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ON KNASTER'S PROBLEM

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Abstract. Dold's theorem gives sufficient conditions for proving that there is no G-equivariant mapping between two spaces. We prove a generalization of Dold's theorem, which requires triviality of homology with some coefficients, up to dimension n, instead of n-connectedness. Then we apply it to a special case of Knaster's famous problem, and obtain a new proof of a result of C. T. Yang, which is much shorter and simpler than previous proofs. Also, we obtain a positive answer to some other cases of Knaster's problem, and improve a result of V. V. Makeev, by weakening the conditions.

1. Introduction

Many problems in topological combinatorics can be reduced to a question about the existence of an equivariant mapping, and it is not surprising that the well-known theorem of A. Dold is a very useful tool. Recall that the result claims that there is no G-equivariant mapping $f : X \to Y$, if G is a finite nontrivial group acting freely on a cell G-complex Y of dimension at most n, and X is an n-connected G-space (see [**12**, 6.2.6]). Volovikov in [**15**] established a generalization of Dold's theorem by relaxing the condition that X is n-connected. Suppose that the group $G = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ is a product of finitely many copies of $\mathbb{Z}_p$, with p prime. Let X and Y be fixed-point free G-spaces such that $\tilde{H}^i(X; \mathbb{Z}_p) = 0$ for all $i \leqslant n$, and Y is finite-dimensional and an n-dimensional cohomology sphere over $\mathbb{Z}_p$. Then Volovikov's theorem says that there is no G-equivariant mapping $f : X \to Y$. In this paper we include a very similar generalization of Dold's theorem, with a slightly simpler proof.

Like Dold's standard theorem, this result can be easily applied to problems of topological combinatorics. One of them is Knaster's problem (from [**9**]) of finding all configurations of points $A_1, \ldots, A_k \in S^{n-1}$, such that for every continuous mapping $f : S^{n-1} \to \mathbb{R}^m$, one can find a rotation ρ of the sphere, such that $f(\rho(A_1)) = \cdots = f(\rho(A_k))$ (k, n and m are fixed). The configurations of points

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which have some kind of symmetry are interesting because they may admit some nice actions of groups. Yang established the case $k = 3$, $m = n - 2$ of this problem for vertices of an equilateral triangle (see [**18**]), and an alternative proof is given in [**16**] using the comparison theorem for spectral sequences. Here we show how the generalization of Dold's theorem could be used to provide a new proof, much shorter and simpler than the previous two.

We also consider two cases of Knaster's problem, studied by Makeev, where the points on the sphere form a regular polygon or a regular simplex. In particular for one of the results from [**11**] we provide here an alternative, short proof. The other case is proved in [**10**] by using Dold's theorem and here we show how the generalization can improve the result, in the sense that the condition of the theorem can be weakened. It seems plausible that the other cases of Knaster's problem, originally established with the use of Dold's theorem, admit a generalization, which can be deduced from the theorem of Volovikov.

2. A generalization of Dold's theorem

This section contains the generalization of Dold's theorem which we need for Knaster's problem. As we have mentioned, Volovikov in [**15**] obtained a similar and even more general result. For the reader's convenience here we include a slightly simplified proof where we rely on an elementary dimension argument. As usual, we consider actions of nontrivial Abelian groups throughout this paper.

THEOREM 2.1. *Let G be a finite group acting freely on a cell G-complex Y of dimension at most n, and let X be a G-space. Let R be a commutative ring with unit such that $H^{n+1}(BG; R) \neq 0$ and $\tilde{H}^k(X; R) = 0$ for $0 \leqslant k \leqslant n$. Then there is no G-equivariant mapping $f : X \to Y$.*

PROOF. Suppose to the contrary that there exists a G-equivariant mapping $f : X \to Y$. Let us consider the Fadell-Husseini cohomological indexes of these spaces, $\mathrm{Ind}_G(X)$ and $\mathrm{Ind}_G(Y)$ (see [**5**]). From the monotonicity property of the index, we have $\mathrm{Ind}_G(X) \supset \mathrm{Ind}_G(Y)$. The idea is to prove the following two facts:

$$H^{n+1}(BG; R) \nsubseteq \mathrm{Ind}_G(X), \text{ and } H^{n+1}(BG; R) \subseteq \mathrm{Ind}_G(Y), \tag{2.1}$$

which will lead to a contradiction.

We know that $\mathrm{Ind}_G(X) = \ker p_X^*$, where $p_X : X_G \to BG$ ($X_G = EG \times_G X$), so $p_X^* : H^*(BG; R) \to H^*(X_G; R)$. Our intention is to find index in dimension $n+1$, i.e. $\mathrm{Ind}_G^{n+1}(X) = \ker(p_X^* : H^{n+1}(BG; R) \to H^{n+1}(X_G; R))$. Consider the cohomology Leray-Serre spectral sequence $\{E_r^{*,*}, d_r\}$ [**13**, Th. 5.2] of the Borel fibration $X \to X_G \xrightarrow{p_X} BG$. It converges to $H^*(X_G; R)$ and has the property: $E_2^{p,q} \cong H^p(BG; \mathcal{H}^q(X; R))$, where $\mathcal{H}^q(X; R)$ are local coefficients.

Look at the E_2-term. From the conditions, X is a path connected space, so we have $H^p(BG; \mathcal{H}^0(X; R)) = H^p(BG; H^0(X; R)) = H^p(BG; R)$, and then $E_2^{p,0} = H^p(BG; R)$, for all p. Above the bottom row, there are n trivial rows. Indeed, for $1 \leqslant q \leqslant n$, $H^q(X; R) = 0$, so we have simple and trivial coefficients.

We know that homomorphism p_X^* is the following composition[**13**, Th. 5.9]:

$$H^{n+1}(BG;R) = E_2^{n+1,0} \twoheadrightarrow E_3^{n+1,0} \twoheadrightarrow \cdots \twoheadrightarrow E_{n+1}^{n+1,0} \\ \twoheadrightarrow E_{n+2}^{n+1,0} = E_\infty^{n+1,0} \subset H^{n+1}(X_G;R).$$

But all epimorphisms in this relation are isomorphisms because all differentials $d_r : E_r^{n+1-r,r-1} \to E_r^{n+1,0}$ are trivial. Therefore, p_X^* is a monomorphism in dimension $n+1$, and $\mathrm{Ind}_G^{n+1}(X) = 0$. Since $H^{n+1}(BG;R) \neq 0$, we have $H^{n+1}(BG;R) \nsubseteq \mathrm{Ind}_G(X)$, so the first relation in (2.1) is proved.

The remaining part is easy. Since G acts on Y freely, $\mathrm{Ind}_G(Y) = \ker p_Y^*$, where $p_Y^* : H^*(BG;R) \to H^*(Y/G;R)$ [**5**, 3.15]. But $H^k(Y/G;R) = 0$ for all $k > n$ because $\dim Y \leqslant n$. It follows that $H^k(BG;R) \subseteq \mathrm{Ind}_G(Y)$, for all $k \geqslant n+1$. Thus (2.1) is proved. Finally, from (2.1) and the monotonicity of index, we obtain a contradiction which proves the theorem. □

3. Knaster's problem

Now we apply the theorem from previous section to Knaster's problem. Consider a finite set of points $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ on the standard sphere S^{n-1}, and fix some $m \in \mathbb{N}$. Knaster's problem asks, for a given arbitrary continuous mapping $f : S^{n-1} \to \mathbb{R}^m$, whether there exists a rotation $\rho \in SO(n)$, such that $f(\rho(A_1)) = f(\rho(A_2)) = \cdots = f(\rho(A_k))$. The problem was originally stated only for $k = n - m + 1$ (it can be shown that for a positive answer, $\mathcal{A}$ should lie in some $(n-m)$-plane). The general answer to the problem is negative; counterexamples were found by Makeev in [**10**], and later by Babenko and Bogatyi in [**1**], and then by Chen in [**4**].

However, for many "nice" configurations of $\mathcal{A}$, the answer to the problem is affirmative. The most famous case is $m = n-1$, $k = 2$, $\mathcal{A} = \{e_1, -e_1\}$, which is the Borsuk–Ulam theorem: for every continuous function $f : S^{n-1} \to \mathbb{R}^{n-1}$, there exists an $x \in S^{n-1}$ such that $f(x) = f(-x)$. This case was generalized by Hopf [**8**] for any two-element subset of S^{n-1}. There are also many positive answers for $m = 1$: when $\mathcal{A}$ is any 3-element subset of S^2 [**6**]; $\mathcal{A} = \{e_1, e_2, \dots, e_n\} \subset S^{n-1}$ being the standard orthonormal basis [**17**]; $\mathcal{A} \subset S^{n-1}$ being the set of vertices of any regular $(n-1)$-simplex [**2**], etc.

In [**18**], Yang proved the positive result for the case when $m = n-2$ and $\mathcal{A}$ is the vertex set of an equilateral triangle on a great circle of S^{n-1} (a great circle of sphere is any circle on the sphere with center at the origin). Here we give much shorter proof for that case, using Theorem 2.1.

THEOREM 3.1. *Let $A_1, A_2, A_3 \in S^{n-1}$ be the vertices of an equilateral triangle on a great circle ($n > 2$). Then for every continuous mapping $f : S^{n-1} \to \mathbb{R}^{n-2}$, there exists a rotation $\rho \in SO(n)$ such that $f(\rho(A_1)) = f(\rho(A_2)) = f(\rho(A_3))$.*

PROOF. Consider the configuration space of all triples of points (X_1, X_2, X_3) that are vertices of some equilateral triangle on some great circle. It can be viewed as Stiefel manifold $V_2(\mathbb{R}^n)$ (the third point X_3 is uniquely determined by the first two, and (X_1, X_2) is any pair of points that make constant angle $\frac{2\pi}{3}$).

Suppose to the contrary that there is a continuous mapping $f : S^{n-1} \to \mathbb{R}^{n-2}$, for which a desired rotation ρ doesn't exist. Since the vertex set of any considered triangle can be obtained from the set $\{A_1, A_2, A_3\}$ by some rotation, it means that there does not exist equilateral triangle on any great circle whose vertices have the same image under f. Define a mapping: $F : V_2(\mathbb{R}^n) \to \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2}$ by

$$F(X_1, X_2, X_3) := (f(X_1), f(X_2), f(X_3)).$$

By assumption, $F(V_2(\mathbb{R}^n))$ has empty intersection with the diagonal Δ in the space $\mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2}$ ($\dim \Delta = n-2$). So we can take smaller codomain and have $F : V_2(\mathbb{R}^n) \to (\mathbb{R}^{n-2})^3 \setminus \Delta$. There are natural $\mathbb{Z}_3$-actions on these spaces. The action of $\mathbb{Z}_3$ cyclically permutes vectors of $V_2(\mathbb{R}^n)$ (with our identification of configuration space with $V_2(\mathbb{R}^n)$, it means that the generator of $\mathbb{Z}_3$ acts as: $g_{\mathbb{Z}_3}(X_1, X_2, X_3) := (X_2, X_3, X_1)$). Also, $\mathbb{Z}_3$ cyclically permutes $(n-2)$-dimensional vectors in $(\mathbb{R}^{n-2})^3 \setminus \Delta$. Each of these actions are free and F is a $\mathbb{Z}_3$-equivariant mapping.

Now take the orthogonal projection $p : (\mathbb{R}^{n-2})^3 \setminus \Delta \to \Delta^{\perp} \setminus \{0\}$, and radial projection r from $\Delta^{\perp} \setminus \{0\}$ onto the unit sphere in $\Delta^{\perp}$, which is S^{2n-5}. Spaces $\Delta^{\perp} \setminus \{0\}$ and S^{2n-5} have inherited $\mathbb{Z}_3$-actions from $(\mathbb{R}^{n-2})^3 \setminus \Delta$, and it is obvious that p and r are $\mathbb{Z}_3$-equivariant deformations. So we have the following composition which is a $\mathbb{Z}_3$-equivariant mapping, and $\mathbb{Z}_3$-actions on the spaces are free:

$$\phi = r \circ p \circ F : V_2(\mathbb{R}^n) \xrightarrow{\mathbb{Z}_3} S^{2n-5}.$$

It remains to show that such equivariant mapping doesn't exist. The domain is $(n-3)$-connected [**7**, 4.53], while the codomain has larger dimension, so we cannot apply Dold's standard theorem. But we can apply Theorem 2.1, for which we need cohomology groups of $V_2(\mathbb{R}^n)$. If n is odd, $H^k(V_2(\mathbb{R}^n); \mathbb{Z}) \cong \mathbb{Z}$ for $k \in \{0, 2n-3\}$, $H^{n-1}(V_2(\mathbb{R}^n); \mathbb{Z}) \cong \mathbb{Z}_2$, and other cohomology groups are trivial. If n is even, $H^k(V_2(\mathbb{R}^n); \mathbb{Z}) \cong H^k(S^{n-1} \times S^{n-2}; \mathbb{Z})$ [**3**, Prop. 10.1]. We have two cases.

1° Let n be odd. From the universal coefficient theorem we derive cohomology groups with $\mathbb{Z}_3$-coefficients: $H^k(V_2(\mathbb{R}^n); \mathbb{Z}_3) \cong \mathbb{Z}_3$ for $k \in \{0, 2n-3\}$, and it is trivial otherwise. Since $H^k(B\mathbb{Z}_3; \mathbb{Z}_3) \neq 0$ for all k [**14**, III 2.5], Theorem 2.1 can be applied to spaces $V_2(\mathbb{R}^n)$ and S^{2n-5}, and $G = R = \mathbb{Z}_3$. It implies that there is no equivariant mapping $V_2(\mathbb{R}^n) \xrightarrow{\mathbb{Z}_3} S^{2n-5}$, which is a contradiction. Thus theorem is proved in this case.

2° Let n be even. Consider the mapping $g : S^n \to \mathbb{R}^{n-1}$, $g := Sf$-the suspension of f (the image of f is compact so the codomain of suspension is a subspace of $\mathbb{R}^{n-1}$). Suppose that there are three points on S^n that are vertices of a great equilateral triangle having the same image under g. From the definition of suspension, those points have to be on the same "level" $S^{n-1} \times \{t\}$ of the suspension S^n. Moreover, in order to be on a great circle, they have to be on the middle level $S^{n-1} \times \{0\}$. Then we obtain the vertices of a great equilateral triangle of sphere S^{n-1} with the same image by f, which is impossible. So, those points cannot exist. Then function g, in the same way as f dues, induces the mapping $G : V_{2,n+1} \xrightarrow{\mathbb{Z}_3} (\mathbb{R}^{n-1})^3 \setminus \Delta$. Since $n+1$ is an odd integer, we obtain a contradiction from 1°. □

The following theorem, which is stated in Makeev's paper [**11**], can be proved in the completely same way for n odd.

THEOREM 3.2. *Let p be an odd prime, $n, m \in \mathbb{N}$, n odd, such that $(p-1)\cdot m < 2n-2$. Let $A_1, A_2, \dots, A_p$ be the vertices of a regular polygon with p sides, on a great circle of sphere S^{n-1}. Then for every continuous mapping $f : S^{n-1} \to \mathbb{R}^m$, there exists a rotation $\rho \in SO(n)$ such that $f(\rho(A_1)) = f(\rho(A_2)) = \cdots = f(\rho(A_p))$.*

The configuration space is again $V_2(\mathbb{R}^n)$ because every regular polygon on a great circle, with p sides, is determined by the first two vertices. Then we consider the mapping $F : V_2(\mathbb{R}^n) \to \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{p}$, $F(X_1, \dots, X_p) := (f(X_1), \dots, f(X_p))$, where we suppose that f is a mapping that contradicts the theorem. Like in the previous theorem, we would obtain a sequence of $\mathbb{Z}_p$-equivariant mappings:

$$V_2(\mathbb{R}^n) \xrightarrow{\mathbb{Z}_p} (\mathbb{R}^m)^p \setminus \Delta \xrightarrow{\mathbb{Z}_p} \Delta^{\perp} \setminus \{0\} \xrightarrow{\mathbb{Z}_p} S^{m(p-1)-1}.$$

Since n is odd, $H^k(V_2(\mathbb{R}^n); \mathbb{Z}_p) \cong \mathbb{Z}_p$ for $k \in \{0, 2n-3\}$, and it is trivial otherwise. Also, $H^k(B\mathbb{Z}_p; \mathbb{Z}_p)$ isn't trivial [**14**, III 2.5]. We have $2n-4 \geqslant m(p-1)-1$, so Theorem 2.1. gives a contradiction again and proves the theorem.

Let us show one more application of Theorem 2.1. By using Dold's standard theorem, Makeev proved the following theorem in [**10**].

THEOREM 3.3 (Makeev). *Let p be an odd prime and $n \in \mathbb{N}$, such that $2p < n+1$. Let $A_1, A_2, \dots, A_p \in S^{n-1}$ be the vertices of a regular $(p-1)$-simplex. Then for every continuous mapping $f : S^{n-1} \to \mathbb{R}$, there exists a rotation $\rho \in SO(n)$ such that $f(\rho(A_1)) = f(\rho(A_2)) = \cdots = f(\rho(A_p))$.*

With the generalization of Dold's theorem, we are able to prove Makeev's result with a weaker condition: $3p \leqslant 2n+2$, for n even. (For n odd, our technique gives the estimate from Makeev's theorem.) The proof follows the idea of Makeev's proof.

THEOREM 3.4. *Let p be an odd prime, $n \in \mathbb{N}$, n even, such that $3p \leqslant 2n+2$. Let $A_1, A_2, \dots, A_p \in S^{n-1}$ be the vertices of a regular $(p-1)$-simplex, whose centre is not at the origin. Then for every continuous mapping $f : S^{n-1} \to \mathbb{R}$, there exists a rotation $\rho \in SO(n)$ such that $f(\rho(A_1)) = f(\rho(A_2)) = \cdots = f(\rho(A_p))$.*

PROOF. Suppose that f is a continuous mapping for which there is no desired rotation. The configuration space of all p-tuples $(X_1, \dots, X_p)$ which form a regular $(p-1)$-simplex congruent with simplex $(A_1, \dots, A_p)$, is the Stiefel manifold $V_p(\mathbb{R}^n)$. (If $A_1, \dots, A_p$ are vertices of an orthonormal frame, it is obvious. Else, every $(X_1, \dots, X_p)$ has its unique corresponding simplex $(X_1', \dots, X_p')$ formed by vertices of an orthonormal frame, and whose centre is collinear with the origin and the centre of $(X_1, \dots, X_p)$.) Further, since vertex set $\{X_1, \dots, X_p\}$ of every considered $(p-1)$-simplex is $\rho(\{A_1, \dots, A_p\})$, for some $\rho \in SO(n)$, then $X_1, \dots, X_p$ never have the same image by f. So the following mapping is well defined: $F : V_p(\mathbb{R}^n) \to \mathbb{R}^p \setminus \Delta$, $F(X_1, \dots, X_p) := (f(X_1), \dots, f(X_p))$. As in Theorem 3.1, now we have reductions by orthogonal and radial projection:

$$V_p(\mathbb{R}^n) \to \mathbb{R}^p \setminus \Delta \to \Delta^{\perp} \setminus \{0\} \to S^{p-2}.$$

Group $\mathbb{Z}_p$ naturally acts on these spaces, on the first two by cyclically permuting the coordinates, and on $\Delta^\perp \setminus \{0\}$ and S^{p-2} actions are inherited from $\mathbb{R}^p$. All these actions are free and all mappings are $\mathbb{Z}_p$-equivariant. It remains to show that there is no $\mathbb{Z}_p$-equivariant mapping $V_p(\mathbb{R}^n) \to S^{p-2}$. Makeev got the contradiction from Dold's standard theorem for $2p < n+1$. For a stronger result, let us consider the cohomology $H^*(V_p(\mathbb{R}^n);\mathbb{Z}_p)$. It is isomorphic to the cohomological algebra (with $\mathbb{Z}_p$-coefficients) of the product $S^{2\overline{n}-3} \times S^{2\overline{n}-7} \times \cdots \times S^{2\overline{q}+1}$, multiplied with S^{n-1} if $2|n$, and multiplied with S^q if $2 \mid q$, where q denotes $n-p$, $\overline{q}$ is the smallest odd integer which is $\geqslant q$ and $\overline{n}$ is the largest odd integer which is $\leqslant n$ [**3**, Prop. 10.2].

In our case, $2 \mid n$, $2 \nmid p$, so $2 \nmid q = \overline{q}$, and we have $H^*(V_p(\mathbb{R}^n);\mathbb{Z}_p) \cong H^*(X;\mathbb{Z}_p)$, where $X = S^{2n-5} \times S^{2n-9} \times \cdots \times S^{2(n-p)+1} \times S^{n-1}$. Then $\widetilde{H}^k(X;\mathbb{Z}_p) = 0$, for all $k \leqslant \min\{2(n-p), n-2\}$. From $3p \leqslant 2n+2$ we conclude that $\min\{2(n-p), n-2\} \geqslant p-2 = \dim S^{p-2}$. Then Theorem 2.1 (for $G = R = \mathbb{Z}_p$) implies nonexistence of Z_p-equivariant mapping $V_p(\mathbb{R}^n) \to S^{p-2}$, which is a contradiction. □

There is an analogous result when the codomain is $\mathbb{R}^m$, $m \geqslant 2$.

THEOREM 3.5. *Let p be an odd prime, $n, m \in \mathbb{N}$, n even and $m \geqslant 2$ such that $(p-1)m + 1 \leqslant n$. Let $A_1, A_2, \ldots, A_p \in S^{n-1}$ be the vertices of a regular simplex. Then for every continuous mapping $f : S^{n-1} \to \mathbb{R}^m$, there exists a rotation $\rho \in SO(n)$ such that $f(\rho(A_1)) = f(\rho(A_2)) = \cdots = f(\rho(A_p))$.*

The proof completely follows from the previous one, so we omit it. We would construct a $\mathbb{Z}_p$-equivariant mapping $V_p(\mathbb{R}^n) \to S^{(p-1)m-1}$, and obtain a contradiction for $\min\{2(n-p), n-2\} \geqslant (p-1)m - 1$, which is equivalent to the condition $(p-1)m + 1 \leqslant n$, for $m \geqslant 2$.

The technique applied throughout this paper provides the following similar result for the vertex set of a regular $(p^k - 1)$-simplex.

THEOREM 3.6. *Let p be a prime, $n, k \in \mathbb{N}$, $k > 1$. Let $A_1, A_2, \ldots, A_{p^k} \in S^{n-1}$ be the vertices of a regular $(p^k - 1)$-simplex, whose center is not at the origin. If numbers p, k and n satisfy one of the following conditions:*

- *p is odd, n is even and $2n + 2 \geqslant 3p^k$,*
- *p is odd, n is odd and $n + 1 \geqslant 2p^k$,*
- *$p = 2$, n is even and $n + 1 \geqslant 2^{k+1}$,*
- *$p = 2$, n is odd and $2n + 2 \geqslant 3 \cdot 2^k$,*

then for every continuous mapping $f : S^{n-1} \to \mathbb{R}$, there exists a rotation $\rho \in SO(n)$ such that $f(\rho(A_1)) = f(\rho(A_2)) = \cdots = f(\rho(A_{p^k}))$.

The proof is analogous to the proof of Theorem 3.4, so it is omitted. We would make obvious modifications (consider the actions of group $(\mathbb{Z}_p)^k$ instead of $\mathbb{Z}_p$), and an application of Volovikov's theorem [**15**], instead of Theorem 2.1, implies a contradiction.

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