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THE INDUCED CONNECTIONS ON TOTAL SPACES OF FIBRED MANIFOLDS

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Abstract. Let $Y \to M$ be a fibred manifold with m-dimensional base and n-dimensional fibres. If $m \geqslant 2$ and $n \geqslant 3$, we classify all linear connections $A(\Gamma, \Lambda, \Theta) : TY \to J^1(TY \to Y)$ in $TY \to Y$ (i.e., classical linear connections on Y) depending canonically on a system $(\Gamma, \Lambda, \Theta)$ consisting of a general connection $\Gamma : Y \to J^1Y$ in $Y \to M$, a torsion free classical linear connection $\Lambda : TM \to J^1(TM \to M)$ on M and a linear connection $\Theta : VY \to J^1(VY \to Y)$ in the vertical bundle $VY \to Y$.

Introduction

All manifolds considered in the paper are assumed to be Hausdorff, second countable, without boundary, finite dimensional and smooth (of class C^∞). Maps between manifolds are assumed to be smooth (infinitely differentiable).

Let $Y \to M$ be a fibred manifold with m-dimensional base M and n-dimensional fibres. Let $\Gamma : Y \to J^1Y$ be a general connection in a fibred manifold $Y \to M$ (i.e., a section of the first jet prolongation $\pi^1_0 : J^1Y \to Y$ of $Y \to M$), $\Lambda : TM \to J^1(TM \to M)$ be a torsion free linear connection in the tangent bundle $TM \to M$ of M (i.e., a torsion free classical linear connection on M) and $\Theta : VY \to J^1(VY \to Y)$ be a linear connection in the vertical bundle $VY \to Y$ of $Y \to M$ (i.e., a vertical classical linear connection on $Y \to M$). More on connections can be found in [**6**].

Here we study how to construct canonically a linear connection $A(\Gamma, \Lambda, \Theta) : TY \to J^1(TY \to Y)$ in $TY \to Y$ (i.e., a classical linear connection on the total space Y) from the system $(\Gamma, \Lambda, \Theta)$ as above.

For example, one can construct a linear connection $\Psi = \Psi(\Gamma, \Lambda, \Theta) : TY \to J^1(TY \to Y)$ in $TY \to Y$ as follows. We decompose $Z \in T_yY$ into the horizontal part $h(Z) = \Gamma(y, Z_0)$, $Z_0 \in T_xM$, $x = p(y)$ and the vertical part vZ. We take a vector field X on M such that $j^1_xX = \Lambda(Z_0)$ and construct its Γ-lift $\Gamma X : Y \to TY$,

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and we take a vertical vector field $\theta Z : Y \to VY$ such that $j^1_y(\theta Z) = \Theta(vZ)$. For every $Z \in T_yY$ we define

$$\Psi(Z) = j^1_y(\Gamma X + \theta Z).$$

The coordinate expression of Ψ can be found in Section 1.

In Section 2, using the torsion of Ψ, we produce 12 tensor fields $\tau_i = \tau_i(\Gamma, \Lambda, \Theta)$ $(i = 1, \dots, 12)$ of type $T^* \otimes T^* \otimes T$ on Y.

The main result of the paper is the following one. If $m \geqslant 2$ and $n \geqslant 3$, then the canonical constructions in question form the 12-parameter family $\Psi + \sum_i \lambda^i \tau_i$ for real numbers λ^i, $i = 1, \dots, 12$.

1. The coordinate expression

Let $x^1, \dots, x^m$ be the usual coordinates on $\mathbf{R}^m$. Let $\mathbf{R}^{m,n}$ be the trivial bundle over $\mathbf{R}^m$ with the standard fiber $\mathbf{R}^n$ and $x^1, \dots, x^m, y^1, \dots, y^n$ be the usual fiber coordinates on $\mathbf{R}^{m,n}$. Let $\eta^1, \dots, \eta^n$ be the additional coordinates on $V\mathbf{R}^{m,n}$ and ξ^i be the additional coordinates in $T\mathbf{R}^m$. Let $(\Gamma, \Lambda, \Theta)$ and $\Psi = \Psi(\Gamma, \Lambda, \Theta)$ be as in Introduction. Let $dy^p = F^p_i(x, y)dx^i$ be the coordinate expression of Γ,

$$d\xi^i = \Lambda^i_{jk}(x)\xi^j dx^k \tag{1.1}$$

be the coordinate expression of Λ, and $d\eta^p = \Theta^p_{qi}(x, y)\eta^q dx^i + \Theta^p_{qs}(x, y)\eta^q dy^s$ be the coordinate expression of Θ. Then we have the following lemma.

LEMMA 1.1. *The coordinate expression of* $\Psi = \Psi(\Gamma, \Lambda, \Theta)$ *is* (1.1) *and*

$$d\eta^p = \Big(\frac{\partial F^p_i}{\partial x^j}\xi^i + F^p_i \Lambda^i_{kj}\xi^k + \Theta^p_{qj}(\eta^q - F^q_i\xi^i)\Big)dx^j + \Big(\frac{\partial F^p_i}{\partial y^s}\xi^i + \Theta^p_{qs}(\eta^q - F^q_i\xi^i)\Big)dy^s. \tag{1.2}$$

where (the same letters) $\xi^1, \dots, \xi^m, \eta^1, \dots, \eta^n$ *denote the usual additional coordinates on* $T\mathbf{R}^{m,n}$.

PROOF. Let $\xi^i = X^i(x)$ and $\eta^p = (\theta Z)^p(x, y)$ be the coordinate expression of X or θZ, respectively. Hence

$$\frac{\partial X^i}{\partial x^j} = \Lambda^i_{kj}X^k, \quad \frac{\partial(\theta Z)^p}{\partial x^j} = \Theta^p_{qj}(\theta Z)^q, \quad \frac{\partial(\theta Z)^p}{\partial y^s} = \Theta^p_{qs}(\theta Z)^q.$$

Then the coordinate expression of $\Gamma X + \theta Z$ is

$$\xi^i = X^i(x) \text{ and } \eta^p = F^p_i(x, y)X^i(x) + (\theta Z)^p(x, y).$$

Differentiating this relation, we obtain (1.2). $\square$

2. Main examples

Let $(\Gamma, \Lambda, \Theta)$ be the triple as in the introduction. According to the usual Γ-decomposition $TY = VY \oplus_Y H^\Gamma Y$ we have the decomposition

$$\begin{aligned} T^*Y \otimes TY = {} & (V^*Y \otimes VY) \oplus_Y (V^*Y \otimes H^\Gamma Y) \\ & \oplus_Y ((H^\Gamma Y)^* \otimes VY) \oplus_Y ((H^\Gamma Y)^* \otimes H^\Gamma Y). \end{aligned}$$

Let id_{HY} be the tensor field of type $T^* \otimes T$ on Y being the $(H^\Gamma Y)^* \otimes H^\Gamma Y$-component of the identity tensor field id_{TY} on Y (the other 3 component of id_{HY} are zero). Let id_{VY} be the tensor field of type $T^* \otimes T$ on Y being the $V^*Y \otimes VY$-component of id_{TY} (the other 3 components of id_{VY} are zero).

Quite similarly, we have the decomposition

$$\begin{aligned} T^*Y \otimes T^*Y \otimes TY = {}& (V^*Y \otimes V^*Y \otimes VY) \oplus_Y (V^*Y \otimes V^*Y \otimes H^\Gamma Y) \\ & \oplus_Y (V^*Y \otimes (H^\Gamma Y)^* \otimes VY) \oplus_Y (V^*Y \otimes (H^\Gamma Y)^* \otimes H^\Gamma Y) \\ & \oplus_Y ((H^\Gamma Y)^* \otimes V^*Y \otimes VY) \oplus_Y ((H^\Gamma Y)^* \otimes V^*Y \otimes H^\Gamma Y) \\ & \oplus_Y ((H^\Gamma Y)^* \otimes H^\Gamma Y)^* \otimes VY) \oplus_Y ((H^\Gamma Y)^* \otimes (H^\Gamma Y)^* \otimes H^\Gamma Y). \end{aligned}$$

Let $\mathsf{Tor}^{H^*\otimes V^*\otimes V}(\Psi)$ be the $(H^\Gamma Y)^* \otimes V^*Y \otimes VY$-component of the torsion tensor field $\mathsf{Tor}(\Psi)$ of the classical linear connection $\Psi = \Psi(\Gamma, \Lambda, \Theta)$ (from Introduction). This components can be treated as the tensor field of type $T^* \otimes T^* \otimes T$ on Y (the other 7 components of it are zero). Taking contraction $C_2^1 : T^*Y \otimes T^*Y \otimes TY \to T^*Y$, $C_2^1(\omega_1 \otimes \omega_2 \otimes v_1) = \langle \omega_2, v_1 \rangle \omega_1$, we produce tensor field $C_2^1 \,\mathsf{Tor}^{H^*\otimes V^*\otimes V}(\Psi)$ of type T^* on Y (horizontal vector field). Similarly, let $\mathsf{Tor}^{H^*\otimes H^*\otimes V)}(\Psi)$ or $\mathsf{Tor}^{V^*\otimes H^*\otimes V}(\Psi)$ or $\mathsf{Tor}^{V^*\otimes V^*\otimes V}(\Psi)$ be the (treated as the tensor field of type $T^*\otimes T^*\otimes T$ on Y) $(H^\Gamma Y)^*\otimes(H^\Gamma Y)^*\otimes VY$- or $V^*Y\otimes(H^\Gamma Y)^*\otimes VY$- or $V^*Y\otimes V^*Y\otimes VY$-component of $\mathsf{Tor}(\Psi)$, respectively. Thus we have the following tensor fields of type $T^* \otimes T^* \otimes T$ on Y canonically depending on $(\Gamma, \Lambda, \Theta)$.

Example 2.1. $\tau_1(\Gamma, \Lambda, \Theta) := \mathsf{Tor}^{H^*\otimes H^*\otimes V}(\Psi)$.

Example 2.2. $\tau_2(\Gamma, \Lambda, \Theta) := \mathsf{Tor}^{H^*\otimes V^*\otimes V}(\Psi)$.

Example 2.3. $\tau_3(\Gamma, \Lambda, \Theta) := \mathrm{id}_{HY} \otimes C_2^1 \,\mathsf{Tor}^{H^*\otimes V^*\otimes V}(\Psi)$.

Example 2.4. $\tau_4(\Gamma, \Lambda, \Theta) := C_2^1 \,\mathsf{Tor}^{H^*\otimes V^*\otimes V}(\Psi) \otimes \mathrm{id}_{HY}$.

Example 2.5. $\tau_5(\Gamma, \Lambda, \Theta) := C_2^1 \,\mathsf{Tor}^{H^*\otimes V^*\otimes V}(\Psi) \otimes \mathrm{id}_{VY}$.

Example 2.6. $\tau_6(\Gamma, \Lambda, \Theta) := \mathrm{id}_{VY} \otimes C_2^1 \,\mathsf{Tor}^{H^*\otimes V^*\otimes V}(\Psi)$.

Example 2.7. $\tau_7(\Gamma, \Lambda, \Theta) = \mathsf{Tor}^{V^*\otimes V^*\otimes V}(\Psi)$.

Example 2.8. $\tau_8(\Gamma, \Lambda, \Theta) := \mathrm{id}_{HY} \otimes C_2^1 \,\mathsf{Tor}^{V^*\otimes V^*\otimes V}(\Psi)$.

Example 2.9. $\tau_9(\Gamma, \Lambda, \Theta) := C_2^1 \,\mathsf{Tor}^{V^*\otimes V^*\otimes V}(\Psi) \otimes \mathrm{id}_{HY}$.

Example 2.10. $\tau_{10}(\Gamma, \Lambda, \Theta) := \mathrm{id}_{VY} \otimes C_2^1 \,\mathsf{Tor}^{V^*\otimes V^*\otimes V}(\Psi)$.

Example 2.11. $\tau_{11}(\Gamma, \Lambda, \Theta) := C_2^1 \,\mathsf{Tor}^{V^*\otimes V^*\otimes V}(\Psi) \otimes \mathrm{id}_{VY}$.

Example 2.12. $\tau_{12}(\Gamma, \Lambda, \Theta) := \mathsf{Tor}^{V^*\otimes H^*\otimes V}(\Psi)$.

3. Natural operators

Let $\mathcal{FM}_{m,n}$ be the category of fibred manifolds with m-dimensional bases and n-dimensional fibres and their fibred (local) diffeomorphisms. The general concept of natural operators can be found in [**6**]. We need the following particular cases of natural operators, only.

Definition 3.1. An $\mathcal{FM}_{m,n}$-natural operator A sending systems $(\Gamma, \Lambda, \Theta)$ as in Introduction on fibred manifolds $Y \to M$ into classical linear connections $A_Y(\Gamma, \Lambda, \Theta)$ on Y is an $\mathcal{FM}_{m,n}$-invariant system of regular operators

$$A_Y : \mathrm{Con}(Y) \times \mathrm{Con}^0_{\mathrm{clas}}(M) \times \mathrm{Con}_{\mathrm{vert-clas}}(Y) \to \mathrm{Con}_{\mathrm{clas}}(Y)$$

for any $\mathcal{FM}_{m,n}$-object $Y = (Y \to M)$, where $\mathrm{Con}(Y)$ is the set of general connections Γ in $Y \to M$, $\mathrm{Con}^0_{\mathrm{clas}}(M)$ is the set of torsion free classical linear connections Λ on M, $\mathrm{Con}_{\mathrm{vert-clas}}(Y)$ is the set of vertical classical linear connections Θ on $Y \to M$ and $\mathrm{Con}_{\mathrm{clas}}(Y)$ is the set of classical linear connections on Y. The $\mathcal{FM}_{m,n}$-invariance of A means that if $(\Gamma, \Lambda, \Theta) \in \mathrm{Con}(Y) \times \mathrm{Con}^0_{\mathrm{clas}}(M) \times \mathrm{Con}_{\mathrm{vert-clas}}(Y)$ is f-related to $(\Gamma_1, \Lambda_1, \Theta_1) \in \mathrm{Con}(Y_1) \times \mathrm{Con}^0_{\mathrm{clas}}(M_1) \times \mathrm{Con}_{\mathrm{vert-clas}}(Y_1)$ for a $\mathcal{FM}_{m,n}$-map $f : Y \to Y_1$ with the base map $\underline{f} : M \to M_1$, then $A_Y(\Gamma, \Lambda, \Theta)$ and $A_{Y_1}(\Gamma_1, \Lambda_1, \Theta_1)$ are f-related. The regularity of A means that A_Y transforms smoothly parametrized families into smoothly parametrized families.

Clearly, the construction of classical linear connection $\Psi(\Gamma, \Lambda, \Theta)$ (from Introduction) determines a natural operator in the above sense.

To classify all natural operators in the sense of Definition 3.1, it suffices to classify all natural operators in the following sense.

Definition 3.2. An $\mathcal{FM}_{m,n}$-natural operator A sending systems $(\Gamma, \Lambda, \Theta)$ as in Introduction on fibred manifolds $Y \to M$ into tensor fields $A_Y(\Gamma, \Lambda, \Theta)$ of type $T^* \otimes T^* \otimes T$ on Y is an $\mathcal{FM}_{m,n}$-invariant system of regular operators

$$A_Y : \mathrm{Con}(Y) \times \mathrm{Con}^0_{\mathrm{clas}}(M) \times \mathrm{Con}_{\mathrm{vert-clas}}(Y) \to \mathsf{Ten}^{(1,2)}(Y)$$

for any $\mathcal{FM}_{m,n}$-object $Y \to M$, where $\mathsf{Ten}^{(1,2)}(Y)$ is the space of tensor fields of type $T^* \otimes T^* \otimes T$ on Y.

Of course, the constructions of tensor fields $\tau_i(\Gamma, \Lambda, \Theta)$ from Examples 2.1–2.12 determine natural operators in the sense of Definition 3.2.

4. Estimation of dimension

We denote the trivial general connection on $\mathbf{R}^{m,n}$ by Γ^0 (i.e., $\Gamma^0 = \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}$), the torsion free flat classical linear connection on $\mathbf{R}^m$ by Λ^0 (i.e., $\Lambda^0 = (0)$) and the trivial vertical classical linear connection on $\mathbf{R}^{m,n}$ by Θ^0 (i.e., $\Theta^0 = \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{p=1}^n dy^p \otimes \frac{\partial}{\partial y^p}$).

We study a natural operators A in the sense of Definition 3.2. For simplicity, we will omit the subscripts Y on A_Y. It is clear that A is determined by the values $A(\Gamma, \Lambda, \Theta)(y) \in T^*_yY \otimes T^*_yY \otimes T_yY$ for fibred manifolds $p_Y : Y \to M$ with m-dimensional bases and n-dimensional fibres, general connections Γ on $Y \to M$, torsion free classical linear connections Λ on M and vertical classical linear connections Θ on $Y \to M$ and $y \in Y_x$, $x \in M$. More, using the invariance of A with respect to (respective) fibred manifold charts, we can assume $Y = \mathbf{R}^{m,n}$, $y = (0,0)$. Further, using Corollary 19.8 in [**6**], we may assume

$$\Gamma = \Gamma^0 + \sum F^p_{j;\alpha\beta} x^\alpha y^\beta dx^j \otimes \frac{\partial}{\partial y^p}, \tag{4.1}$$

where the sum is over all m-tuples α and all n-tuples β of non-negative integers and $j = 1, \dots, m$ and $p = 1, \dots, n$ with $1 \leqslant |\alpha| + |\beta| \leqslant K$ (we can assume $F^p_{j;(0)(0)} = 0$ by the existence of respective "adapted" (for Γ) fibred coordinates),

$$\Lambda = \Big(\sum \Lambda^i_{jk;\gamma} x^\gamma\Big)_{i,j,k=1,\dots,m}, \quad \Lambda^i_{jk;\gamma} = \Lambda^i_{kj;\gamma}, \tag{4.2}$$

where the sums are over all m-tuples γ of non-negative integers with $1 \leqslant |\gamma| \leqslant K$ (we can assume $\Lambda^i_{jk;(0)} = 0$ by the existence of Λ-normal coordinates on the base and the fact that torsion free classical linear connection has vanishing symbols in the center of normal coordinates),

$$\Theta = \Theta^0 + \sum \Theta^r_{ip;\delta\sigma} x^\delta y^\sigma \eta^p dx^i \otimes \frac{\partial}{\partial \eta^r} + \sum \Theta^r_{sp;\delta\sigma} x^\delta y^\sigma \eta^p dy^s \otimes \frac{\partial}{\partial \eta^r}, \tag{4.3}$$

where the first sum is over all m-tuples δ and all n-tuples σ of non-negative integers and $i = 1, \dots, m$ and $r, p = 1, \dots, n$ with $0 \leqslant |\delta| + |\sigma| \leqslant K$ and the second sum is over all m-tuples δ and n-tuples σ of non-negative integers and $r, s, p = 1, \dots, n$ with $0 \leqslant |\delta| + |\sigma| \leqslant K$, where K is an arbitrary positive integer.

(More precisely, from Corollary 19.8 in [**6**], given $(\Gamma, \Lambda, \Theta) \in \mathrm{Con}(\mathbf{R}^{m,n}) \times \mathrm{Con}^0_{\mathrm{clas}}(\mathbf{R}^m) \times \mathrm{Con}_{\mathrm{vert-clas}}(\mathbf{R}^{m,n})$, there exists a finite number $r = r(\Gamma, \Lambda, \Theta)$ such that for any $\Gamma_1 \in \mathrm{Con}(\mathbf{R}^{m,n})$ we have the following implication

$$j^r_{(0,0)}\Gamma_1 = j^r_{(0,0)}\Gamma \Rightarrow A(\Gamma_1, \Lambda, \Theta)(0,0) = A(\Gamma, \Lambda, \Theta)(0,0),$$

i.e., we may replace Γ by Γ^1 being polynomial. Next, by the quite similar argument, we can replace Λ by Λ^1 being polynomial. Next, by the quite similar argument, we can replace Θ by Θ^1 being polynomial.)

So, A is determined by the collection of smooth maps $A_K : \mathbf{R}^{n(K)} \to \mathbf{R}^q = T^*_{(0,0)}\mathbf{R}^{m,n} \otimes T^*_{(0,0)}\mathbf{R}^{m,n} \otimes T_{(0,0)}\mathbf{R}^{m,n}$ $(K = 1, 2, \dots)$ given by

$$A_K((F^p_{j;\alpha\beta}), (\Lambda^i_{jk;\gamma}), (\Theta^r_{ip;\delta\sigma}), (\Theta^r_{sp;\delta\sigma})) := A(\Gamma, \Lambda, \Theta)(0,0),$$

where Γ, Λ, Θ are as in (4.1),(4.2) and (4.3).

Using the invariance of A with respect to $\varphi_t \times \phi_t$, $\varphi_t = t\,\mathrm{id}_{\mathbf{R}^m}$, $\phi_t = t\,\mathrm{id}_{\mathbf{R}^n}$, $t > 0$, we get the homogeneous condition

$$\begin{aligned} &tA_K((F^p_{j;\alpha\beta}), (\Lambda^i_{jk;\gamma}), (\Theta^r_{ip;\delta\sigma}), (\Theta^r_{sp;\delta\sigma})) \\ &\quad = A_K((t^{|\alpha|+|\beta|}F^p_{j;\alpha\beta}), (t^{|\gamma|+1}\Lambda^i_{jk;\gamma}), (t^{|\delta|+|\sigma|+1}\Theta^r_{ip;\delta\sigma}), (t^{|\delta|+|\sigma|+1}\Theta^r_{sp;\delta\sigma})). \end{aligned}$$

By the homogeneous function theorem [**6**], from this homogeneity condition we obtain.

Lemma 4.1. *A_K is independent of $F^p_{j;\alpha\beta}$ with $|\alpha|+|\beta|\geqslant 2$, A is independent of $\Lambda^i_{jk;\gamma}$ with $|\gamma|\geqslant 1$, A_K is independent of $\Theta^r_{ip;\delta\sigma}$ with $|\delta|+|\sigma|\geqslant 1$ and A_K is independent of $\Theta^r_{sp;\delta\sigma}$ with $|\delta|+|\sigma|\geqslant 1$. Even, A_K is a linear combination with real coefficients of $\Theta^r_{ip;(0)(0)}$, $\Theta^r_{sp;(0)(0)}$ and $F^p_{j;\alpha\beta}$ with $|\alpha|+|\beta|=1$, $i,j=1,\dots,m$, $p,r,s=1,\dots,n$. In particular, $A_K(\Gamma^0,\Lambda^0,\Theta^0)(0,0)=0$.*

Now, we prove the following two lemmas.

Lemma 4.2. *Let $m\geqslant 2$ and $n\geqslant 2$. Any natural operator A in the sense of Definition 3.2 is fully determined by the collection of values*

$$A^1 := A\Big(\Gamma^0 + x^2dx^1\otimes\frac{\partial}{\partial y^1},\Lambda^0,\Theta^0\Big)(0,0),$$

$$A^2 := A\Big(\Gamma^0 + y^1dx^1\otimes\frac{\partial}{\partial y^1},\Lambda^0,\Theta^0\Big)(0,0),$$

$$A^3 := A\Big(\Gamma^0,\Lambda^0,\Theta^0 + \eta^1dy^2\otimes\frac{\partial}{\partial \eta^1}\Big)(0,0),$$

where $\Gamma^0,\Lambda^0,\Theta^0$ are the trivial connections.

Proof. We know that the collection of maps A_K for $K=1,2,\dots$ determines A. Then, using Lemma 4.1, it remains to prove that:

(a) $A(\Gamma^0+x^{j_0}dx^{i_0}\otimes\frac{\partial}{\partial y^{p_0}},\Lambda^0,\Theta^0)(0,0)$ is determined by A^1,

(b) $A(\Gamma^0+y^{q_0}dx^{i_0}\otimes\frac{\partial}{\partial y^{p_0}},\Lambda^0,\Theta^0)(0,0)$ is determined by A^2,

(c) $A(\Gamma^0,\Lambda^0,\Theta^0+\eta^{q_0}dx^{i_0}\otimes\frac{\partial}{\partial \eta^{p_0}})(0,0)$ is determined by A^2, and

(d) $A(\Gamma^0,\Lambda^0,\Theta^0+\eta^{q_0}dy^{r_0}\otimes\frac{\partial}{\partial \eta^{p_0}})(0,0)$ is determined by A^3

We start with the proof of (a). By the invariance of A with respect to the (local) $\mathcal{FM}_{m,n}$-map $k=(x^1,\dots,x^m,y^1,\dots,y^{p_0-1},y^{p_0}+\frac{1}{2}(x^{i_0})^2,y^{p_0+1},\dots,y^n)$, from $A(\Gamma^0,\Lambda^0,\Theta^0)(0,0)=0$, we get $A(\Gamma^0+x^{i_0}dx^{i_0}\otimes\frac{\partial}{\partial y^{p_0}},\Lambda^0,\Theta^0)(0,0)=0$ as k preserves Λ^0 and Θ^0 and sends Γ^0 into $\Gamma^0+x^{i_0}dx^{i_0}\otimes\frac{\partial}{\partial y^{p_0}}$. If $i_0\neq j_0$, there exists a permutation of coordinates sending A^1 into $A(\Gamma^0+x^{i_0}dx^{j_0}\otimes\frac{\partial}{\partial y^{p_0}},\Lambda^0,\Theta^0)(0,0)$. So, (a) is complete.

Now, we prove (b). By the invariance of A with respect to the $\mathcal{FM}_{m,n}$-map $f:=(x^1,\dots,x^m,y^1+y^2,y^2,\dots,y^n)$ we see that $A^0=A(\Gamma^0+(y^1-y^2)dx^1\otimes\frac{\partial}{\partial y^1},\Lambda^0,\Theta^0)(0,0)$ is the image of A^2 by f. Therefore $A(\Gamma^0+y^2dx^1\otimes\frac{\partial}{\partial y^1},\Lambda^0,\Theta^0)(0,0)$ $=A^2-A^0$ is determined by A^2. So, using the invariance of A with respect to respective permutations of coordinates, we complete (b).

Next we prove (c). By the invariance of A with respect to the (local) $\mathcal{FM}_{m,n}$-map $g:=(x^1,\dots,x^m,y^1+x^1y^1,y^2,\dots,y^n)$ (then $Vg=(g,\eta^1+x^1\eta^1,\eta^2,\dots,\eta^n)$) from $A(\Gamma^0,\Lambda^0,\Theta^0)=0$ we get that $A(\Gamma^0+y^1dx^1\otimes\frac{\partial}{\partial y^1},\Lambda^0,\Theta^0+\eta^1dx^1\otimes\frac{\partial}{\partial\eta^1})(0,0)=0$. More precisely, g preserves Λ^0 and transforms Γ^0 and Θ^0 into $\Gamma^0+\frac{y^1}{1+x^1}dx^1\otimes\frac{\partial}{\partial y^1}=\Gamma^0+y^1dx^1\otimes\frac{\partial}{\partial y^1}+\dots$ and $\Theta^0+\frac{\eta^1}{1+x^1}dx^1\otimes\frac{\partial}{\partial\eta^1}=\Theta^0+\eta^1dx^1\otimes\frac{\partial}{\partial\eta^1}+\dots$ (where the dots have the 1-jets at $(0,0)$ equal to 0) which can be replaced by $\Gamma^0+y^1dx^1\otimes\frac{\partial}{\partial y^1}$ and $\Theta^0+\eta^1dx^1\otimes\frac{\partial}{\partial\eta^1}$ in A in account of Lemma 4.1. Then

$A' = A(\Gamma^0, \Lambda^0, \Theta^0 + \eta^1 dx^1 \otimes \frac{\partial}{\partial \eta^1})(0,0)$ is determined by A^2 (it is $-A^2$). Then using the invariance of A with respect to f (the one of case (b)) we see that $A'' = A(\Gamma^0, \Lambda^0, \Theta^0 + (\eta^1 - \eta^2) dx^1 \otimes \frac{\partial}{\partial \eta^1})(0,0)$ is determined by A^2 (it is the image of A' by f). Then $A(\Gamma^0, \Lambda^0, \Theta^0 + \eta^2 dx^1 \otimes \frac{\partial}{\partial \eta^1})(0,0)$ is determined by A^2 (it is $A' - A''$). Now, using the invariance of A with respect to respective permutations of coordinates, we complete (c).

Finally, we prove (d). By the invariance of A with respect to the (local) $\mathcal{FM}_{m,n}$-map $H = (x^1, \dots, x^m, y^1 + \frac{1}{2}(y^1)^2, y^2, \dots, y^n)$ (then $VH = (H, \eta^1 + y^1\eta^1, \eta^2, \dots, \eta^n)$), from $A(\Gamma^0, \Lambda^0, \Theta^0)(0,0) = 0$ and Lemma 4.1 we get (using similar arguments as in (c)) that $A''' = A(\Gamma^0, \Lambda^0, \Theta^0 + \eta^1 dy^1 \otimes \frac{\partial}{\partial \eta^1})(0,0) = 0$. If $q_0 \geqslant 2$, by the invariance of A with respect to the $\mathcal{FM}_{m,n}$-map $h =: (x^1, \dots, x^m, y^1 + y^{q_0}, y^2, \dots, y^n)$, we see that $A'''' = A(\Gamma^0, \Lambda^0, \Theta^0 + (\eta^1 - \eta^{q_0}) dy^2 \otimes \frac{\partial}{\partial \eta^1})(0,0)$ is determined by A^3 (it is the image of A^3 by h). Then $A(\Gamma^0, \Lambda^0, \Theta^0 + \eta^{q_0} dy^2 \otimes \frac{\partial}{\partial \eta^1})(0,0)$ is determined by A^3 (it is $A^3 - A''''$). In particular, for $q_0 = 2$, $A''''' = A(\Gamma^0, \Lambda^0, \Theta^0 + \eta^2 dy^2 \otimes \frac{\partial}{\partial \eta^1})(0,0)$ is determined by A^3. Then using the invariance of A with respect to f (the one of case (b)) from $A''' = 0$ we get

$$A\Big(\Gamma^0, \Lambda^0, \Theta^0 + (\eta^1 - \eta^2)(dy^1 - dy^2) \otimes \frac{\partial}{\partial \eta^1}\Big)(0,0) = 0,$$

and consequently $A(\Gamma^0, \Lambda^0, \Theta^0 + \eta^2 dy^1 \otimes \frac{\partial}{\partial \eta^1})(0,0) = A''''' - A^3 + A''' = A''''' - A^3$ is determined by A^3. So, using the invariance of A with respect to respective permutations of coordinates, we complete (d). □

LEMMA 4.3. *Let $m \geqslant 2$ and $n \geqslant 3$. Let A^1, A^2, A^3 be the values from the last lemma. There are real numbers $a_1, \dots, a_{12}$ such that*

$$\begin{aligned}
A^1 &= a_1\Big(d_{(0,0)}x^2 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} - d_{(0,0)}x^1 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial y^1}_{|(0,0)}\Big),\\
A^2 &= a_2 \sum_{p=1}^n d_{(0,0)}x^1 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p}_{|(0,0)} + a_3 \sum_{p=1}^n d_{(0,0)}y^p \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^p}_{|(0,0)}\\
&\quad + a_4 d_{(0,0)}x^1 \otimes d_{(0,0)}y^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} + a_5 d_{(0,0)}y^1 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)}\\
&\quad + a_6 \sum_{i=1}^m d_{(0,0)}x^1 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i}_{|(0,0)} + a_7 \sum_{i=1}^m d_{(0,0)}x^i \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial x^i}_{|(0,0)},\\
A^3 &= a_8 \sum_{p=1}^n d_{(0,0)}y^p \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial y^p}_{|(0,0)} + a_9 \sum_{p=1}^n d_{(0,0)}y^2 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p}_{|(0,0)}\\
&\quad + a_{10} \sum_{i=1}^m d_{(0,0)}x^i \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial x^i}_{|(0,0)} + a_{11} \sum_{i=1}^m d_{(0,0)}y^2 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i}_{|(0,0)}\\
&\quad + a_{12}\Big(d_{(0,0)}y^2 \otimes d_{(0,0)}y^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} - d_{(0,0)}y^1 \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial y^1}_{|(0,0)}\Big).
\end{aligned}$$

Proof. We start with the proof of the first formula. By the invariance of A with respect to the $\mathcal{FM}_{m,n}$-maps $a_{t,\tau} := (t^1x^1, \dots, t^mx^m, \tau^1y^1, \dots, \tau^ny^n)$ for $t^1 > 0, \dots, t^m > 0$ and $\tau^1 > 0, \dots, \tau^n > 0$ we get easily

$$A^1 = b_1 d_{(0,0)}x^2 \otimes d_{(0,0)}x^1 \frac{\partial}{\partial y^1}_{|(0,0)} + b_2 d_{(0,0)}x^1 \otimes d_{(0,0)}x^2 \frac{\partial}{\partial y^1}_{|(0,0)}$$

for some real numbers b_1, b_2. Then (by the invariance of A with respect to permuting x^1 and x^2)

$$A(\Gamma^0 + x^1dx^2 \otimes \frac{\partial}{\partial y^1}, \Lambda^0, \Theta^0)(0,0) = b_1 d_{(0,0)}x^1 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} + b_2 d_{(0,0)}x^2 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)}.$$

But by the invariance of A with respect to the (local) $\mathcal{FM}_{m,n}$-map $(x^1, \dots, x^m, y^1 + x^1x^2, y^3, \dots, y^n)$ from $A(\Gamma^0, \Lambda^0, \Theta^0)(0,0) = 0$, we get (using the similar arguments as in (a) of the proof of Lemma 4.2) that $A(\Gamma^0 + x^2dx^1 \otimes \frac{\partial}{\partial y^1} + x^1dx^2 \otimes \frac{\partial}{\partial y^1}, \Lambda^0, \Theta^0)(0,0) = 0$. Therefore $b_1 = -b_2$. That is why, the first formula of the lemma is complete.

Now, we prove the second formula of the lemma. By the invariance of A with respect to $a_{t,\tau}$ (the same as above), we get immediately

$$A^2 = \sum_{p=1}^{n} b_p d_{(0,0)}y^p \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^p}_{|(0,0)} + \sum_{p=1}^{n} c_p d_{(0,0)}x^1 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p}_{|(0,0)} + \sum_{i=1}^{m} d_i d_{(0,0)}x^i \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial x^i}_{|(0,0)} + \sum_{i=1}^{m} e_i d_{(0,0)}x^1 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i}_{|(0,0)}.$$

Then by the invariance of A with respect to respective permutation of coordinates, we deduce $b_2 = \dots = b_n$, $c_2 = \dots = c_n$, $d_2 = \dots = d_m$, $e_2 = \dots = e_m$. Then $A^2 =$ the right-hand side of the second formula of the lemma $+bdx^1 \otimes dx^1 \otimes \frac{\partial}{\partial x^1}$. Now, by the invariance of A with respect to $(x^1, x^2 + x^1, x^3, \dots, x^m, y^1, \dots, y^n)$ one can obtain that $b = 0$. That is why, the second formula of the lemma is true.

Finally we prove the last formula of the lemma. By the invariance of A with respect to $a_{t,\tau}$ (the same as above) we get immediately

$$A^3 = \sum_{p=1}^{n} b_p d_{(0,0)}y^2 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p}_{|(0,0)} + \sum_{p=1}^{n} c_p d_{(0,0)}y^p \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial y^p}_{|(0,0)} + \sum_{i=1}^{m} d_i d_{(0,0)}y^2 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i}_{|(0,0)} + \sum_{i=1}^{m} e_i d_{(0,0)}x^i \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial x^i}_{|(0,0)}.$$

Then by the invariance of A with respect to respective permutation of coordinates, we deduce $b_3 = \dots = b_n$, $c_3 = \dots = c_n$, $d_1 = \dots = d_m$ and $e_1 = \dots = e_m$. Then

$$(*) \quad A^3 = \lambda_1 d_{(0,0)}y^2 \otimes d_{(0,0)}y^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)}$$

$$+\lambda_2 d_{(0,0)}y^1\otimes d_{(0,0)}y^2\otimes \frac{\partial}{\partial y^1}_{|(0,0)} + \lambda_3 d_{(0,0)}y^2\otimes d_{(0,0)}y^2\otimes \frac{\partial}{\partial y^2}_{|(0,0)}$$
$$+\lambda_4\sum_{p=1}^{n} d_{(0,0)}y^p\otimes d_{(0,0)}y^2\otimes \frac{\partial}{\partial y^p}_{|(0,0)} + \lambda_5\sum_{p=1}^{n} d_{(0,0)}y^2\otimes d_{(0,0)}y^p\otimes \frac{\partial}{\partial y^p}_{|(0,0)}$$
$$+\lambda_6\sum_{i=1}^{m} d_{(0,0)}x^i\otimes d_{(0,0)}y^2\otimes \frac{\partial}{\partial x^i}_{|(0,0)} + \lambda_7\sum_{i=1}^{m} d_{(0,0)}y^2\otimes d_{(0,0)}x^i\otimes \frac{\partial}{\partial x^i}_{|(0,0)}.$$

Then by invariance of A with respect to $\mathcal{FM}_{m,n}$-map $(x^1,\dots,x^m,y^1-y^2,y^2,\dots,y^n)$, from (*) we deduce

$$A^3 + A\Big(\Gamma^0,\Lambda^0,\Theta^0+\eta^2 dy^2\otimes\frac{\partial}{\partial\eta^1}\Big)(0,0) = A^3 + \lambda_1 d_{(0,0)}y^2\otimes d_{(0,0)}y^2\otimes \frac{\partial}{\partial y^1}_{|(0,0)}$$
$$+\lambda_2 d_{(0,0)}y^2\otimes d_{(0,0)}y^2\otimes \frac{\partial}{\partial y^1}_{|(0,0)} - \lambda_3 d_{(0,0)}y^2\otimes d_{(0,0)}y^2\otimes \frac{\partial}{\partial y^1}_{|(0,0)}.$$

On the other hand, from the invariance of A with respect to the (local) $\mathcal{FM}_{m,n}$-map $G=(x^1,\dots,x^m,y^1+\frac12(y^2)^2,y^2,\dots,y^n)$ (then $VG=(G,\eta^1+y^2\eta^2,\eta^2,\dots,\eta^n)$) from $A(\Gamma^0,\Lambda^0,\Theta^0)(0,0)=0$ and Lemma 4.1 (using similar arguments as in (c) of the proof of Lemma 4.2), we get $A(\Gamma^0,\Lambda^0,\Theta^0+\eta^2dy^2\otimes\frac{\partial}{\partial\eta^1})(0,0)=0$. So, $\lambda_1+\lambda_2-\lambda_3=0$. Further, from the invariance of A with respect to $(x^1,\dots,x^m,y^1-y^3,y^2,\dots,y^n)$ (we assume $n\geqslant 3$) from (*) we get (after cancelling A^3)

$$A\Big(\Gamma^0,\Lambda^0,\Theta^0+\eta^3 dy^2\otimes\frac{\partial}{\partial\eta^1}\Big)(0,0) = \lambda_1 d_{(0,0)}y^2\otimes d_{(0,0)}y^3\otimes \frac{\partial}{\partial y^1}_{|(0,0)}$$
$$+\lambda_2 d_{(0,0)}y^3\otimes d_{(0,0)}y^2\otimes \frac{\partial}{\partial y^1}_{|(0,0)}.$$

Then from the invariance of A with respect to the switching y^2 and y^3 we get

$$A\Big(\Gamma^0,\Lambda^0,\Theta^0+\eta^2 dy^3\otimes\frac{\partial}{\partial\eta^1}\Big)(0,0) = \lambda_1 d_{(0,0)}y^3\otimes d_{(0,0)}y^2\otimes \frac{\partial}{\partial y^1}_{|(0,0)}$$
$$+\lambda_2 d_{(0,0)}y^2\otimes d_{(0,0)}y^3\otimes \frac{\partial}{\partial y^1}_{|(0,0)}.$$

On the other hand from the invariance of A with respect to $K=(x^1,\dots,x^m,y^1+y^2y^3,y^2,\dots,y^n)$ (then $VK=(K,\eta^1+y^2\eta^3+y^3\eta^2,\eta^2,\dots,\eta^n)$) from Lemma 4.1 and $A(\Gamma^0,\Lambda^0,\Theta^0)(0,0)=0$, we get (using similar arguments as in (c) of the proof of Lemma 4.2) that $A(\Gamma^0,\Lambda^0,\Theta^0+\eta^3dy^2\otimes\frac{\partial}{\partial\eta^1}+\eta^2dy^3\otimes\frac{\partial}{\partial\eta^1})(0,0)=0$. So, $\lambda_1=-\lambda_2$ (and then $\lambda_3=0$). That is why, the last formula of the lemma holds. □

From Lemmas 4.2 and 4.3 it follows immediately the following proposition.

Proposition 4.1. *If $m\geqslant 2$ and $n\geqslant 3$, the vector space of all natural operators in the sense of Definition 3.2 is of dimension not more than 12.*

5. Linear independence

We prove the following proposition.

Proposition 5.1. *Let $m \geqslant 2$ and $n \geqslant 3$. The natural operators τ_i ($i = 1, \dots, 12$) in the sense of Definition* 3.2 *from Examples* 2.1–2.12 *are linearly independent.*

Proof. By Lemma 4.2, it is sufficient to study the values A^1, A^2, A^3 from Lemma 4.2 for $A = \tau_i$, $i = 1, \dots, 12$. To compute these values, we use Lemma 1.1.

The case $\Sigma = (\Gamma^0 + x^2 dx^1 \otimes \frac{\partial}{\partial y^1}, \Lambda^0, \Theta^0)$. In this case, we have (in the notation of Lemma 1.1) $F^1_1(x,y) = x^2$ and other $F^p_i(x,y) = 0$, $\Lambda^k_{ij} = 0$, $\Theta^p_{sj} = 0$, $\Theta^p_{qs} = 0$. Then (by Lemma 1.1) $d\eta^1 = \xi^1 dx^2$ and other $d\eta^p = 0$, and $d\xi^i = 0$. Then (modulo signum)

$$\mathsf{Tor}(\Psi(\Sigma))(0,0) = d_{(0,0)}x^1 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} - d_{(0,0)}x^2 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)}.$$

Hence (modulo signum)

$$\tau_1(\Sigma)(0,0) = d_{(0,0)}x^1 \otimes d_{(0,0)}x^2 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} - d_{(0,0)}x^2 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)}$$

and $\tau_i(\Sigma)(0,0) = 0$ for $i = 2, \dots, 12$.

The case $\Sigma = (\Gamma^0 + y^1 dx^1 \otimes \frac{\partial}{\partial y^1}, \Lambda^0, \Theta^0)$. In this case, by Lemma 1.1, $d\eta^1 = \xi^1 dy^1$ and other $d\eta^p = 0$, and $d\xi^i = 0$. Then (modulo signum)

$$\mathsf{Tor}(\Psi(\Sigma))(0,0) = d_{(0,0)}x^1 \otimes d_{(0,0)}y^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} - d_{(0,0)}y^1 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)}.$$

Then $\tau_1(\Sigma)(0,0) = 0$ and (modulo signum)

$$\tau_2(\Sigma)(0,0) = d_{(0,0)}x^1 \otimes d_{(0,0)}y^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)},$$

$$\tau_3(\Sigma)(0,0) = \sum_{i=1}^{m} d_{(0,0)}x^i \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial x^i}_{|(0,0)},$$

$$\tau_4(\Sigma)(0,0) = \sum_{i=1}^{m} d_{(0,0)}x^1 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i}_{|(0,0)},$$

$$\tau_5(\Sigma)(0,0) = \sum_{p=1}^{n} d_{(0,0)}x^1 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p}_{|(0,0)},$$

$$\tau_6(\Sigma)(0,0) = \sum_{p=1}^{n} d_{(0,0)}y^p \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^p}_{|(0,0)},$$

$\tau_i(\Sigma)(0,0) = 0$ for $i = 7, \dots, 11$ and (modulo signum) $\tau_{12}(\Sigma)(0,0) = d_{(0,0)}y^1 \otimes d_{(0,0)}x^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)}$.

The case $\Sigma = (\Gamma^0, \Lambda^0, \Theta^0 + \eta^1 dy^2 \otimes \frac{\partial}{\partial \eta^1})$. In this case, by Lemma 1.1, $d\eta^1 = \eta^1 dy^2$ and $d\eta^p = 0$ for other p, and $d\xi^i = 0$. Then (modulo signum)

$$\mathsf{Tor}(\Psi(\Sigma))(0,0) = d_{(0,0)}y^1 \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} - d_{(0,0)}y^2 \otimes d_{(0,0)}y^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)}$$

. Then $\tau_i(\Sigma)(0,0) = 0$ for $i = 1, \dots, 6$, and (modulo signum)

$$\tau_7(\Sigma)(0,0) = d_{(0,0)}y^2 \otimes d_{(0,0)}y^1 \otimes \frac{\partial}{\partial y^1}_{|(0,0)} - d_{(0,0)}y^1 \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial y^1}_{|(0,0)},$$

$$\tau_8(\Sigma)(0,0) = \sum_{i=1}^{m} d_{(0,0)}x^i \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial x^i}_{|(0,0)},$$

$$\tau_9(\Sigma)(0,0) = \sum_{i=1}^{m} d_{(0,0)}y^2 \otimes d_{(0,0)}x^i \otimes \frac{\partial}{\partial x^i}_{|(0,0)},$$

$$\tau_{10}(\Sigma)(0,0) = \sum_{p=1}^{n} d_{(0,0)}y^p \otimes d_{(0,0)}y^2 \otimes \frac{\partial}{\partial y^p}_{|(0,0)},$$

$$\tau_{11}(\Sigma)(0,0) = \sum_{p=1}^{n} d_{(0,0)}y^2 \otimes d_{(0,0)}y^p \otimes \frac{\partial}{\partial y^p}_{|(0,0)}$$

and $\tau_{12}(\Sigma)(0,0) = 0$.

Now, it is easily seen that the natural operators $\tau_1, \dots, \tau_{12}$ are linearly independent. Proposition 5.1 is complete. □

6. The main result

From Propositions 4.1 and 5.1 the main theorem of the paper follows immediately.

THEOREM 6.1. *Let $m \geqslant 2$ and $n \geqslant 3$. Any natural operator A in the sense of Definition 3.1 is of the form $A(\Gamma, \Lambda, \Theta) = \Psi(\Gamma, \Lambda, \Theta) + \sum_i \lambda^i \tau_i(\Gamma, \Lambda, \Theta)$ for some (uniquely determined by A) real numbers λ^i, $i = 1, \dots, 12$, where τ_i are the natural operators described in Examples 2.1–2.12 and $\Psi(\Gamma, \Lambda, \Theta)$ is the connection from Introduction.*

7. Final remarks

Let $Y \to M$ be a fibred manifold with m-dimensional basis and n-dimensional fibres. Let $E \to M$ be a vector bundle with the same base M and n-dimensional fibres (the same n). A vertical parallelism on $Y \to M$ is a vector bundle isomorphism $\Phi : Y \times_M E \to VY$ covering the identity map of Y, i.e., a system of parallelism $\Phi_x : Y_x \times E_x \to TY_x$, $x \in M$. Let Γ be a general connection on $Y \to M$, Λ be a classical linear connection on M, $\Phi : Y \times_M E \to VY$ be a vertical parallelism on $Y \to M$ and Δ be a linear connection on $E \to M$.

In [**5**], Kolář constructed the classical linear connection $(\Gamma, \Lambda, \Phi, \Delta)$ on Y depending canonically on $(\Gamma, \Lambda, \Phi, \Delta)$. Using our connection $\Psi(\Gamma, \Lambda, \Theta)$ from Introduction, we can reobtain the connection $(\Gamma, \Lambda, \Phi, \Delta)$ by Kolář as follows. The system (Δ, Φ) determines a vertical classical linear connection $\Theta = \Theta(\Delta, \Phi) : VY \to$

$J^1(VY \to Y)$ on $Y \to M$. Indeed, for any point $v = \Phi(y, v_0) \in V_yY$, $y \in Y_x$, $v_0 \in E_x$, $x = p(y)$, we take a section $\sigma : M \to E$ such that $j^1_x\sigma = \Delta(v_0)$ and we define $\Theta(v) = j^1_y(\varphi(\sigma))$, where $\varphi(\sigma) : Y \to VY$ is a vertical vector field given by $\varphi(\sigma)(y) = \Phi(y, \sigma(p(y)))$. Now, if we additionally use the connections Γ and Λ we can produce the classical linear connection $\Psi(\Gamma, \Lambda, \Theta(\Delta, \Phi))$, which (as easily to see) coincides with the Kolář connection $(\Gamma, \Lambda, \Phi, \Delta)$. So, the construction of $\Psi(\Gamma, \Lambda, \Theta)$ from Introduction is a generalization of the construction of $(\Gamma, \Lambda, \Phi, \Delta)$ by Kolář [**5**].

If $Y = E \to M$, $\Gamma = \Delta$ and Φ is the canonical vertical parallelism, the connection by Kolář (and then the connection Ψ from Introduction) is a generalization of the classical linear connection on E from Δ by means of Λ presented by Gancarzewicz [**3**]. If $P = Y \to M$ is a principal G-bundle, Γ is principal (i.e., right invariant), $E \to M$ is the usual $\mathcal{L}(G)$-algebra bundle of P and Φ is the canonical vertical parallelism, the connection by Kolář (and then the connection Ψ from Introduction) is a generalization of the classical linear connection $N(\Gamma, \Lambda)$ on P considered in [**6**, p. 415], see [**5**].

In [**7**], we described all classical linear connections $A(\Gamma, \Lambda, \Phi, \Delta)$ on Y canonically depending on the system $(\Gamma, \Lambda, \Phi, \Delta)$. Thus the present paper can be treated as the generalization of [**7**]. In [**7**], we showed that all $A(\Gamma, \Lambda, \Phi, \Delta)$ form the 12-parameter family, too.

Let us also remark, why we must use an auxiliary object (in the paper we use Θ) to construct from Γ and Λ a classical linear connection $A(\Gamma, \Lambda)$ on Y. In the other case we would have a classical linear connection $A(\Gamma^0, \Lambda^0)$ on $\mathbf{R}^m \times \mathbf{R}^n$ which would be $GL(m) \times \mathrm{Diff}(\mathbf{R}^n, \mathbf{R}^n)$ invariant. But this is impossible as the group of affine transformations of a classical linear connection is a (finite dimensional) Lie group.

Classifications of constructions on connections has been studied in many papers, e.g., [**1, 2, 4, 6, 7**], e.t.c.

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