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FIRST INTEGRALS AND EXACT SOLUTIONS OF THE GENERALIZED MODELS OF MAGNETIC INSULATION

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Abstract. We suggest generalizations of the mathematical model of magnetic insulation, described by multidimensional quasi potential ODE system or PDE system with two-dimensional Laplace operator. Existence conditions of the first integrals of a certain type for the class of nonlinear quasi potential systems, including the model vacuum diode are obtained. Integrability of the vacuum diode models is justified. We find for PDE system the class of exact radially symmetric solutions given by fractional-rational functions. The class of systems with variable density, reduced to a similar system with the constant current density by special transformations is specified. The class of exact solutions of the non-singular boundary-value problem in annular domain is found.

1. Introduction

During the recent decade, the international scientific community was involved in the process of fostering plasma-and nano-technologies. In this connection, investigations bound up with understanding interactions between electromagnetic fields and charged particles have been stimulated [**1**]. Solution of such problems has necessitated an interdisciplinary approach, in which mathematical modeling plays an important role. This approach allows the researcher to bring together and coordinate various principles of physics characteristic of the interacting objects of diverse physical nature. Modeling plasma represented as a flow of charged particles interacting in vacuum usually necessitates application of the Vlasov–Maxwell or Vlasov–Poisson equations [**2**–**8**]. When solving these nonlinear systems of partial differential equations (PDEs) with initial and boundary value conditions, it is

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necessary to find the solutions and ascertain their properties for a number of additional conditions (positiveness, monotony, singularity, etc.). And this is a special mathematical problem.

Transition to a new, simpler model described by a system of ordinary differential equations (ODEs) with boundary-value conditions and, nevertheless, retaining the principal physical and other properties typical of the initial (more complicated) model represents a constructive and more efficient way to overcome possible mathematical difficulties. In this way the limit model of magnetic insulation for the plane vacuum diode, which is a system of two nonlinear ODEs of second order is obtained [**9**]. In this connection, it is necessary to obtain a solution for the respective singular boundary value problem and study its properties.

Some results related to this limit model and the boundary-value problem were obtained earlier with the application and combining analytical and numerical methods [**9**,**10**]. Integrability of the limit model is established, a complete system of four first integrals is constructed and a method for solution of the singular boundary-value problem is suggested [**11**]. Moreover, a parametric family of exact solutions is constructed and it is shown that solutions to singular boundary-value problems under certain conditions can be found. Therefore, the limit problem of magnetic insulation [**9**] has been quite interesting mathematical object for the study.

In this paper we consider a generalization of the mathematical model of magnetic insulation [**9**] in two distinct directions. The first one concerns the transition from two-dimensional unknown function to the vector of arbitrary dimension: as a result we obtain the ODE system. The principal questions that we study here are: 1) construct explicit first integrals of a certain class of functions (for example, linear to the given arguments); 2) search integrability conditions. Let us note that as it is specified in the surveys [**12**, **13**], integrable systems have very much importance though they are met rather seldom.

The second one concerns the transition from one-dimensional argument of unknown function to two-dimensional argument, which leads to the system of two PDEs with two-dimensional Laplace operator. The main question here is to build parametric families of exact solutions and satisfy the boundary conditions. The knowledge of explicit exact solutions of the nonlinear PDE is very useful for the qualitative analysis [**14**]. The obtained analytical solution can be usefully applied for testing the other, experimental or numerical results for adequate modeling of modern technologies [**15**].

The generalized models of magnetic insulation, as mentioned above, with the main objectives of the study are given in Section 2. In Sections 3, 4, existence conditions of the first integrals of the "energy" type and linear in "velocity" are presented respectively. As an example Section 5 shows to which integrals we come under the proposed approach in the case of the basic model of magnetic insulation [**9**]. In the next Section 6, we consider systems with the potential of a special type, for which we can prove existence of a family linear with respect to "velocity" integrals. In Section 7, integrability of 12-dimensional model, which is in some sense analogous to the classic two-body problem is justified. The questions of model representation in the Hamiltonian form and integrability are considered in Section

8. Section 9 is devoted to exact solutions of PDEs system with two-dimensional Laplace operator in the case of a constant current density. The same questions for the case of a variable current density are discussed in Section 10. In the last Section 11, the conditions of existence and the explicit form of exact solutions of the boundary-value problems in the annular domain are given.

2. The generalized mathematical models of magnetic insulation and the problem statement

The limit model of a plane vacuum diode has been proposed by a group of mathematicians from the University of Toulouse [**9**]. The model consists of two second-order nonlinear ODEs

$$\frac{d^2\varphi}{dx^2} = j\frac{(1+\varphi)}{\sqrt{(1+\varphi)^2 - a^2 - 1}}, \quad \frac{d^2a}{dx^2} = j\frac{a}{\sqrt{(1+\varphi)^2 - a^2 - 1}}. \tag{2.1}$$

Here the independent variable $x \in [0,1]$ denotes the relative distance from the cathode, and $x = 1$ corresponds to the anode. The function $\varphi(x)$ describes the distribution of the electric potential in the process of moving from cathode to anode; $a(x)$ is the potential of the magnetic field; the unique constructive model's parameter j is the density of current through the diode. System (2.1) describes the electric and magnetic fields inside the diode, and its solution shall satisfy the following boundary conditions

$$\varphi(0) = 0, \quad a(0) = 0, \quad \varphi'(0) = \frac{d\varphi}{dx}(0) = 0, \tag{2.2}$$

$$\varphi(1) = \varphi_1, \quad a(1) = a_1. \tag{2.3}$$

Note that boundary-value problem (2.1)–(2.3) is singular: after substituting conditions (2.2) into equations (2.1) for $x = 0$, the denominator vanishes. Moreover, in solving (2.1)–(2.3) it is assumed [**9**] that the parameter j is free and should be found together with the solution of the boundary value problem. Together with singular problem (2.1)–(2.3), nonsingular boundary-value problems (2.1) also may be of interest, when instead of (2.2), the solution must satisfy the conditions of the type (2.3), given for various values of the independent variable and does not lead to conversion of denominator to zero in (2.1). Now, we describe the multidimensional quasi potential system of ODEs generalizing model (2.1).

Let us suppose that $\Omega \subset \mathbf{R}^n$ is a domain, in which the continuously differentiable scalar function $\Pi(q)$, $q \in \Omega$ is given, hereinafter called the potential. The independent real variable $t \in \mathbf{R}$ to be called "time", and the first and second order derivatives of $q(t)$ in t will be denoted by one or two dots over the letter.

Let us consider in a domain $\Gamma = \{(q, \dot{q}) \in \Omega \times \mathbf{R}^n\}$ the system of ODEs of the second order

$$A\ddot{q} + B\frac{\partial \Pi(q)}{\partial q} = 0, \tag{2.4}$$

where A, B are the constant square matrices of the dimension $n \times n$ with real elements.

The classic Peano theorem guarantees existence of the solution $q(t, q_0, \dot{q}_0)$, $\dot{q}(t, q_0, \dot{q}_0)$ in the case of a non-singular matrix A for every initial point $(q_0, \dot{q}_0) \in \Gamma$ of the system (2.4) (generally speaking non-unique) on the "time" interval $t \in (\alpha, \omega)$ determined by the initial conditions $(q_0, \dot{q}_0)$ and the choice of a particular solution in the nonuniqueness case. In the case of a singular matrix A, the system (2.4) presents the differential-algebraic system [**16**] and there are no solutions for any initial conditions.

Note that model (2.1) is written in the form (2.4), therefore it suffices to put

$$t = x \in \mathbf{R}, \quad q = \operatorname{col}(q_1, q_2) \in \mathbf{R}^2, \quad \Pi(q) = j\sqrt{(1+q_1)^2 - 1 - q_2},$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Omega = \big\{ q \in \mathbf{R}^2 : (1+q_1)^2 - 1 - q_2 > 0 \big\}.$$

Therefore, system (2.4) can be regarded as a generalized mathematical model of magnetic insulation. On the other hand system (2.4) can describe many other physical objects such as the famous problem of the motion of planets in the solar system under gravity [**17**].

The function $F(q, \dot{q})$, which does not reduce to the identical constant, is called the first integral of (2.4) in the domain Γ if it preserves the constant values along any real solutions of the system, which does not leave this domain.

Checking of the property of the first integral for continuously differentiable functions can be performed without actual knowledge of solutions by calculating the derivative along the system.

One of the main goals of this paper is to obtain the conditions that guarantee existence of the first integrals of (2.4) of a certain type, whose derivatives are identically equal to zero by (2.4). We will consider two types of integrals : integral of "energy", and integral linear in "velocity" $\dot{q}$. The system (2.4) with the identity matrix $B = E$ and the symmetric positive definite matrix A is considered [**17**] in analytical mechanics. In this case, the terms "time", "velocity", and "energy" are basic and can be used without quotation marks. But, as shown above, the system (2.4) can act as the mathematical model of an object having a nonmechanical nature, such as a vacuum diode. While the independent variable has no relation to the physical time. Therefore, the terms ŞenergyŤ integral, etc. are only nominal ones and they have mechanical analogs. Existence of solutions of (2.4) in the case of a singular matrix A is not guaranteed. Thus, in this situation, the constructed integrals should be considered only as formal ones providing zero derivative along the system in the considered domain. The question of existence of real solutions of the system is not considered here. Existence of the classic solutions of the differential-algebraic systems has been considered in [**16**], and generalized in [**18**].

Therefore, one of the main goals of this paper is to obtain the conditions of existence and effective construction of the formal first integrals of system (2.4). For the case of a nonsingular matrix A, the found integrals are guaranteed to be the first integrals in the precise sense of the above definition.

Now, we give a generalization of model (2.1) in another direction. Replacing the unknown function in (2.1) by the formula $\psi = 1 + \varphi$ and the second derivatives by two-dimensional Laplace operator, we obtain the system of two nonlinear PDEs of the second order

$$\begin{aligned}\Delta_{xy}\psi &\equiv \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=j\frac{\psi}{\sqrt{\psi^2-a^2-1}}, \quad \psi \triangleq \psi(x,y),\\ \Delta_{xy}a &\equiv \frac{\partial^2 a}{\partial x^2}+\frac{\partial^2 a}{\partial y^2}=j\frac{a}{\sqrt{\psi^2-a^2-1}}, \quad a \triangleq a(x,y).\end{aligned} \tag{2.5}$$

The goal of this part of the paper is: 1) to construct the exact solutions of system (2.5), and 2) to obtain the exact solutions of the boundary value problems of system (2.5) in annular domains.

We assume that, along with the search of the solution of system (2.5), just as for the boundary-value problem (2.1)–(2.3) according to [**9**], it can be useful to build the function $j(x,y)$. As we established earlier [**19**](Theorem 2), the set of solutions of system (2.1) is certainly contained in the set of solutions of (2.5) and in this sense it is possible to consider system in partial derivatives (2.5) as a generalized model of magnetic insulation with respect to model (2.1) offered in [**9**].

3. "Energy" integral

THEOREM 3.1. *If the matrix B is nondegenerate, and the matrix $S=B^{-1}A$ is symmetric, then the function*

$$V(q,\dot q)=0.5\dot q^T S\dot q+\Pi(q) \tag{3.1}$$

is the first integral of (2.4) *in the domain* Γ.

PROOF. Calculate derivative (3.1) with respect to system (2.4)

$$\begin{aligned}\frac{dV}{dt}\Big|_{(2.4)} &= 0.5\ddot q^T S\dot q+0.5\dot q^T S\ddot q+\frac{\partial\Pi^T}{\partial q}\dot q\\ &=0.5\ddot q^T B^{-1}A\dot q+0.5\dot q^T B^{-1}A\ddot q+\frac{\partial\Pi^T}{\partial q}\dot q\\ &=0.5\ddot q^T(B^{-1}A)^T\dot q+0.5\dot q^T B^{-1}\Big(-B\frac{\partial\Pi}{\partial q}\Big)+\frac{\partial\Pi^T}{\partial q}\dot q\\ &=0.5\ddot q^T A^T(B^{-1})^T\dot q+0.5\dot q^T B^{-1}\Big(-B\frac{\partial\Pi}{\partial q}\Big)+\frac{\partial\Pi^T}{\partial q}\dot q\\ &=-0.5\frac{\partial\Pi^T}{\partial q}B^T(B^{-1})^T\dot q-0.5\dot q^T\frac{\partial\Pi}{\partial q}+\frac{\partial\Pi^T}{\partial q}\dot q\equiv 0.\end{aligned}$$ □

Note that nondegeneracy of the matrix A is not required during the proof as well as its symmetry. Function (3.1) takes the form $V(q,\dot q)=0.5\dot q^T A\dot q+\Pi(q)$ in the case of a mechanical system and expresses the total energy while A is symmetric and positive definite. Therefore, integral (3.1) is naturally called "energy" integral.

4. Integral linear in the "velocity"

THEOREM 4.1. *Let a vector $h\in\mathbf{R}^n$ exist and the $n\times n$ matrices C, $M(q)$ such that*

1) $Cq + h = M(q)\dfrac{\partial \Pi}{\partial q}$ *for all* $q \in \Omega$*;*

2) *the matrix* $M^T(q)B$ *is skew-symmetric for all* $q \in \Omega$*;*

3) *the matrix* $C^T A$ *is skew-symmetric.*

Then the function

$$J(q, \dot q) = (Cq + h)^T A\dot q \tag{4.1}$$

is the first integral of (2.4) *in the domain* Γ*.*

Proof. Calculate derivative (4.1) with respect to system (2.4)

$$\frac{dJ}{dt}\Big|_{(2.4)} = \dot q^T C^T A\dot q + (Cq + h)^T\Big(-B\frac{\partial \Pi}{\partial q}\Big) = -\frac{\partial \Pi^T}{\partial q} M^T(q) B\frac{\partial \Pi}{\partial q} \equiv 0. \qquad \square$$

Note that nondegeneracy of the matrices A, B is not required in the proof, as well as their symmetry. In the case of a mechanical system, $\dot q$ represents a vector of generalized velocities. Therefore, integral (4.1) is naturally called the integral linear in "velocities". It is also important to note that in some cases (4.1) may contain more than one integral even a whole family.

5. Integrals of the system modeling vacuum diode

Applying Theorem 3.1 to system (2.1), we obtain the expression of the "energy" integral

$$V = 0.5(\dot q_2^2 - \dot q_1^2) + j\sqrt{(1 + q_1)^2 - 1 - q_2^2}. \tag{5.1}$$

We put

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad M(q) = \frac{\sqrt{(1 + q_1)^2 - 1 - q_2^2}}{j}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Applying Theorem 4.1 to system (2.1), we obtain the expression of the integral linear in "velocity"

$$J = (1 + q_1)\dot q_2 - q_2\dot q_1. \tag{5.2}$$

Note that integrals (5.1), (5.2) for the diode model (2.1) can be also obtained with direct observation of (2.1), and not as a consequence of integrals of general system (2.4).

6. Systems with a potential of the special form

Let us suppose the matrix $S = B^{-1}A$ to be symmetric and consider (2.4) in the case of a potential of special form

$$\Pi(q) = f(\sigma), \quad \sigma = \frac{1}{2}\, q^T S q + q^T k, \tag{6.1}$$

where $k \in \mathbf{R}^n$ is a constant vector, and $f(\sigma)$ is a continuously differentiable scalar function of scalar argument. It is easy to see that we can take $f(\sigma) = \sqrt{-2\sigma}$, $k = \mathrm{col}(-1, 0)$ for diode model (2.1), i.e., this model is included in considered class (6.1).

THEOREM 6.1. *If the matrices A, B are nondegenerate, the matrix $S = B^{-1}A$ is symmetric, and the potential has the form* (6.1)*, then the function*

$$K(q, \dot{q}) = \dot{q}^T(Cq + h), \quad h = CA^{-1}Bk \tag{6.2}$$

is the first integral of (2.4) *in the domain* Γ *for any constant skew-symmetric matrix* C.

PROOF. Calculate derivative (6.2) with respect to system (2.4)

$$\begin{aligned} \frac{dK}{dt}\Big|_{(2.4)} &= \dot{q}^T C\dot{q} + \ddot{q}^T(Cq + h) = -\Big(A^{-1}B\frac{\partial \Pi}{\partial q}\Big)^T(Cq + h) \\ &= -f'(\sigma)(Sq + k)^T(A^{-1}B)^T(Cq + h). \end{aligned} \tag{6.3}$$

The domain Ω splits into two disjoint subsets: Ω_0, where $f'(0.5q^TSq+q^Tk) = 0$, and Ω_1 with $f'(0.5q^TSq+q^Tk) \neq 0$. From (6.3), it follows that the equality $\frac{dK}{dt}\big|_{(2.4)} \equiv 0$ will be satisfied on the set Ω_0. To establish the same equality for the subset Ω_1, we define the matrix $M(q)$ on it by the following way $M(q) = \frac{1}{f'(\sigma)}CA^{-1}B$. We have the chain of equalities

$$\begin{aligned} M(q)\frac{\partial \Pi}{\partial q} &= \frac{1}{f'(\sigma)}CA^{-1}Bf'(\sigma)(Sq + k) \\ &= CA^{-1}BB^{-1}Aq + CA^{-1}Bk = Cq + h. \end{aligned}$$

We express $Cq + h = CA^{-1}B(Sq + k)$ from this chain and we substitute it in (6.3), then, taking into account the skew-symmetric matrix C, we obtain

$$\frac{dK}{dt}\Big|_{(2.4)} = -f'(\sigma)(Sq + k)^T(A^{-1}B)^TCA^{-1}B(Sq + k) \equiv 0$$

for the subset Ω_1. Thus, Theorem 6.1 is proved completely. □

Since the skew-symmetric matrix C can be considered arbitrary, then, in fact, (6.2) gives not one, but a whole family of integrals.

Note that if the conditions of Theorem 6.1 and Theorem 3.1 are satisfied, then along with family of integrals (6.1), the system has "energy" integral (3.1).

7. Generalized model of diode with 6 degrees of freedom

In this section we consider system (2.4) with the vector $q \in \mathbf{R}^6$, the identity matrix $A = E$, nondegenerate diagonal matrix $B = \mathrm{diag}\,(b_i,\ i = \overline{1, 6})$ and potential of the form

$$\Pi(q) = f(\sigma), \quad \sigma = \frac{1}{2}\sum_{i=1}^{6} b_i^{-1}q_i^2 + \sum_{i=1}^{6} k_iq_i, \tag{7.1}$$

where b_i, k_i are some constants, and $f(\sigma)$ is an arbitrary scalar continuously differentiable function of scalar argument. It is easy to see that system (2.4), (7.1) is a special case of (2.4), (6.1). At the same time, (2.4), (7.1) can be considered as a generalization of diode model (2.1), because it has three times as much coordinates, not fixed values of parameters b_i, k_i, $i = \overline{1, 6}$, and in the arbitrary scalar function in potential.

To construct the integrals of system (2.4), (7.1), we apply Theorem 6.1, whose conditions are satisfied. Choosing only one element above the principal diagonal of the matrix C from (6.2) as a nonzero one and applying Theorem 6.1, we obtain 15 first integrals

$$K_{ij}(q,\dot q) = (q_i + b_i k_i)\dot q_j - (q_j + b_j k_j)\dot q_i = \text{const}, \quad i = \overline{1,6}, \quad j = \overline{i+1,6}. \tag{7.2}$$

Certainly, all these integrals cannot be independent. Only 9 are, because the rank of the Jacobian matrix for (7.2) is 9. For example, one of the minors of the 9-th order can be written as

$$\Delta = \dot q_6((q_5 + b_5 k_5)\dot q_6 - (q_6 + b_6 k_6)\dot q_5)((q_1 + b_1 k_1)\dot q_2 - (q_2 + b_2 k_2)\dot q_1)^3$$

and does not vanish in some open sub-domains of Γ.

Note that (2.4), (7.1) is analogous in some sense to the classic problem of celestial mechanics of two gravitating material points [**17**] (two-body problem): both problems are described by six differential equations of the second order (or by twelve equations of the first order), in both there are 10 first integrals of this type (1 “energy” integral +9 linear in “velocities”).

The classic two-body problem is known to be integrable [**17**], so that the generalized vacuum diode model (2.4), (7.1) must also be integrable.

8. Representation in Hamiltonian form and integrability

Let us assume that the matrices A, B are nondegenerate, and the matrix $S = B^{-1}A$ is symmetric. We show that in this case system (2.4) can be represented in the Hamiltonian form. We set $p = S\dot q$, $H(q,p) = 0.5p^T S^{-1} p + \Pi(q)$. Thus, the system (2.4) can be written in the new coordinates in the canonical form

$$\dot q = \frac{\partial H}{\partial p}, \quad \dot p = -\frac{\partial H}{\partial q}. \tag{8.1}$$

Because diode model (2.1) satisfies the conditions of nondegeneracy of A, B, and $S = B^{-1}A$ is diagonal for it, then (2.1) can be written in the form (8.1) with Hamiltonian $H(q,p) = 0.5(-p_1^2 + p_2^2) + j\sqrt{(1+q_1)^2 - 1 - q_2^2}$. Integrals (5.1), (5.2) in the canonical variables are presented as $H(q,p) = \text{const}$, $J = (1+q_1)p_2 + q_2 p_1 = \text{const}$. They are independent, independent of time and are in involution. Therefore, due to Liouville's theorem [**17**], the diode model (2.1) is integrable. Integration was done in [**11**], where using Liouville theorem, a complete system of 4-th first integrals is constructed.

Representation in the canonical form for generalized model (2.4), (7.1) is also possible, and integrals (3.1), (7.2) are now written as

$$\begin{gathered} H(q,p) = \frac{1}{2}\sum_{i=1}^{6} b_i p_i^2 + \Pi(q) = \text{const}, \\ K_{ij}(q,p) = (q_i + b_i k_i) b_j p_j - (q_j + b_j k_j) b_i p_i = \text{const}, \\ i = \overline{1,6}, \quad j = \overline{i+1,6}. \end{gathered} \tag{8.2}$$

There will be only four independent integrals in involution as follows: $H(q,p) = \text{const}$, $b_1p_1(q_2+b_2k_2)+b_2p_2(q_1+b_1k_1) = \text{const}$, $b_3p_3(q_4+b_4k_4)+b_4p_4(q_3+b_3k_3) = \text{const}$, $b_5p_5(q_6+b_6k_6)+b_6p_6(q_5+b_5k_5) = \text{const}$. They are not sufficient to justify the integrability by Liouville theorem [**17**].

However, as noted above, the generalized diode model represented in the canonical form, has 10 independent first integrals (1 "energy" integral +9 linear in "momentum" p). Using the linear momentum integral, we can transform the system while preserving its Hamiltonian form so that the transformed system would have a cyclic coordinate [**17**] and reduce the order of the system by two units. The energy integral reduces the system order by two units more [**17**]. Still unused 8 independent integrals of family (8.2) represented in new coordinates will provide integrability of reduced Hamiltonian system with 4 degrees of freedom.

Thus, generalized diode model (2.4), (7.1) is integrable as well as the model (2.1).

9. Exact solutions of the system of equations related to magnetic insulation of the vacuum diode in two-dimensional coordinate space

In this section, we consider the problem of finding the exact solutions of system of nonlinear elliptic equations (2.5). First of all, we are interested in a radially symmetric solutions of (2.5) as the simplest class of multi-dimensional solutions, i.e., the functions

$$\psi(x,y) \triangleq \psi(r),\ a(x,y) \triangleq a(r),\ \text{where } r^2 = x^2+y^2.$$

In this case, the system (2.5) is transformed to a nonlinear system of the second order ODEs

$$\begin{aligned} \psi'' + \frac{1}{r}\,\psi' &= j\frac{\psi}{\sqrt{\psi^2-a^2-1}}, \quad \psi \triangleq \psi(r), \\ a'' + \frac{1}{r}\,a' &= j\frac{a}{\sqrt{\psi^2-a^2-1}}, \quad a \triangleq a(r). \end{aligned} \tag{9.1}$$

Hereinafter in this section, the prime sign denotes the derivative with respect to the argument r. One can easily make sure that system (9.1) possesses the following first integral

$$\mathbf{I} \equiv r(\psi a' - a\psi') = \text{const}\,. \tag{9.2}$$

We seek a solution of system (9.1) in the form of the following functional ansatz

$$\begin{aligned} \psi(r) &= \frac{1}{2\gamma}z(r)\big(e^{\omega(r)} + \gamma^2 e^{-\omega(r)}\big), \\ a(r) &= \frac{1}{2\gamma}z(r)\big(e^{\omega(r)} - \gamma^2 e^{-\omega(r)}\big), \end{aligned} \tag{9.3}$$

where $\gamma \neq 0$ is an arbitrary real constant, and new functions $z \triangleq z(r)$, $\omega \triangleq \omega(r)$ have to be determined. If the constant γ is chosen to be equal to one then instead of formulas (9.3) we obtain an ansatz of the form

$$\psi(r) = z(r)\cosh\omega(r), \quad a(r) = z(r)\sinh\omega(r).$$

As it will be shown below, system (9.1) may be decomposed with the ansatz (9.3), that is to reduce it to two equations: one for the function $z(r)$, and the other for the function $\omega(r)$.

Substituting ansatz (9.3) into (9.1) and calculating the required derivatives, we group the terms in such a way to get the following algebraic system

$$X\Big(\frac{1}{2\gamma}e^{\omega(r)} + \frac{\gamma}{2}e^{-\omega(r)}\Big) + Y\Big(\frac{1}{2\gamma}e^{\omega(r)} - \frac{\gamma}{2}e^{-\omega(r)}\Big) = 0,$$
$$X\Big(\frac{1}{2\gamma}e^{\omega(r)} - \frac{\gamma}{2}e^{-\omega(r)}\Big) + Y\Big(\frac{1}{2\gamma}e^{\omega(r)} + \frac{\gamma}{2}e^{-\omega(r)}\Big) = 0.$$

The following notations are introduced

$$X = z'' + \frac{1}{r}z' + z\omega'^2 - j\frac{z}{\sqrt{z^2-1}},$$
$$Y = \Big(2z' + \frac{1}{r}z\Big)\omega' + z\omega''.$$

With respect to the variables X, Y, the resulting system of algebraic equations is linear and homogeneous, its determinant is equal to one, therefore, it has only one trivial solution $X = 0$, $Y = 0$. Hence, system (9.1) is reduced to the following two ODEs:

$$z'' + \frac{1}{r}z' + z\omega'^2 - j\frac{z}{\sqrt{z^2-1}} = 0, \tag{9.4}$$

$$\Big(2z' + \frac{1}{r}z\Big)\omega' + z\omega'' = 0. \tag{9.5}$$

Integrating relation (9.5) we obtain

$$\omega' = \frac{B_0}{rz^2}, \tag{9.6}$$

where $B_0 \geqslant 0$ is a constant of integration. Substitution of (9.6) into (9.4) gives

$$z'' + \frac{1}{r}z' + \frac{B_0^2}{r^2}z^{-3} - j\frac{z}{\sqrt{z^2-1}} = 0. \tag{9.7}$$

Therefore, solvability of system (9.1) in the form of ansatz (9.3) has been reduced to the solvability of one nonlinear nonautonomous second order ODE (9.7) for the function $z(r)$, because the function $\omega(r)$ can be found by (9.6) by simple integration. My means of the substitution $z = \sqrt{y^2+1}$, $y \stackrel{\triangle}{=} y(r)$, equation (9.7) can be transformed into the form without radicals via simple transformations

$$y^2(y^2+1)\Big(y'' + \frac{1}{r}y'\Big) + yy'^2 + \frac{B_0^2}{r^2}y - j(y^2+1)^2 = 0.$$

This equation has an exact solution in the form of exponential function with $B_0 = 2$

$$y(r) = \frac{1}{4}jr^2,$$

whence it is easy to obtain an explicit solution of (9.7) of the form

$$z(r) = \frac{1}{4}\sqrt{j^2 r^4 + 16}. \tag{9.8}$$

Integrating (9.6), with the account of (9.8), we obtain

$$\omega(r) = \ln\left(\frac{C_0 r^2}{\sqrt{j^2 r^4 + 16}}\right),$$

where $C_0 > 0$ is a constant of integration. Thus, we obtain the following exact solution of system (9.1) from (9.3) (9.1)

$$\begin{aligned} \psi_1(r) &= \frac{\gamma_0^2 + j^2}{8\gamma_0}\, r^2 + \frac{2}{\gamma_0}\, r^{-2}, \\ a_1(r) &= \frac{\gamma_0^2 - j^2}{8\gamma_0}\, r^2 - \frac{2}{\gamma_0}\, r^{-2}. \end{aligned} \tag{9.9}$$

We introduce the new constant $\gamma_0 = C_0/\gamma$. In turn of the analysis of first integral (9.2), we propose another exact solution of (9.4), which is distinct from ((9.9) but close in structure to it

$$\begin{aligned} \psi_2(r) &= \frac{\gamma_0^2 + j^2}{8\gamma_0}\, r^2 + \frac{2\gamma_0}{j^2}\, r^{-2}, \\ a_2(r) &= \frac{\gamma_0^2 - j^2}{8\gamma_0}\, r^2 + \frac{2\gamma_0}{j^2}\, r^{-2}. \end{aligned} \tag{9.10}$$

Functions (9.9), (9.10) for $\gamma_0 \neq j$ represent linearly independent solutions. If we choose a constant γ_0 such that $\gamma_0 = j$, then, in this particular case, solutions (9.9), (9.10) coincide and may be written as

$$\psi(r) = \frac{j}{4}\, r^2 + \frac{2}{j}\, r^{-2}, \quad a(r) = \frac{2}{j}\, r^{-2}.$$

Finally, in terms of the initial variables x, y, from (9.9), (9.10), we obtain the following exact radial-symmetric solutions of system (2.5):

$$\begin{aligned} \psi_1(x,y) &= \frac{\gamma_0^2 + j^2}{8\gamma_0}(x^2+y^2) + \frac{2}{\gamma_0}(x^2+y^2)^{-1}, \\ a_1(x,y) &= \frac{\gamma_0^2 - j^2}{8\gamma_0}(x^2+y^2) - \frac{2}{\gamma_0}(x^2+y^2)^{-1}, \end{aligned} \tag{9.11}$$

$$\begin{aligned} \psi_2(x,y) &= \frac{\gamma_0^2 + j^2}{8\gamma_0}(x^2+y^2) + \frac{2\gamma_0}{j^2}(x^2+y^2)^{-1}, \\ a_2(x,y) &= \frac{\gamma_0^2 - j^2}{8\gamma_0}(x^2+y^2) + \frac{2\gamma_0}{j^2}(x^2+y^2)^{-1}. \end{aligned} \tag{9.12}$$

Having resumed the above considerations, we can conclude the following.

THEOREM 9.1. *System of nonlinear elliptic equations* (2.5) *with a constant current density possesses the exact radial-symmetric solutions* (9.11), (9.12).

The validity of this statement may be confirmed by direct substitution of formulas (2.5), (9.11), into (9.12), which transforms it into identity.

10. Exact solutions with a variable current density

In this section, we consider the problem of constructing the exact multi dimensional solutions of the system of nonlinear elliptic equations with a variable current density:

$$\begin{aligned} \Delta_{xy}\psi &= j(x,y)\frac{\psi}{\sqrt{\psi^2-a^2-1}}, \quad \psi \triangleq \psi(x,y), \\ \Delta_{xy}a &= j(x,y)\frac{a}{\sqrt{\psi^2-a^2-1}}, \quad a \triangleq a(x,y). \end{aligned} \tag{10.1}$$

Definition 10.1. The harmonic functions $\xi(x,y)$, $\eta(x,y)$ are called conjugate in the single-connected domain D if the function $F(z) = \xi(x,y) + i\eta(x,y)$, is an analytical function of complex argument $z = x + iy$ in the domain D.

The conjugate harmonic functions are related by the Cauchy–Riemann equations

$$\frac{\partial\xi}{\partial x} = \frac{\partial\eta}{\partial y}, \quad \frac{\partial\xi}{\partial y} = -\frac{\partial\eta}{\partial x},$$

and define each other everywhere in D within an additive constant and consequently, have the following properties

$$(\nabla\xi, \nabla\eta) = 0, \quad |\nabla\xi|^2 = |\nabla\eta|^2. \tag{10.2}$$

Here $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^T$ is nabla operator.

Theorem 10.1. *If the current density $j(x,y)$ in system* (10.1) *is a square of gradient of the arbitrary harmonic function, i.e.,*

$$j(x,y) = J_0\,|\nabla\xi|^2, \quad J_0 = \text{const} > 0,$$

then by transformation

$$\psi(x,y) = \psi(\xi,\eta), \quad a(x,y) = a(\xi,\eta), \tag{10.3}$$

where $\xi \triangleq \xi(x,y)$, $\eta \triangleq \eta(x,y)$ are conjugate harmonic functions, system (10.1) *is reduced to the equation of a similar form with a constant current density*

$$\begin{aligned} \Delta_{\xi\eta}\psi &= J_0\frac{\psi}{\sqrt{\psi^2-a^2-1}}, \quad \psi \triangleq \psi(\xi,\eta), \\ \Delta_{\xi\eta}a &= J_0\frac{a}{\sqrt{\psi^2-a^2-1}}, \quad a \triangleq a(\xi,\eta). \end{aligned} \tag{10.4}$$

Proof. Due to the properties of (10.2) of conjugate harmonic functions, it is easy to show that by transformation (10.3) the following relations are obtained

$$\Delta_{x\,y}\,\psi(x,y) = |\nabla\xi|^2\Delta_{\xi\eta}\,\psi(\xi,\eta), \quad \Delta_{x\,y}\,a(x,y) = |\nabla\xi|^2\Delta_{\xi\eta}\,a(\xi,\eta).$$

Here $\Delta_{\xi\eta}\cdot = \left(\frac{\partial^2}{\partial\xi^2}\cdot + \frac{\partial^2}{\partial\eta^2}\cdot\right)$ is a two-dimensional Laplace operator in the space of the variables $(\xi,\, \eta)$. In this case, the system (10.1) assumes the form

$$
\begin{aligned}
|\nabla\xi|^2\Delta_{\xi\eta}\psi &= j(x,y)\frac{\psi}{\sqrt{\psi^2 - a^2 - 1}},\\
|\nabla\xi|^2\Delta_{\xi\eta}a &= j(x,y)\frac{a}{\sqrt{\psi^2 - a^2 - 1}}.
\end{aligned} \tag{10.5}
$$

Hence, assuming that the current density $j(x,y)$ satisfies the condition of the theorem, we immediately obtain the system of equations (10.4). □

Example 10.1. Let us construct the exact solutions for the following system of equations:

$$
\begin{aligned}
\Delta_{xy}\psi &= j_0(x^2+y^2)^{k-1}\frac{\psi}{\sqrt{\psi^2 - a^2 - 1}},\\
\Delta_{xy}a &= j_0(x^2+y^2)^{k-1}\frac{a}{\sqrt{\psi^2 - a^2 - 1}},
\end{aligned} \tag{10.6}
$$

where $j_0 = \mathrm{const} > 0$. Note that the following relation

$$(x^2+y^2)^{k-1} = \frac{1}{k^2}|\nabla\phi(x,y)|^2,$$

is satisfied, where the function $\phi(x,y)$ is a harmonic polynomial of degree $k \in \mathbf{N}$. Therefore, the current density in equations (10.6) satisfies the condition of Theorem 10.1. Hence, by transformation of (10.3), they are reduced to the system of equations (10.4) to the functions $\psi(\xi,\eta)$, $a(\xi,\eta)$. Furthermore, the new variables ξ, η are conjugate harmonic polynomials, i.e.,

$$\xi(x,y) = (x^2+y^2)^{\frac{k}{2}}\cos(k\varphi(x,y)), \quad \eta(x,y) = (x^2+y^2)^{\frac{k}{2}}\sin(k\varphi(x,y)),$$

where the functions $\varphi(x,y)$ can be represented in the form of the following two types

$$\varphi(x,y) = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right) \quad \text{or} \quad \varphi(x,y) = \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right).$$

Now, in order to write out the exact solutions of (10.6) with the variable current density, it is necessary to obtain explicit non-trivial solutions of (10.4) with a constant current density. In the previous section, we have found the exact radial-symmetric solutions (9.11), (9.12) for the model of magnetic insulation (2.5), which we present in terms of the variables ξ, η for system (10.4)

$$
\begin{aligned}
\psi_1(\xi,\eta) &= \frac{\gamma_0^2 + J_0^2}{8\gamma_0}(\xi^2+\eta^2) + \frac{2}{\gamma_0}(\xi^2+\eta^2)^{-1},\\
a_1(\xi,\eta) &= \frac{\gamma_0^2 - J_0^2}{8\gamma_0}(\xi^2+\eta^2) - \frac{2}{\gamma_0}(\xi^2+\eta^2)^{-1},
\end{aligned} \tag{10.7}
$$

$$\begin{aligned} \psi_2(\xi,\eta) &= \frac{\gamma_0^2+J_0^2}{8\gamma_0}(\xi^2+\eta^2)+\frac{2\gamma_0}{J_0^2}(\xi^2+\eta^2)^{-1}, \\ a_2(\xi,\eta) &= \frac{\gamma_0^2-J_0^2}{8\gamma_0}(\xi^2+\eta^2)+\frac{2\gamma_0}{J_0^2}(\xi^2+\eta^2)^{-1}. \end{aligned} \tag{10.8}$$

Here $\gamma_0 \neq 0$ is an arbitrary constant. Making the transition to the variables x, y in (10.7), (10.8) and taking into account the equality $J_0 = j_0/k^2$, we can write down the exact radial-symmetric solutions of (10.6)

$$\begin{aligned} \psi_1(x,y) &= \frac{\gamma_0^2k^4+j_0^2}{8\gamma_0k^4}(x^2+y^2)^k+\frac{2}{\gamma_0}(x^2+y^2)^{-k}, \\ a_1(x,y) &= \frac{\gamma_0^2k^4-j_0^2}{8\gamma_0k^4}(x^2+y^2)^k-\frac{2}{\gamma_0}(x^2+y^2)^{-k}, \end{aligned} \tag{10.9}$$

$$\begin{aligned} \psi_2(x,y) &= \frac{\gamma_0^2k^4+j_0^2}{8\gamma_0k^4}(x^2+y^2)^k+\frac{2\gamma_0k^4}{j_0^2}(x^2+y^2)^{-k}, \\ a_2(x,y) &= \frac{\gamma_0^2k^4-j_0^2}{8\gamma_0k^4}(x^2+y^2)^k+\frac{2\gamma_0k^4}{j_0^2}(x^2+y^2)^{-k}. \end{aligned} \tag{10.10}$$

By direct verification, we can see that the obtained solutions (10.9), (10.10) satisfy (10.6) for any real $k \in \mathbf{R}$, $k \neq 0$, despite the fact that the number k is natural. Note that the current density for k in system (10.6) becomes constant. Therefore, in this case it is obvious that formulas (10.9), (10.10) define the solution of (10.4) in the space of the variables x, y and coincide with formulas (10.7), (10.8).

If $k = 0$, then, in this interesting particular case, system of equations (10.1) takes the form

$$\begin{aligned} \Delta_{xy}\psi &= \frac{j_0}{x^2+y^2}\frac{\psi}{\sqrt{\psi^2-a^2-1}}, \\ \Delta_{xy}a &= \frac{j_0}{x^2+y^2}\frac{a}{\sqrt{\psi^2-a^2-1}}, \end{aligned}$$

and possesses the following exact solutions

$$\begin{aligned} \psi_1(x,y) &= \frac{\gamma_0^2+j_0^2}{8\gamma_0}\sigma(x,y)+\frac{2}{\gamma_0\sigma(x,y)}, \quad & a_1(x,y) &= \frac{\gamma_0^2-j_0^2}{8\gamma_0}\sigma(x,y)-\frac{2}{\gamma_0\sigma(x,y)}, \\ \psi_2(x,y) &= \frac{\gamma_0^2+j_0^2}{8\gamma_0}\sigma(x,y)+\frac{2\gamma_0}{j_0^2\sigma(x,y)}, \quad & a_2(x,y) &= \frac{\gamma_0^2-j_0^2}{8\gamma_0}\sigma(x,y)+\frac{2\gamma_0}{j_0^2\sigma(x,y)}, \end{aligned}$$

where the following notation is accepted

$$\sigma(x,y) = \frac{1}{4}\ln^2(x^2+y^2)+\arctan^2\frac{y}{x}.$$

11. On exact solutions of the boundary value problems in annular domain

We consider the boundary value problem for the system of two nonlinear differential equations of the second order with partial derivatives (2.5) in annular domain

$\Theta = \{(x,y) : 0 < \rho_1^2 \leqslant x^2 + y^2 \leqslant \rho_2^2 < +\infty\}$

$$\psi(x,y) = \bar{\psi}_1, \quad a(x,y) = \bar{a}_1 \quad \text{for } x^2 + y^2 = \rho_1^2, \tag{11.1}$$

$$\psi(x,y) = \bar{\psi}_2, \quad a(x,y) = \bar{a}_2 \quad \text{for } x^2 + y^2 = \rho_2^2, \tag{11.2}$$

where ρ_1, $\bar{\psi}_1$, $\bar{a}_1$, ρ_2, $\bar{\psi}_2$, $\bar{a}_2$ are some given positive numbers $0 < \rho_1 < \rho_2 < +\infty$. Using the presented exact solutions in Section 9, one can specify the conditions to the numbers ρ_1, $\bar{\psi}_1$, $\bar{a}_1$, ρ_2, $\bar{\psi}_2$, and $\bar{a}_2$, under which there is a value $j > 0$ such that boundary value problem (2.5), (11.1), (11.2) has a radial-symmetric solution in the annular domain Θ given by the explicit formulas.

Theorem 11.1. *If the conditions*

$$\bar{\psi}_i^2 - \bar{a}_i^2 - 1 > 0, \quad (i = 1, 2),$$
$$\rho_2^2(\bar{\psi}_1 + \bar{a}_1) = \rho_1^2(\bar{\psi}_2 + \bar{a}_2),$$
$$\rho_2^4(\bar{\psi}_1^2 - \bar{a}_1^2 - 1) = \rho_1^4(\bar{\psi}_2^2 - \bar{a}_2^2 - 1)$$

are satisfied, then the boundary value problem (2.5), (11.1), (11.2) *has a solution for* $j = \dfrac{4}{\rho_1^2}\sqrt{\bar{\psi}_1^2 - \bar{a}_1^2 - 1}$

$$\psi(x,y) = \frac{k^2 + j^2}{8k}(x^2 + y^2) + \frac{2}{k(x^2 + y^2)}, \tag{11.3}$$

$$a(x,y) = \frac{k^2 - j^2}{8k}(x^2 + y^2) - \frac{2}{k(x^2 + y^2)}, \tag{11.4}$$

where $k = \frac{4}{\rho_1^2}(\bar{\psi}_1 + \bar{a}_1)$.

Proof. According to Theorem 9.1, system (2.5) has an exact solution of (11.3), (11.4). Consider equalities (11.3), (11.4) on the circle $x^2 + y^2 = \rho_1^2$. Adding them we get $k = \frac{4}{\rho_1^2}(\bar{\psi}_1 + \bar{a}_1)$. Substituting this value into any of equalities (11.3), (11.4), we obtain $j^2 = \frac{16}{\rho_1^4}(\bar{\psi}_1^2 - \bar{a}_1^2 - 1)$. Likewise, for the second circle $x^2 + y^2 = \rho_2^2$, one can obtain $k = \frac{4}{\rho_2^2}(\bar{\psi}_2 + \bar{a}_2)$ and $j^2 = \frac{16}{\rho_2^4}(\bar{\psi}_2^2 - \bar{a}_2^2 - 1)$. Since the values of the parameters k, j must be the same for both circles, we come to the theorem and the proof is complete. $\square$

In the same way we prove the following theorem.

Theorem 11.2. *If the conditions*

$$\bar{\psi}_i^2 - \bar{a}_i^2 - 1 > 0, \quad (i = 1, 2),$$
$$\rho_2^2\Big(\bar{\psi}_1 + \bar{a}_1 - \frac{1}{\bar{\psi}_1 - \bar{a}_1}\Big) = \rho_1^2\Big(\bar{\psi}_2 + \bar{a}_2 - \frac{1}{\bar{\psi}_2 - \bar{a}_2}\Big),$$
$$\rho_2^4(\bar{\psi}_1^2 - \bar{a}_1^2 - 1) = \rho_1^4(\bar{\psi}_2^2 - \bar{a}_2^2 - 1)$$

are satisfied, then boundary value problem (2.5), (11.1), (11.2) *for* $j = \dfrac{4}{\rho_1^2}\sqrt{\bar{\psi}_1^2 - \bar{a}_1^2 - 1}$ *has the solution of the form*

$$\psi(x,y) = \frac{k^2+j^2}{8k}(x^2+y^2) + \frac{2k}{j^2(x^2+y^2)},$$
$$a(x,y) = \frac{k^2-j^2}{8k}(x^2+y^2) - \frac{2k}{j^2(x^2+y^2)},$$

where $k = \frac{4}{\rho_1^2}\big(\bar{\psi}_1 + \bar{a}_1 - \frac{1}{\bar{\psi}_1 - \bar{a}_1}\big)$.

Thus, if the conditions of at least one of Theorems 11.1 or 11.2 are satisfied, the current density can be assigned to a uniform and, thus, the solution of the boundary value problem (2.5), (11.1), (11.2) is written explicitly.

12. Conclusion

Proposed in [**9**], the limit model of magnetic insulation was a quite interesting object for study, with a rich set of features. In this paper, we examined some mathematical generalizations of the limit problem described by a quasi potential multi-dimensional ODE system and a PDE system with two-dimensional Laplace operator. The results (the first integrals constructed in explicit form, and the exact solutions) may be useful not only for the problems of magnetic insulation and plasma physics but also in mechanics, fluid dynamics, and other areas.

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