\tilde{M}$ be a Riemannian manifold with a metric tensor $\tilde{g}$. A submanifold M is said to be a *twisted product submanifold* of $\tilde{M}$ if it satisfies

(i) M is a Riemannian submanifold of $\tilde{M}$,

(ii) M is a twisted product manifold of two submanifolds M_1 and M_2 of $\tilde{M}$,

(iii) for a certain Riemannian metric g_1 (resp. g_2) of M_1 (resp. M_2),

$$g(U, V) = e^{f^2} g_1(\pi_{1*}U, \pi_{1*}V) + g_2(\pi_{2*}U, \pi_{2*}V)$$

is an induced metric of $\tilde{g}$ for any $U, V \in T\tilde{M}$ and a positive differentiable function f on M, where π_1 (resp. π_2) is the projection operator of $\tilde{M}$ to M_1 (resp. M_2), and π_{1*} (resp. π_{2*}) is the differential of π_1 (resp. π_2).

(iv) the submanifolds M_1 and M_2 are orthogonal, that is, $\tilde{g}(X, Z) = 0$ for any $X \in TM_1$ and $Z \in TM_2$.

4. Twisted product CR-submanifolds in a locally conformal Kaehler manifold

In this section, we consider a special twisted product submanifold in an l.c.K.-manifold.

Definition 4.1. A submanifold M in an l.c.K.-manifold $\tilde{M}$ is said to be the *first* (resp. *second*) *kind twisted product CR-submanifold* in $\tilde{M}$ if it satisfies

(i) M is a product manifold of a holomorphic submanifold $M_{\top}$ and a totally real submanifold $M_{\perp}$,

(ii) for a certain Riemannian metric tensor g_1 (resp. g_2) on $M_{\top}$ (resp. $M_{\perp}$) and a positive differentiable function f on M,

$$g(V, U) = e^{f^2} g_1(\pi_* V, \pi_* U) + g_2(\eta_* V, \eta_* U) \tag{4.1}$$

$$\Big(\text{resp.} \quad g(V, U) = g_1(\pi_* V, \pi_* U) + e^{f^2} g_2(\eta_* V, \eta_* U)\Big) \tag{4.2}$$

is a induced metric of $\tilde{g}$, that is, $\tilde{g}(V, U) = g(i_* V, i_* U)$, for any $V, U \in TM$, where π (resp. η) is a projection operator of M to $M_{\top}$ (resp. $M_{\perp}$) and i is an identity map of M to $\tilde{M}$.

Then we write the first (resp. second) kind twisted product CR-submanifold $M = M_{\top} \times_f M_{\perp}$ (resp. $M = M_{\perp} \times_f M_{\top}$).

Remark 4.1. We write $\mathcal{D}$ (resp. $\mathcal{D}^{\perp}$) instead of $TM_{\top}$ (resp. $TM_{\perp}$).

REMARK 4.2. In our submanifold, since the holomorphic distribution $\mathcal{D}$ is integrable, we have to assume that the second fundamental form σ satisfies (3.1).

REMARK 4.3. About warped product and doubly warped product CR-submanifolds in an l.c.K.-manifold, we can find in [3, 4, 5].

In a CR-submanifold M of an l.c.K.-manifold $\tilde{M}$, let be $\dim \mathcal{D} = 2p$, $\dim \mathcal{D}^{\perp} = q$, $\dim M = n$, $\dim \nu = 2s$ and $\dim \tilde{M} = m$. Then we know $2p + q = n$ and $2(p + q + s) = m$.

Now we recall an adapted frame on $\tilde{M}$. We take a following local orthonormal frame on $\tilde{M}$,

(i) $\{e_1, e_2, \dots, e_p, e^*{}_1, e^*{}_2, \dots, e^*{}_p\}$ is an orthonormal frame of $\mathcal{D}$,

(ii) $\{e_{2p+1}, e_{2p+2}, \dots, e_{2p+q}\}$ is an orthonormal frame of $\mathcal{D}^{\perp}$,

(iii) $\{e_{n+q+1}, e_{n+q+2}, \dots, e_{n+q+s}, e^*{}_{n+q+1}, e^*{}_{n+q+2}, \dots, e^*{}_{n+q+s}\}$ is an orthonormal frame of ν. Then we know

(a) $\{e_1, \dots, e_p, e^*{}_1, \dots, e^*{}_p, e_{2p+1}, \dots, e_{2p+q}\}$ is an orthonormal frame of TM,

(b) $\{e^*{}_{2p+1}, \dots, e^*{}_{2p+q}, e_{n+q+1}, \dots, e_{n+q+s}, e^*{}_{n+q+1}, \dots, e^*{}_{n+q+s}\}$ is an orthonormal frame of $T^{\perp}M$, where $e^*{}_i = Je_i$ for $i \in \{1, 2, \dots, p\}$, $e^*{}_{2p+a} = Je_{2p+a}$ for any $a \in \{1, 2, \dots, q\}$ and $e^*{}_{n+q+\alpha} = Je_{n+q+\alpha}$ for any $\alpha \in \{1, 2, \dots, s\}$. We call such an orthonormal frame $\{e_1, \dots, e^*{}_{n+q+s}\}$, an *adapted frame* of $\tilde{M}$.

First of all, we consider the first kind twisted product CR-submanifold M in an l.c.K.-manifold $\tilde{M}$. Then, by the definition, the induced metric g on M is defined by (4.1).

Then we have

$$\begin{aligned} \nabla_Y X &= \nabla_{1Y} X + f^2\{(Y \log f)X + (X \log f)Y \\ &\qquad - f^2 g_1(Y, X)\{(\partial_1{}^l \log f)\partial_l + e^{f^2}(\partial_2{}^e \log f)\partial_e\}, \\ \nabla_X Z &= \nabla_Z X = f^2(Z \log f)X, \quad \nabla_Z W = \nabla_{2Z} W \end{aligned} \tag{4.3}$$

for any $Y, X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^{\perp}$, where ∇_1 (resp. ∇_2) denotes the covariant differentiation with respect to g_1 (resp. g_2).

PROPOSITION 4.1. *For a proper first kind twisted product CR-submanifold* $M = M_{\top} \times_f M_{\perp}$ *in an l.c.K.-manifold* $\tilde{M}$, *we have*

(1) $\tilde{g}(\sigma(X, JY), JZ) = \tilde{g}(\alpha^{\sharp}, Z)\tilde{g}(X, Y) - \tilde{g}(\alpha^{\sharp}, JZ)\tilde{g}(X, JY) - f^2(Z \log f)\tilde{g}(X, Y)$,

(2) $\tilde{g}(\sigma(X, Y), JZ) = \tilde{g}(\alpha^{\sharp}, JZ)\tilde{g}(X, Y)$ *and* $\tilde{g}(\alpha^{\sharp}, Z) = f^2(Z \log f)$,

(3) $\tilde{g}(\sigma(JX, Z), JW) = -\tilde{g}(\alpha^{\sharp}, X)\tilde{g}(Z, W)$

for any $Y, X \in \mathcal{D}$.

PROOF. For any $Y, X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$, we have from (2.2) and (4.3)

$$\begin{aligned} \tilde{g}(\sigma(X, JY), JZ) &= \tilde{g}(\tilde{\nabla}_X(JY), JZ) = -\tilde{g}(\tilde{\nabla}_X(JZ), JY) \\ &= -\tilde{g}((\tilde{\nabla}_X J)Z, JY) - \tilde{g}(J\tilde{\nabla}_X Z, JY) \\ &= \tilde{g}(\alpha^{\sharp}, Z)\tilde{g}(X, Y) - \tilde{g}(\alpha^{\sharp}, JZ)\tilde{g}(X, JY) - \tilde{g}(\nabla_X Z, Y). \end{aligned}$$

Using (4.3) and the above equation, we have (1).

In (1), if we put JX instead of X, then we have

$$\tilde{g}(\sigma(X,Y),JZ) - \tilde{g}((\alpha^\sharp, JZ)\tilde{g}(X,Y) = \{f^2(Z\log f) - \tilde{g}(\alpha^\sharp, Z)\}\tilde{g}(X,JY).$$

In the above equation, the left-hand side is symmetric and the right-hand side is skew symmetric with respect to X and Y. So, we have (2).

From (2), (1) is written as

$$\tilde{g}(\sigma(X,JY),JZ) = -\tilde{g}(\alpha^\sharp, JZ)\tilde{g}(X,JY). \tag{1'}$$

Finally, for (3), we have from (2.2) and (4.3)

$$\begin{aligned}\tilde{g}(\sigma(JX,Z),JW) &= \tilde{g}(\tilde{\nabla}_Z(JX),JW) = \tilde{g}((\tilde{\nabla}_Z J)X,JW) + \tilde{g}(J\tilde{\nabla}_Z X,JW)\\ &= \tilde{g}(-\tilde{g}(\alpha^\sharp,X)JZ + \tilde{g}(Z,X)\beta^\sharp + \tilde{g}(JX,Z)\alpha^\sharp\\ &\quad - \tilde{g}(\beta^\sharp,X)Z,JW) + \tilde{g}(\tilde{\nabla}_Z X,W) = -\tilde{g}(\alpha^\sharp,X)\tilde{g}(Z,W)\end{aligned}$$

which means (3). □

By virtue of (2) in the above proposition, we know

Proposition 4.2. *There does not exist a proper first kind of twisted product CR-submanifold in an l.c.K.-manifold whose Lee vector field $\alpha^\sharp$ is normal to $\mathcal{D}^\perp$.*

Proof. By the assumption, we easily know the function f is in $M_\top$ which means our proposition. □

Next, we consider the second kind of twisted product CR-submanifold $M = M_\perp \times_f M_\top$ in an l.c.K.-manifold $\tilde{M}$. Then, (4.2) means

$$\begin{aligned}(g_{\mu\lambda}) &= \begin{pmatrix} g_{ba} & 0\\ 0 & g_{ji}\end{pmatrix} = \begin{pmatrix} e^{f^2}g_{2ba} & 0\\ 0 & g_{1ji}\end{pmatrix}\\ (g^{\mu\lambda}) &= \begin{pmatrix} g^{ba} & 0\\ 0 & g^{ji}\end{pmatrix} = \begin{pmatrix} e^{-f^2}{g_2}^{ba} & 0\\ 0 & {g_1}^{ji}\end{pmatrix}.\end{aligned}$$

In the similar way with a first kind case, we obtain

$$\begin{aligned}&\{{}_c{}^a{}_b\} = \{{}_c{}^a{}_b\}_2 + f^2\big\{(\partial_c\log f)\delta_b{}^a + (\partial_b\log f)\delta_i{}^a - ({\partial_2}^a\log f)g_{2cb}\big\},\\ &\{{}_i{}^a{}_b\} = f^2(\partial_i\log f)\delta_b{}^a, \quad \{{}_i{}^a{}_h\} = 0,\\ &\{{}_b{}^h{}_a\} = -f^2e^{f^2}({\partial_1}^h\log f)g_{2ba}, \quad \{{}_b{}^h{}_i\} = 0, \quad \{{}_j{}^h{}_i\} = \{{}_j{}^h{}_i\}_1,\end{aligned} \tag{4.4}$$

Equations (4.4) mean

$$\begin{aligned}\nabla_Z W &= \nabla_{2Z}W + f^2\{(Z\log f)W + (W\log f)Z\\ &\qquad - f^2g_2(Z,W)\{({\partial_2}^e\log f)\partial_e + e^{f^2}({\partial_1}^l\log f)\partial_l\},\\ \nabla_Z X &= \nabla_X Z = f^2(X\log f)Z, \quad \nabla_Y X = \nabla_{1Y}X\end{aligned} \tag{4.5}$$

for any $Y, X \in \mathcal{D}$, $Z, W \in \mathcal{D}^\perp$, where ∇_1 (resp. ∇_2) denotes the covariant differentiation with respect to g_1 (resp. g_2).

By virtue of (2.2) and (4.5), we obtain

Proposition 4.3. *For the second kind twisted product CR-submanifold $M = M_\perp \times_f M_\top$ in an l.c.K.-manifold $\tilde{M}$, we have*

$$\tilde{g}(\sigma(Y, JX), JZ) = \tilde{g}(\alpha^\sharp, Z)\tilde{g}(X, Y) + \tilde{g}(\alpha^\sharp, JZ)\tilde{g}(X, JY), \tag{1}$$

$$\tilde{g}(\sigma(X, Y), JZ) = -\tilde{g}(\alpha^\sharp, JZ)\tilde{g}(X, Y) \text{ and } \tilde{g}(\alpha^\sharp, Z) = 0, \tag{2}$$

$$\tilde{g}(\sigma(JX, Z), JW) = \{-\tilde{g}(\alpha^\sharp, X) + f^2 X \log f\}\tilde{g}(Z, W) \tag{3}$$

for any $Y, X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\perp$.

The proof of Proposition 4.3 is similar to Proposition 4.1. So, we omit it.

By virtue of (2) in Proposition 4.3, we know

Proposition 4.4. *For the second kind twisted product CR-submanifold $M = M_\perp \times_f M_\top$ in an l.c.K.-manifold $\tilde{M}$, the Lee vector field $\alpha^\sharp$ is orthogonal to the totally real distribution $\mathcal{D}^\perp$, automatically.*

5. The length of the second fundamental form

In this section, we consider the length $\|\sigma\|$ of the second fundamental form σ of twisted product CR-submanifolds $M = M_\top \times_f M_\perp$ and $M = M_\perp \times_f M_\top$ in an l.c.K.-manifold $\tilde{M}$.

Using the adapted frame, the length $\|\sigma\|$ of the second fundamental form σ is defined as

$$\|\sigma\|^2 = \sum_{r=n+1}^{m} \sum_{\mu,\lambda=1}^{n} \{\tilde{g}(\sigma(e_\lambda, e_\mu), e_r)\}^2. \tag{5.1}$$

The equation (5.1) is separated as

$$\begin{aligned}
\|\sigma\|^2 &= \sum_{r=n+1}^{n+q} \sum_{\mu,\lambda=1}^{n} \{\tilde{g}(\sigma(e_\lambda, e_\mu), e_r)\}^2 + \sum_{r=n+q+1}^{m} \sum_{\mu,\lambda=1}^{n} \{\tilde{g}(\sigma(e_\lambda, e_\mu), e_r)\}^2 \\
&= \sum_{a=1}^{q} \sum_{\mu,\lambda=1}^{n} \{\tilde{g}(\sigma(e_\lambda, e_\mu), Je_{2p+a})\}^2 + \sum_{r=n+q+1}^{m} \sum_{\mu,\lambda=1}^{n} \{\tilde{g}(\sigma(e_\lambda, e_\mu), e_r)\}^2 \\
&= \sum_{a=1}^{q} \sum_{j,i=1}^{2p} \{\tilde{g}(\sigma(e_j, e_i), e^*_{2p+a})\}^2 + 2\sum_{i=1}^{2p} \sum_{b,a=1}^{q} \{\tilde{g}(\sigma(e_i, e_{2p+b}), e^*_{2p+a})\}^2 \\
&\quad + \sum_{c,b,a=1}^{q} \{\tilde{g}(\sigma(e_{2p+b}, e_{2p+b}), e^*_{2p+a})\}^2 + \sum_{r=n+q+1}^{m} \sum_{\mu,\lambda=1}^{n} \{\tilde{g}(\sigma(e_\lambda, e_\mu), e_r)\}^2,
\end{aligned}$$

that is,

$$\begin{aligned}
\|\sigma\|^2 &= \sum_{a=1}^{q} \sum_{j,i=1}^{2p} \{\tilde{g}(\sigma(e_j, e_i), e^*_{2p+a})\}^2 + 2\sum_{i=1}^{2p} \sum_{b,a=1}^{q} \{\tilde{g}(\sigma(e_i, e_{2p+b}), e^*_{2p+a})\}^2 \\
&\quad + \sum_{r=n+q+1}^{m} \sum_{\mu,\lambda=1}^{n} \{\tilde{g}(\sigma(e_\lambda, e_\mu), e_r)\}^2 + \sum_{c,b,a=1}^{q} \{\tilde{g}(\sigma(e_{2p+c}, e_{2p+b}), e^*_{2p+a})\}^2.
\end{aligned} \tag{5.2}$$

Now, assume that our submanifold is the first kind of twisted product CR-submanifold $M_\top \times_f M_\perp$ in an l.c.K.-manifold $\tilde{M}$. Then we have from Proposition 4.1,

$$\begin{aligned}
\tilde{g}(\sigma(e^*{}_i, e_{2p+a}), e^*{}_{2p+b}) &= -\tilde{g}(\alpha^\sharp, e_i)\delta_{ba},\\
\tilde{g}(\sigma(e_i, e_{2p+a}), e^*_{2p+b}) &= \tilde{g}(\alpha^\sharp, e^*_i)\delta_{ba},\\
\tilde{g}(\sigma(e_i, e^*{}_j), e_{2p+a}) &= \{\tilde{g}(\alpha^\sharp, e_{2p+a}) - f^2(e_{2p+a}\log f)\}\delta_{ji}.
\end{aligned}$$

for any $j, i \in \{1, 2, \ldots, p\}$ and $a, b \in \{1, 2, \ldots, q\}$.

Using the above equation, we obtain

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{b,a=1}^{q}\tilde{g}(\sigma(e^*_i, e_{2p+a}), e^*_{2p+b}) = -q\sum_{i=1}^{p}\tilde{g}(\alpha^\sharp, e_i),\\
&\sum_{i=1}^{p}\sum_{b,a=1}^{q}\tilde{g}(\sigma(e_i, e_{2p+a}), e^*_{2p+b}) = q\sum_{i=1}^{p}\tilde{g}(\alpha^\sharp, e^*_j),\\
&\sum_{j,i=1}^{2p}\tilde{g}(\sigma(e_i, e^*_j), e^*_{2p+a}) = 2p\{\tilde{g}(\alpha^\sharp, e_{2p+a}) - f^2(e_{2p+a}\log f)\}.
\end{aligned}\tag{5.3}$$

Substituting (5.3) into (5.2), we have

$$\begin{aligned}
\|\sigma\|^2 = {}& 2\{p\|\alpha^\sharp_{\mathcal{D}^\perp}\|^2 + q\|\alpha^\sharp_{\mathcal{D}}\|^2\} + 2p\sum_{a=1}^{q}\{\tilde{g}(\alpha^\sharp, e_{2p+a}) - f^2(e_{2p+a}\log f)\}^2\\
&+ \sum_{c,b,a=1}^{q}\{\tilde{g}(\sigma(e_{2p+c}, e_{2p+b}), e^*_{2p+a})\}^2 + \sum_{r=n+q+1}^{m}\sum_{\mu,\lambda=1}^{n}\{\tilde{g}(\sigma(e_\lambda, e_\mu), e_r)\}^2.
\end{aligned}\tag{5.4}$$

where $\|\alpha^\sharp_{\mathcal{D}^\perp}\|$ (resp. $\|\alpha^\sharp_{\mathcal{D}}\|$) denotes the length of $\alpha^\sharp$ in $\mathcal{D}^\perp$ (resp. $\mathcal{D}$)-part. Hence, we have

$$\|\sigma\|^2 \geqslant 2\{p\|\alpha^\sharp_{\mathcal{D}^\perp}\|^2 + q\|\alpha^\sharp_{\mathcal{D}}\|^2\} + 2p\sum_{a=1}^{q}\{\tilde{g}(\alpha^\sharp, e_{2p+a}) - f^2(e_{2p+a}\log f)\}^2.\tag{5.5}$$

Thus we have

THEOREM 5.1. *In the first kind of twisted product CR-submanifold $M = M_\top \times_f M_\perp$ in an l.c.K.-manifold $\tilde{M}$, we have* (5.4). *The equality of* (5.5) *is satisfied if and only if the second fundamental form σ satisfies $\sigma(\mathcal{D}^\perp, \mathcal{D}^\perp) \subset \nu$ and $\sigma(TM, TM) \subset J\mathcal{D}^\perp$.*

COROLLARY 5.1. *In the first kind of twisted product CR-submanifold $M = M_\top \times_f M_\perp$ in an l.c.K.-manifold $\tilde{M}$, inequality* (5.5) *satisfies the equality, then the submanifold M is $\mathcal{D}^\perp$-geodesic.*

Next, we consider the second kind of twisted product CR-submanifold $M = M_\perp \times_f M_\top$. Then we have from Proposition 4.3

$$\begin{aligned}
\tilde{g}(\sigma(e_j, e_i), e^*{}_{2p+a}) &= \tilde{g}(\alpha^\sharp, e_{2p+a})\delta_{ji} + \tilde{g}(\alpha^\sharp, e^*{}_{2p+a})\tilde{g}(e_j, e^*{}_i),\\
\tilde{g}(\sigma(e_j, e_i), e^*{}_{2p+a}) &= -\tilde{g}(\alpha^\sharp, e^*{}_{2p+a})\delta_{ji},
\end{aligned}\tag{5.6}$$

$$\tilde{g}(\sigma(e^*{}_i, e_{2p+b}), e^*{}_{2p+a}) = \{-\tilde{g}((\alpha^\sharp, e_i) + f^2(e_i \log f)\}\delta_{ba}$$

for any $j, i \in \{1, 2, \ldots, 2p\}$ and $b, a \in \{1, 2, \ldots, q\}$.

Using Proposition 4.4 and equation (5.6), we obtain

$$\sum_{a=1}^{q}\sum_{j,i=1}^{2p}\{\tilde{g}(\sigma(e_j, e_i), e^*{}_{2p+a})\}^2 = 2p\sum_{a=1}^{q}\{\tilde{g}((\alpha^\sharp, e^*{}_{2p+a})\}^2$$

$$\begin{aligned}&2\sum_{i=1}^{p}\sum_{b,a=1}^{q}[\tilde{g}(\sigma(e_i, e_{2p+b}), e^*{}_{2p+a})\}^2\\ &\quad= 2q\sum_{i=1}^{p}[\{\tilde{g}(\alpha^\sharp, e^*{}_i) - f^2(e^*{}_i \log f)\}^2 + \{\tilde{g}(\alpha^\sharp, e_i) - f^2(e_i \log f)\}^2].\end{aligned}$$

Hence the length $\|\sigma\|$ satisfies

$$\begin{aligned}\|\sigma\|^2 &= 2p\sum_{a=1}^{q}\{\tilde{g}(\alpha^\sharp, e^*{}_{2p+a})\}^2\\ &\quad+ q\sum_{i=1}^{p}[\{\tilde{g}(\alpha^\sharp, e^*{}_i) - f^2(e^*{}_i \log f)\}^2 + \{\tilde{g}(\alpha^\sharp, e_i) - f^2(e_i \log f)\}^2]\\ &\quad+ \sum_{c,b,a=1}^{q}\{\tilde{g}(\sigma(e_{2p+c}, e_{2p+b}), e^*{}_{2p+a})\}^2 + \sum_{r=n+q+1}^{m}\sum_{\mu,\lambda=1}^{n}\{\tilde{g}(\sigma(e_\mu, e_\lambda), e_r)\}^2\\ &= 2p\|\alpha^\sharp{}_{J\mathcal{D}^\perp}\|^2 + 2q[\|\alpha^\sharp_{\mathcal{D}}\|^2 - f^2\sum_{i=1}^{2p}\tilde{g}(\alpha^\sharp, e_i)(e_i \log f)\\ &\quad+ f^4\sum_{i=1}^{p}\{(e_i \log f)(e_i^* \log f)\}^2] + \sum_{c,b,a=1}^{q}\{\tilde{g}(\sigma(e_{2p+c}, e_{2p+b}), e^*{}_{2p+a})\}^2\\ &\quad+ \sum_{r=n+q+1}^{m}\sum_{\mu,\lambda=1}^{n}\{\tilde{g}(\sigma(e_\mu, e_\lambda), e_r)\}^2,\end{aligned}$$

where $\alpha^\sharp{}_{J\mathcal{D}^\perp}$ denotes the $J\mathcal{D}^\perp$-component of $\alpha^\sharp$. Thus we have

$$\begin{aligned}\|\sigma\|^2 &\geqslant 2p\sum_{a=1}^{q}\{\tilde{g}(\alpha^\sharp, e^*{}_{2p+a}\}^2\\ &\quad+ 2q\sum_{i=1}^{p}[\{\tilde{g}(\alpha^\sharp, e^*{}_i) - f^2e^*{}_i \log f\}^2 + \{\tilde{g}(\alpha^\sharp, e_i) - f^2e_i \log f\}^2]\\ &= 2p\|\alpha^\sharp{}_{j\mathcal{D}^\perp}\|^2 + 2q[\|\alpha^\sharp_{\mathcal{D}}\|^2 - f^2\sum_{i=1}^{2p}\tilde{g}(\alpha^\sharp, e_i)(e_i \log f)\\ &\quad+ f^4\sum_{i=1}^{p}\{(e_i \log f)(e_i^* \log f)\}^2.\end{aligned} \tag{5.7}$$

By virtue of (5.7), we obtain

THEOREM 5.2. *In a second kind twisted product CR-submanifold $M = M_\perp \times_f M_\top$ in an l.c.K.-manifold $\tilde{M}$, the length $\|\sigma\|$ satisfies inequality* (5.7) *and equality of* (5.7) *is satisfied if and only if $\sigma(\mathcal{D}^\perp, \mathcal{D}^\perp) \subset \nu$. and $\sigma(TM, TM) \subset J\mathcal{D}^\perp$ and then the submanifold M is $\mathcal{D}^\perp$-geodesic.*

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