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A CONVERGENCE THEOREM OF MULTI-STEP ITERATIVE SCHEME FOR NONLINEAR MAPS

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Dedicated to Professor Z. Xue for his unique style of mentoring.

Abstract. Let K be a nonempty closed convex subset of a real Banach space X, $T : K \to K$ a nearly uniformly L-Lipschitzian (with sequence $\{r_n\}$) asymptotically generalized Φ-hemicontractive mapping (with sequence $k_n \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$) such that $F(T) = \{\rho \in K : T\rho = \rho\}$. Let $\{\alpha_n\}_{n\geqslant 0}$, $\{\beta_n^k\}_{n\geqslant 0}$ be real sequences in $[0, 1]$ satisfying the conditions:

(i) $\sum_{n\geqslant 0} \alpha_n = \infty$

(ii) $\lim_{n\to\infty} \alpha_n, \beta_n^k = 0, \quad k = 1, 2, \ldots, p-1$.

For arbitrary $x_0 \in K$, let $\{x_n\}_{n\geqslant 0}$ be a multi-step sequence iteratively defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n^1, \quad n \geqslant 0,\\ y_n^k &= (1-\beta_n^k)x_n + \beta_n^k T^n y_n^{k+1}, \quad k = 1, 2, \ldots, p-2,\\ y_n^{p-1} &= (1-\beta_n^{p-1})x_n + \beta_n^{p-1} T^n x_n, \quad n \geqslant 0, p \geqslant 2. \end{aligned} \tag{0.1}$$

Then, $\{x_n\}_{n\geqslant 0}$ converges strongly to $\rho \in F(T)$. The result proved in this note significantly improve the results of Kim et al. [**2**].

1. Introduction

Let X be a real Banach space and J the normalized duality mapping from X into 2^{X^*} defined by $J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$, where X^* denotes the dual space of real Banach space X and $\langle ., .\rangle$ denotes the generalized duality pairing between elements of X and X^*. We first recall and define some concepts as follows. Let K be a nonempty subset of real Banach space X.

Definitions 1. Let $T : K \to K$ be a mapping.

(1) T is said to be uniformly L-Lipschitzian [**1, 5**] if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \leqslant L\|x - y\|$, for any $x, y \in K$ and $\forall n \geqslant 1$.

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(2) T is said to be asymptotically generalized Φ-hemicontractive with sequence $\{k_n\}_{n\geqslant 0}$ if $F(T) \neq \emptyset$ and for each $n \in N$ and $x \in K$, $x^* \in F(T)$, there exists constant $k_n \geqslant 1$ with $\lim_{n\to\infty} k_n = 1$, strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle T^n x - T^n x^*, j(x - x^*)\rangle \leqslant k_n\|x - x^*\|^2 - \Phi(\|x - x^*\|).$$

The class of asymptotically generalized Φ-hemicontractive mapping is the most general among those defined in [**5**].

(3) A mapping $T : K \to X$ is called Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leqslant L\|x - y\|, \text{ for all } x, y \in K$$

and is called generalized Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leqslant L(\|x - y\| + 1), \text{ for all } x, y \in K.$$

It is obvious that the class of generalized Lipschitzian map includes the class of Lipschitz map. Sahu [**5**] introduced the following new class of nonlinear mappings which are more general than the class of generalized Lipschitzian mappings and the class of uniformly L-Lipschitzian mappings. Fix a sequence $\{r_n\}_{n\geqslant 0}$ in $[0,\infty]$ with $r_n \to 0$.

(4) A mapping $T : K \to K$ is called nearly Lipschitzian with respect to $\{r_n\}$ if for each $n \in N$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leqslant k_n(\|x - y\| + r_n), \text{ for all } x, y \in K.$$

A nearly Lipschitzian mapping T with sequence $\{r_n\}_{n\geqslant 0}$ is said to be nearly uniformly L-Lipschitzian if $k_n = L$ for all $n \in N$.

Observe that the class of nearly uniformly L-Lipschitzian mapping is more general than the class of uniformly L-Lipschitzian mappings. We establish a strong convergence theorem for a more general class of map in real Banach space. It is worth noting that comparing [**2**, Theorem 2.1] our result have the following features:
(i) The modified Mann iterative process is replaced by Multi-step iterative process.
(ii) We removed the condition that $\{r_n/\alpha_n\}$ is bounded.
(iii) Our restriction imposed on α_n is much weaker than those in [**2**, Theorem 2.1].

Furthermore, our result also improves and extends the corresponding results in [**1, 3**]. For this, we need the following Lemmas.

Lemma 1.1. [**1**] *Let X be real Banach Space and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then, for any $x, y \in X$*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

Lemma 1.2. [**4**] *Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing function with $\Phi(x) = 0 \Leftrightarrow x = 0$ and let $\{b_n\}_{n=0}^{\infty}$ be a positive real sequence satisfying*

$$\sum_{n=0}^{\infty} b_n = +\infty \text{ and } \lim_{n\to\infty} b_n = 0.$$

Suppose that $\{a_n\}_{n=0}^{\infty}$ *is a nonnegative real sequence. If there exists an integer* $N_0 > 0$ *satisfying*

$$a_{n+1}^2 \leqslant a_n^2 + o(b_n) - b_n\Phi(a_{n+1}), \quad \forall n \geqslant N_0,$$

where $\lim_{n\to\infty} \frac{o(b_n)}{b_n} = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

2. Main results

THEOREM 2.1. *Let* K *be a nonempty closed convex subset of a real Banach space* X, $T : K \to K$ *a nearly uniformly* L*-Lipschitzian (with sequence* $\{r_n\}_{n\geqslant 0}$*) asymptotically generalized* $\Phi-$*hemicontractive map (with sequence* $k_n \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$*) such that* $F(T) = \{\rho \in K : T\rho = \rho\}$. *Let* $\{\alpha_n\}_{n\geqslant 0}$, $\{\beta_n^k\}_{n\geqslant 0}$ *be real sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\sum_{n\geqslant 0} \alpha_n = \infty$

(ii) $\lim_{n\to\infty} \alpha_n = 0 = \beta_n^k, \quad k = 1, 2, \dots, p-1.$

For arbitrary $x_0 \in K$, *let* $\{x_n\}_{n\geqslant 0}$ *be iteratively defined by* (0.1). *Then,* $\{x_n\}_{n\geqslant 0}$ *converges strongly to* $\rho \in F(T)$.

PROOF. Since $T : K \to K$ is an asymptotically generalized Φ-hemicontractive mapping, there exists a strictly increasing continuous function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle T^n x - T^n \rho, j(x-\rho)\rangle \leqslant k_n\|x-\rho\|^2 - \Phi(\|x-\rho\|), \tag{2.1}$$

for $x \in K$, $\rho \in F(T)$, that is

$$\langle (T^n - k_nI)x - (T^n - k_nI)\rho, j(x-\rho)\rangle \leqslant -\Phi(\|x-\rho\|). \tag{2.2}$$

Choose an $x_0 \in K$ and $x_0 \neq Tx_0$ such that $\|x_0 - T^nx_0\|\|x_0-\rho\| + (k_n-1)\|x_0-\rho\|^2 \in R(\Phi)$ and denote $a_0 = \|x_0 - T^nx_0\|\|x_0-\rho\| + (k_n-1)\|x_0-\rho\|^2$. Indeed, if $\Phi(a) \to +\infty$ as $a \to \infty$, then $a_0 \in R(\Phi)$; if $\sup\{\Phi(a) : a \in [0,\infty]\} = a_1 < +\infty$ with $a_1 < a_0$ Then for $\rho \in K$, there exists a sequence $\{u_n\}$ in K such that $u_n \to \rho$ as $n \to \infty$ with $u_n \neq \rho$. Clearly, $Tu_n \to T\rho$ as $n \to \infty$ thus $\{u_n - Tu_n\}$ is a bounded sequence. Therefore, there exists an n_0 such that $\|u_n - T^nu_n\|\|u_n-\rho\| + (k_n-1)\|u_n-\rho\|^2 < \frac{a_1}{2}$ for $n \geqslant n_0$. Then we redefine $x_0 = u_{n_0}$ and $\|x_0 - T^nx_0\|\|x_0-\rho\| + (k_n-1)\|x_0-\rho\|^2 \in R(\Phi)$. This is to ensure that $\Phi^{-1}(a_0)$ is well defined.

We first show that $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence.

Set $R = \Phi^{-1}(a_0)$; then from (2.2), we obtain that $\|x_n - \rho\| \leqslant R$. Denote

$$B_1 = \{x \in K : \|x-\rho\| \leqslant R\}, \quad B_2 = \{x \in K : \|x-\rho\| \leqslant 2R\}.$$

Now, we want to prove that $x_n \in B_1$. If $n = 0$, then $x_0 \in B_1$. Now assume that it holds for some n, that is, $x_n \in B_1$. Suppose that, it is not the case, then $\|x_{n+1} - \rho\| > R > \frac{R}{2}$.

Since $\{r_n\} \in [0,\infty]$ with $r_n \to 0$. Let $M = \sup\{r_n : n \in N\}$ and denote

$$\tau_0 = \min\Big\{1, \frac{\Phi(R/2)}{24R^2}, \frac{\Phi(R/2)}{12R[(2R+M)L+R]}, \frac{\Phi(R/2)}{12R[2((2R+M)L+R)+M]L}\Big\}.$$

Since $\lim_{n\to\infty} \alpha_n = 0 = \beta_n^k$, for $k = 1, 2, \ldots, p-1$ and $\lim_{n\to\infty} k_n = 1$. Without loss of generality, let $0 \leqslant \alpha_n, \beta_n^k, k_n - 1 \leqslant \tau_0$ for any $n \geqslant 0$. Then, we have the following estimates from (2.1) for $k = 1, 2, \ldots, p-1$.

$$\|y_n^{p-1} - \rho\| \leqslant (1 - \beta_n^{p-1})\|x_n - \rho\| + \beta_n^{p-1}\|T^n x_n - \rho\| \leqslant R + \tau_0 L(R + M) \leqslant 2R,$$

then $y^{p-1} \in B_2$. Similarly,

$$\|y_n^{p-2} - \rho\| \leqslant (1 - \beta_n^{p-2})\|x_n - \rho\| + \beta_n^{p-2}\|T^n y_n^{p-1} - \rho\| \leqslant R + \tau_0 L(2R + M) \leqslant 2R,$$

then $y^{p-2} \in B_2 \ldots$, we have

$$\|y_n^1 - \rho\| \leqslant (1 - \beta_n^1)\|x_n - \rho\| + \beta_n^1\|T^n y_n^2 - \rho\| \leqslant R + \tau_0 L(2R + M) \leqslant 2R,$$

then $y^1 \in B_2$. Therefore, we get

$$\|x_{n+1} - \rho\| \leqslant (1 - \alpha_n)\|x_n - \rho\| + \alpha_n\|T^n y_n^1 - \rho\| \leqslant R + \tau_0 L(2R + M) \leqslant 2R.$$

Also we have the following relations,

$$\begin{aligned}\|x_{n+1} - x_n\| \leqslant \alpha_n\|T^n y_n^1 - x_n\| &\leqslant \alpha_n(\|T^n y_n^1 - \rho\| + \|x_n - \rho\|)\\ &\leqslant \tau_0(L(2R + M) + R).\end{aligned}$$

$$\begin{aligned}\|y_n^1 - x_{n+1}\| &\leqslant \beta_n\|T^n y_n^2 - x_n\| + \alpha_n\|T^n y_n^1 - x_n\|\\ &\leqslant \beta_n(\|T^n y_n^2 - \rho\| + \|x_n - \rho\|) + \alpha_n(\|T^n y_n^1 - \rho\| + \|x_n - \rho\|)\\ &\leqslant 2\tau_0(L(2R + M) + R).\end{aligned}$$

Using Lemma 1.1 and the above relations, we have

$$\begin{aligned}\|x_{n+1} - \rho\|^2 &\leqslant \|x_n - \rho\|^2 + 2\alpha_n\langle T^n y_n^1 - x_n, j(x_{n+1} - \rho)\rangle\\ &= \|x_n - \rho\|^2 + 2\alpha_n\langle T^n x_{n+1} - x_{n+1}, j(x_{n+1} - \rho)\rangle\\ &\quad + \langle x_{n+1} - x_n, j(x_{n+1} - \rho)\rangle\\ &\quad + \langle T^n y_n^1 - T^n x_{n+1}, j(x_{n+1} - \rho)\rangle\\ &\leqslant \|x_n - \rho\|^2 + 2\alpha_n(k_n\|x_{n+1} - \rho\|^2 - \Phi(\|x_{n+1} - \rho\|))\\ &\quad - 2\alpha_n\|x_{n+1} - \rho\|^2 + 2\alpha_n L(\|y_n^1 - x_{n+1}\|)\|x_{n+1} - \rho\|\\ &\quad + 2\alpha_n\|x_{n+1} - x_n\|\|x_{n+1} - \rho\|\\ &\leqslant \|x_n - \rho\|^2 + 2\alpha_n(k_n - 1)\|x_{n+1} - \rho\|^2\\ &\quad - 2\alpha_n\Phi(\|x_{n+1} - \rho\|))\\ &\quad + 2\alpha_n L(\|y_n^1 - x_{n+1}\|)\|x_{n+1} - \rho\|\\ &\quad + 2\alpha_n\|x_{n+1} - x_n\|\|x_{n+1} - \rho\|\\ &\leqslant \|x_n - \rho\|^2 - 2\alpha_n\Phi(R/2) + 2\alpha_n\frac{\Phi(R/2)}{6}\\ &\quad + 2\alpha_n\frac{\Phi(R/2)}{12R}2R + 2\alpha_n\frac{\Phi(R/2)}{12R}2R\\ &\leqslant \|x_n - \rho\|^2 - \alpha_n\Phi(R/2) \leqslant R^2.\end{aligned} \tag{2.3}$$

which is a contradiction. Hence $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence.

We next prove that $\|x_n - \rho\| \to 0$ as $n \to \infty$.

Since $\lim_{n\to\infty} \alpha_n = 0 = \beta_n^k$, $\lim_{n\to\infty} k_n = 1$ and $\{x_n\}$ is bounded. Clearly,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \quad \lim_{n\to\infty} L\|y_n^1 - x_{n+1}\| = 0.$$

Thus from (2.3), we have

$$\begin{aligned}
\|x_{n+1} - \rho\|^2 &\leqslant \|x_n - \rho\|^2 + 2\alpha_n \langle T^n y_n^1 - x_n, j(x_{n+1} - \rho)\rangle \\
&= \|x_n - \rho\|^2 + 2\alpha_n \langle T^n x_{n+1} - x_{n+1}, j(x_{n+1} - \rho)\rangle \\
&\quad + \langle x_{n+1} - x_n, j(x_{n+1} - \rho)\rangle \\
&\quad + \langle T^n y_n^1 - T^n x_{n+1}, j(x_{n+1} - \rho)\rangle \\
&\leqslant \|x_n - \rho\|^2 + 2\alpha_n (k_n \|x_{n+1} - \rho\|^2 - \Phi(\|x_{n+1} - \rho\|)) \\
&\quad - 2\alpha_n \|x_{n+1} - \rho\|^2 + 2\alpha_n L(\|y_n^1 - x_{n+1}\|)\|x_{n+1} - \rho\| \\
&\quad + 2\alpha_n \|x_{n+1} - x_n\|\|x_{n+1} - \rho\| \\
&\leqslant \|x_n - \rho\|^2 + 2\alpha_n (k_n - 1)\|x_{n+1} - \rho\|^2 \\
&\quad - 2\alpha_n \Phi(\|x_{n+1} - \rho\|)) + 2\alpha_n L(\|y_n^1 - x_{n+1}\|)\|x_{n+1} - \rho\| \\
&\quad + 2\alpha_n \|x_{n+1} - x_n\|\|x_{n+1} - \rho\| \\
&= \|x_n - \rho\|^2 - 2\alpha_n \Phi(\|x_{n+1} - \rho\|) + o(\alpha_n),
\end{aligned}$$

where

$$\begin{aligned}
2\alpha_n (k_n - 1)\|x_{n+1} - \rho\|^2 + 2\alpha_n L(\|y_n^1 - x_{n+1}\|)\|x_{n+1} - \rho\| \\
+2\alpha_n \|x_{n+1} - x_n\|\|x_{n+1} - \rho\| = o(\alpha_n).
\end{aligned}$$

By Lemma 1.2, we obtain $\lim_{n\to\infty} \|x_n - \rho\| = 0$. □

REMARK 2.1. If we set $p = 0$ and $\beta_n^1 = 0$, then the modified version of the result of [**2**] holds as a special case of our theorem.

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