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APPROXIMATIONS OF PERIODIC FUNCTIONS BY ANALOGUE OF ZYGMUND SUMS IN THE SPACES $L^{p(\cdot)}$

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Abstract. We found order estimates for the upper bounds of the deviations of analogue of Zygmund's sums on the classes of $(\psi;\beta)$-differentiable functions in the metrics of generalized Lebesgue spaces with variable exponent.

1. Definition and formulation of the problem

Let $p = p(x)$ be a 2π-periodic measurable and essentially bounded function and let $L^{p(\cdot)}$ be space of measurable 2π-periodic functions f such that

$$\int_{-\pi}^{\pi} |f(x)|^{p(x)}\, dx < \infty.$$

If $\underline{p} := \operatorname{ess\,inf}_x |p(x)| \geqslant 1$ and $\bar{p} := \operatorname{ess\,sup}_x |p(x)| < \infty$, then $L^{p(\cdot)}$ are Banach spaces [**16**] (see also [**8**]) with the norm, which can be given by

$$\|f\|_{p(\cdot)} := \inf\Big\{\alpha > 0 : \int_{-\pi}^{\pi} \Big|\frac{f(x)}{\alpha}\Big|^{p(x)}\, dx \leqslant 1\Big\}.$$

Here are some definitions which will be used in the statement and proof of the results of this article.

Definition 1.1. It is said that a function $p = p(x)$ satisfies the Dini–Lipschitz condition of order γ, if $\omega(p;\delta)\big(\ln\frac{1}{\delta}\big)^{\gamma} \leqslant K = \text{const}$, $0 < \delta < 1$, where

$$\omega(p;\delta) = \sup_{x_1,x_2\in[-\pi;\pi]} \big\{|p(x_1) - p(x_2)| : \quad |x_1 - x_2| \leqslant \delta\big\}.$$

The set of 2π-periodic exponents $p = p(x) > 1$, satisfying the Dini–Lipschitz condition of order $\gamma \geqslant 1$ in the period, is denoted by $\mathcal{P}^{\gamma}$. Obviously, if $p \in \mathcal{P}^{\gamma}$, then $\underline{p} > 1$ and $\bar{p} < \infty$. In the work [**16**] shown that when $1 < \underline{p}$, $\bar{p} < \infty$, space $L^{q(\cdot)}$,

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where $q(x) = \frac{p(x)}{p(x)-1}$, is conjugate for $L^{p(\cdot)}$ and for arbitrary functions $f \in L^{p(\cdot)}$ and $g \in L^{q(\cdot)}$ an analogue of the classical Hölder inequality is true:

$$\int_{-\pi}^{\pi} |f(x)g(x)|\,dx \leqslant K_{p,q}\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}, \quad (K_{p,q} \leqslant 1/\underline{p} + 1/\underline{q}),$$

which, in particular, implies embedding: $L^{p(\cdot)} \subset L$, where L is space of 2π-periodic Lebesgue integrable on the period functions.

The spaces $L^{p(\cdot)}$ are called generalized Lebesgue spaces with variable exponent. It is clear, that if $p = p(x) \equiv \text{const} > 0$, spaces $L^{p(\cdot)}$ coincide with the classical Lebesgue spaces L_p. In its turn, if $\bar{p} < \infty$, spaces $L^{p(\cdot)}$ are a special case of the so-called Orlicz–Musielak spaces [**10**].

For the first time, a Lebesgue space with variable exponent appeared in the literature in the article of W. Orlicz [**12**]. In the work [**11**] spaces $L^{p(\cdot)}$ considered as an example of the more general function spaces and, furthermore, have been studied by many authors in different directions. The basic results of the theory of these spaces are available, for example, in [**1**, **2**, **4**, **8**, **10**–**12**, **14**, **16**, **18**, **19**]. Note also that the generalized Lebesgue spaces with variable exponent used in the theory elastic mechanics, the theory of differential operators, variations calculus [**3**,**13**,**15**].

Next, we need the definitions of the $(\psi;\beta)$-derivative and the sets L^{ψ}_{β}, which belongs to A. I. Stepanetz [**20**, pp. 142–143].

Definition 1.2. Let $f \in L$ and

$$S[f] = \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \big(a_k(f)\cos kx + b_k(f)\sin kx\big) \equiv \sum_{k=0}^{\infty} A_k(f,x)$$

be its Fourier series. Let, further, $\psi(k)$ be arbitrary function of natural argument and $\beta \in \mathbb{R}$. Assume that the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)}\Big(a_k(f)\cos\Big(kx + \frac{\beta\pi}{2}\Big) + b_k(f)\sin\Big(kx + \frac{\beta\pi}{2}\Big)\Big)$$

is the Fourier series of some function from L. This function is denoted by $f^{\psi}_{\beta}(\cdot)$ (or $(D^{\psi}_{\beta}f)(\cdot)$) and called $(\psi;\beta)$-derivative of a function $f(\cdot)$. The set of functions $f(\cdot)$, satisfying this condition is denoted by L^{ψ}_{β}.

Denote by $\hat{Z}_n(f;x)$ (see [**5**]) the trigonometric polynomials of the form

$$\hat{Z}_n(f;x) := \frac{a_0(f)}{2} + \sum_{k=1}^{n-1}\Big(1 - \frac{\psi(n)}{\psi(k)}\Big)A_k(f;x). \tag{1.1}$$

Note that in the case $\psi(k) = 1/k^r,\ r > 0$, the sums (1.1) are Zygmund's well known sums

$$Z_n(f;x) := \frac{a_0(f)}{2} + \sum_{k=1}^{n-1}\Big(1 - \Big(\frac{k}{n}\Big)^r\Big)A_k(f;x).$$

In this paper, we study the value

$$\mathcal{E}(L^{\psi}_{\beta,p(\cdot)};\hat{Z}_n)_{s(\cdot)} := \sup_{f\in L^{\psi}_{\beta,p(\cdot)}} \|f-\hat{Z}_n(f)\|_{s(\cdot)}$$

upper bounds of deviations analogues of Zygmund's sums $\hat{Z}_n(f;x)$ on the classes $L^{\psi}_{\beta,p(\cdot)} := \{f \in L^{\psi}_{\beta} : \ f^{\psi}_{\beta} \in U_{p(\cdot)}\}$, where $U_{p(\cdot)} := \{\varphi \in L^{p(\cdot)} : \ \|\varphi\|_{p(\cdot)} \leqslant 1\}$ is the unit ball of $L^{p(\cdot)}$.

2. Auxiliary results

In the proof of the main assertions of this work we use the following well-known results.

Theorem 2.1. [**19**] *If $p \in \mathcal{P}^{\gamma}$, then for an arbitrary function $f \in L^{p(\cdot)}$ the inequalities hold*

$$\|S_n(f)\|_{p(\cdot)} \leqslant C_p\|f\|_{p(\cdot)}, \tag{2.1}$$

$$\|\tilde{f}\|_{p(\cdot)} \leqslant K_p\|f\|_{p(\cdot)}, \tag{2.2}$$

where

$$S_n(f;x) = \frac{a_0(f)}{2} + \sum_{k=1}^{n-1}(a_k(f)\cos kx + b_k(f)\sin kx), \quad n = 0,1,\ldots,$$

is the Fourier partial sums of the order n of function f, $\tilde{f}(\cdot)$ is trigonometric conjugate to $f(\cdot)$ functions, and C_p, K_p are positive constants which don't depend on n and f.

From inequality (2.1), it follows that for an arbitrary function $f \in L^{p(\cdot)}$, on condition $p \in \mathcal{P}^{\gamma}$, its Fourier series converges to f in the metric of the spaces $L^{p(\cdot)}$, that is $\|f - S_n(f)\|_{p(\cdot)} \to 0$, as $n \to \infty$, and the relation holds

$$E_n(f)_{p(\cdot)} \leqslant \|f - S_{n-1}(f)\|_{p(\cdot)} \leqslant K_pE_n(f)_{p(\cdot)},$$

where

$$E_n(\varphi)_{p(\cdot)} := \inf_{t_{n-1}\in\mathcal{T}_{2n-1}} \|\varphi - t_{n-1}\|_{p(\cdot)}, \quad \varphi \in L^{p(\cdot)},$$

is the best approximation of function φ by subspace $\mathcal{T}_{2n-1}$ of trigonometric polynomials of order, not higher than $n-1$, and K_p is the value which depends only on $p = p(x)$.

Lemma 2.1. [**7**] *Let the sequence $\mu(k)$, $k = 0,1,2,\ldots$, satisfies the conditions*

$$\nu_0 = \nu_0(\mu) = \sup_k |\mu(k)| \leqslant C, \qquad \sigma_0 = \sigma_0(\mu) = \sup_{m\in\mathbb{N}} \sum_{k=2^m}^{2^{m+1}} |\mu(k+1) - \mu(k)| \leqslant C,$$

where C is the value which does not depend on k and m. Then, if $p \in \mathcal{P}^{\gamma}$, for a given function $f \in L^{p(\cdot)}$ there exists a function $F \in L^{p(\cdot)}$ such that the series

$$\frac{\mu(0)a_0(f)}{2} + \sum_{k=1}^{\infty} \mu(k)\big(a_k(f)\cos kx + b_k(f)\sin kx\big)$$

is the Fourier series of F and the inequality is true

$$\|F\|_{p(\cdot)} \leqslant K\lambda\|f\|_{p(\cdot)}, \quad \lambda = \max\{\nu_0, \sigma_0\}, \tag{2.3}$$

where the value K does not depend on the function f.

In the case $p = p(x) \equiv \text{const}$ this statement is a well-known lemma of Marcinkievicz for multipliers [**9**].

We will also use the following theorem of Hardy–Littlewood.

Theorem 2.2. [**6**] *Let $1 < p < s < \infty$, $p, s = \text{const}$, $\alpha = p^{-1} - s^{-1}$ and $D_\alpha(t) := \sum_{k=1}^{\infty} k^{-\alpha}\cos kt$. Then, for an arbitrary function $\varphi \in L_p$ the convolution*

$$\Phi_\alpha(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(x+t)D_\alpha(t)dt$$

belongs to L_s, and $\|\Phi_\alpha\|_s \leqslant C_{s,p}\|\varphi\|_p$, where $C_{s,p}$ depends on s and p only.

Note that if $\varphi \in L_p$ and $S[\varphi] = \sum_{k=0}^{\infty} A_k(\varphi; x)$, then

$$S[\Phi_\alpha] = \sum_{k=0}^{\infty} k^{-\alpha}A_k(\varphi; x),$$

that is $\Phi_\alpha = M_\alpha(\varphi)$, where M_α is an operator-multiplier, which is determined by the sequence of $\mu_\alpha(k) = k^{-\alpha}$, $k = 0, 1, 2, \ldots$, and it acts from L_p to L_s, where indicators $1 < p < s < \infty$, $p, s = \text{const}$, are related by the equation $p^{-1} - s^{-1} = \alpha$.

3. Approximation by analogue of Zygmund's sums

We define the sequences $\mu(k)$ and $\tilde{\mu}(k)$, $k \in \mathbb{N}$, as follows:

$$\mu(k) = \mu_{n,\alpha}(k) := \begin{cases} k^\alpha\psi(n)\cos\frac{\beta\pi}{2}, & 1 \leqslant k \leqslant n-1, \\ k^\alpha\psi(k)\cos\frac{\beta\pi}{2}, & n \leqslant k, \end{cases} \tag{3.1}$$

$$\tilde{\mu}(k) = \tilde{\mu}_{n,\alpha}(k) := \begin{cases} k^\alpha\psi(n)\sin\frac{\beta\pi}{2}, & 1 \leqslant k \leqslant n-1, \\ k^\alpha\psi(k)\sin\frac{\beta\pi}{2}, & n \leqslant k, \end{cases} \tag{3.2}$$

where $n \in \mathbb{N}$, $\alpha \geqslant 0$, $\beta \in \mathbb{R}$. For each fixed $\alpha \geqslant 0$ we denote by $\Upsilon_{\alpha,n}$ the set of pairs $(\psi; \beta)$, such that for any positive number n the conditions

$$\nu_\alpha(\psi; \beta; n) := \sup_k |\mu_{n,\alpha}(k)| \leqslant C\nu(n)n^\alpha < K, \tag{3.3}$$

$$\sigma_\alpha(\psi; \beta; n) := \sup_{m\in\mathbb{N}} \sum_{k=2^m}^{2^{m+1}} |\mu_{n,\alpha}(k+1) - \mu_{n,\alpha}(k)| \leqslant C\nu(n)n^\alpha < K, \tag{3.4}$$

and similar conditions for the function $\tilde{\mu}_{n,\alpha}(k)$ hold, where $\nu(n) = \sup_{k\geqslant n}|\psi(k)|$, C and K are positive constants uniformly bounded on n.

At first we consider the case when the functions $p = p(x)$ and $s = s(x)$ on the period satisfy the inequality $s(x) \leqslant p(x)$. In our notation, the following assertion is true.

THEOREM 3.1. *Let* $(\psi;\beta) \in \Upsilon_{0,n}$ $p, s \in \mathcal{P}^\gamma$, $s(x) \leqslant p(x)$, $x \in [0;2\pi]$. *Then, for all* $n \in \mathbb{N}$ *the inequality*

$$C_{p,s}\nu(n) \leqslant \mathcal{E}(L^{\psi}_{\beta,p(\cdot)};\hat{Z}_n)_{s(\cdot)} \leqslant K_{p,s}\nu(n),$$

holds, where $C_{p,s}$ *and* $K_{p,s}$ *are some constants depending on* $p = p(x)$ *and* $s = s(x)$ *only.*

PROOF. Suppose first that $p(x) \equiv s(x)$, $x \in [0;2\pi]$. For an arbitrary function $f \in L^{\psi}_{\beta,p(\cdot)}$, the equality is true

$$\begin{aligned} f(x) - \hat{Z}_n(f;x) &= \sum_{k=1}^{n-1} \frac{\psi(n)}{\psi(k)} A_k(f;x) + \sum_{k=n}^{\infty} A_k(f;x) \\ &= \sum_{k=1}^{\infty} \mu_{n,0} A_k(f^{\psi}_{\beta};x) + \sum_{k=1}^{\infty} \tilde{\mu}_{n,0} \tilde{A}_k(f^{\psi}_{\beta};x) := M_0(f^{\psi}_{\beta}) + \tilde{M}_0(\tilde{f}^{\psi}_{\beta}), \end{aligned} \tag{3.5}$$

where M_0 and $\tilde{M}_0$ are operators-multipliers, which are defined by the sequences (3.1) and (3.2) respectively, $\alpha = 0$. According to the conditions of the theorem, the couples $(\psi;\beta)$ belong to the set $\Upsilon_{0,n}$, therefore, sequence (3.1) and (3.2) satisfy the conditions of Lemma 2.1. Applying this lemma, given inequalities (2.2), (3.3) and (3.4), for an arbitrary function $f \in L^{\psi}_{\beta,p(\cdot)}$ on the basis of the equality (3.5) we find

$$\begin{aligned} \|f(\cdot) - \hat{Z}_n(f;\cdot)\|_{p(\cdot)} &= \|M_0(f^{\psi}_{\beta}) + \tilde{M}_0(\tilde{f}^{\psi}_{\beta})\|_{p(\cdot)} \\ &\leqslant K\nu(n)(\|f^{\psi}_{\beta}\|_{p(\cdot)} + \|\tilde{f}^{\psi}_{\beta}\|_{p(\cdot)}) \leqslant C_p\nu(n), \end{aligned} \tag{3.6}$$

where C_p is a positive constant, which depends only on the function $p = p(x)$. In the article [**17**] it was shown that if $1 \leqslant s(x) \leqslant p(x) \leqslant \bar{p} < \infty$, then for an arbitrary function $f \in L^{p(\cdot)}$ the inequality hold

$$\|f\|_{s(\cdot)} \leqslant K_{s,p}\|f\|_{p(\cdot)}. \tag{3.7}$$

From relations (3.6) and (3.7) we obtain the estimate

$$\mathcal{E}(L^{\psi}_{\beta,p(\cdot)};\hat{Z}_n)_{s(\cdot)} \leqslant \mathcal{E}(L^{\psi}_{\beta,s(\cdot)};\hat{Z}_n)_{s(\cdot)} \leqslant C_{p,s}\nu(n),$$

where $C_{p,s}$ is a positive constant, which depends only on the functions $p = p(x)$ and $s = s(x)$. We now obtain the lower estimate for the value of $\mathcal{E}(L^{\psi}_{\beta,p(\cdot)};\hat{Z}_n)_{s(\cdot)}$. If for given sequence $\psi(k)$ and the number $n \in \mathbb{N}$, there exists the natural number k_n, for which the equality

$$\nu(n) = \sup_{k \geqslant n} |\psi(k)| = \psi(k_n), \tag{3.8}$$

is true, then the corresponding lower estimate can be obtained with help of the function

$$f_n(x) = \frac{\psi(k_n)}{\|\cos k_n x\|_{p(\cdot)}} \cos\Big(k_n x - \frac{\beta\pi}{2}\Big). \tag{3.9}$$

Indeed, since

$$\|(f_n(x))^{\psi}_{\beta}\|_{p(\cdot)} = \left\| \frac{\cos k_n x}{\|\cos k_n x\|_{p(\cdot)}} \right\|_{p(\cdot)} = 1,$$

then $f_n \in L^{\psi}_{\beta,p(\cdot)}$ and

$$\mathcal{E}(L^{\psi}_{\beta,p(\cdot)}; \hat{Z}_n)_{s(\cdot)} \geqslant \|f_n - \hat{Z}(f_n)\|_{s(\cdot)} = \frac{\psi(k_n)}{\|\cos k_n x\|_{p(\cdot)}} \|\cos k_n x\|_{s(\cdot)} = C_{p,s}\nu(n).$$

But if for the sequence $\psi(k)$ and the number $n \in \mathbb{N}$, there exists no natural number k_n, for which equality (3.8) holds, due to the limitation of the set $\{|\psi(k)|\}$ of values of the function $\psi(k)$ we will have

$$\nu(n) = \sup_{k \geqslant n} |\psi(k)| = \sup_{k \geqslant n} \{|\psi(k)|\}.$$

In this case, there exists a sequence k_j, $j \in \mathbb{N}$ such that $k_j \geqslant n$ and the numbers $\psi(k_j)$ don't decrease and converge to $\nu(n)$. Let $\mathbb{A} = \cup_j f_j(x)$, where the function $f_j(x)$ is defined by equality (3.9). Since $f_j \in L^{\psi}_{\beta,p(\cdot)}$ for any $j \in \mathbb{N}$, then

$$\begin{aligned}\mathcal{E}(L^{\psi}_{\beta,p(\cdot)}; \hat{Z}_n)_{s(\cdot)} &= \sup_{f \in L^{\psi}_{\beta,p(\cdot)}} \|f - \hat{Z}_n(f)\|_{s(\cdot)} \geqslant \sup_{f \in \mathbb{A}} \|f - \hat{Z}_n(f)\|_{s(\cdot)} \\ &= \sup_{j \in \mathbb{N}} \frac{\psi(k_j)}{\|\cos k_j x\|_{p(\cdot)}} \|\cos k_j x\|_{s(\cdot)} = C_{p,s}\nu(n). \qquad \square\end{aligned}$$

We now obtain an estimate of the sequence $\mathcal{E}(L^{\psi}_{\beta,p(\cdot)}; \hat{Z}_n)_{s(\cdot)}$ in the case when the function $p = p(x)$ and $s = s(x)$ on the period satisfies the inequality $p(x) < s(x)$. The following result gives the upper estimate.

THEOREM 3.2. *Let $p, s \in \mathcal{P}^{\gamma}$, $p(x) < s(x)$, $x \in [0; 2\pi]$ and $(\psi; \beta) \in \Upsilon_{\alpha,n}$, $\alpha = 1/\underline{p} - 1/\overline{s}$. Then, for all $n \in \mathbb{N}$ the following inequality*

$$\mathcal{E}(L^{\psi}_{\beta,p(\cdot)}; \hat{Z}_n)_{s(\cdot)} \leqslant C_{p,s} n^{\alpha} \nu(n),$$

holds, where $C_{p,s}$ is a positive constant depending on $p = p(x)$ and $s = s(x)$ only.

PROOF. For an arbitrary function $f \in L^{\psi}_{\beta,p(\cdot)}$, the equality

$$\begin{aligned}f(x) - \hat{Z}_n(f; x) &= \sum_{k=1}^{n-1} \frac{\psi(n)}{\psi(k)} A_k(f; x) + \sum_{k=n}^{\infty} A_k(f; x) \\ &= \sum_{k=1}^{\infty} \mu_{n,\alpha} k^{-\alpha} A_k(f^{\psi}_{\beta}; x) + \sum_{k=1}^{\infty} \tilde{\mu}_{n,\alpha} k^{-\alpha} \tilde{A}_k(f^{\psi}_{\beta}; x) \\ &:= M_{\alpha}(g_{\alpha}) + \tilde{M}_{\alpha}(\tilde{g}_{\alpha}),\end{aligned} \tag{3.10}$$

holds, where M_{α} and $\tilde{M}_{\alpha}$ are operators-multipliers, which are defined by the sequences (3.1) and (3.2) respectively, $\alpha = 1/\underline{p} - 1/\overline{s}$ and

$$g_{\alpha}(x) := \sum_{k=1}^{\infty} k^{-\alpha} A_k(f^{\psi}_{\beta}; x) = \frac{1}{\pi} \int_0^{2\pi} f^{\psi}_{\beta}(x+t) D_{\alpha}(t) dt,$$

$$\tilde{g}_\alpha(x) := \sum_{k=1}^{\infty} k^{-\alpha} \tilde{A}_k(f_\beta^\psi; x) = \frac{1}{\pi} \int_0^{2\pi} \tilde{f}_\beta^\psi(x+t) D_\alpha(t) dt,$$

$D_\alpha(t)$ is function defined in Theorem 2.2.

Since $f \in L^\psi_{\beta,p(\cdot)}$, then $f_\beta^\psi \in L^{p(\cdot)}$, and moreover $f_\beta^\psi \in L^{\underline{p}}$. By Theorem 2.2 we conclude that the convolution $g_\alpha(x)$ belongs $L^{\overline{s}}$, and moreover $g_\alpha \in L^{s(\cdot)}$. From the condition $(\psi;\beta) \in \Upsilon_{n,\alpha}$ by Lemma 2.1 we conclude that the operator-multiplier M_α acts from $L^{s(\cdot)}$ to $L^{s(\cdot)}$ for any $s \in \mathcal{P}^\gamma$. Using analogous arguments for the function $\tilde{g}_\alpha(x)$, taking into account inequalities (2.2), (2.3), (3.3) and (3.4), for an arbitrary function $f \in L^\psi_{\beta,p(\cdot)}$ on the basis of equality (3.10) we find

$$\begin{aligned} \|f - \hat{Z}_n(f)\|_{s(\cdot)} &\leqslant \|M_\alpha(g_\alpha)\|_{s(\cdot)} + \|\tilde{M}_\alpha(\tilde{g}_\alpha)\|_{s(\cdot)} \leqslant K n^\alpha \nu(n)(\|g_\alpha\|_{s(\cdot)} + \|\tilde{g}_\alpha\|_{s(\cdot)}) \\ &\leqslant C_{p,s} n^\alpha \nu(n)(\|f_\beta^\psi\|_{p(\cdot)} + \|\tilde{f}_\beta^\psi\|_{p(\cdot)}) \leqslant C_{p,s} n^\alpha \nu(n). \end{aligned}$$ □

To make formulate the following assertion, which gives a lower estimate for the quantity $\mathcal{E}(L^\psi_{\beta,p(\cdot)}; \hat{Z}_n)_{s(\cdot)}$ in the case, if the function $p = p(x)$ and $s = s(x)$ satisfies the inequality $p(x) < s(x)$ on the period, we need the following definition.

Denoted by $\mathfrak{B}$ the set of pairs $(\psi;\beta)$, such that for any $n \in \mathbb{N}$ the relations are true:

$$\sup_{n \leqslant k \leqslant 2n} \left| \frac{\nu(n)}{\psi(k)} \right| \leqslant C, \quad \sup_{m \in \mathbb{N}} \sum_{k=2^m}^{2^{m+1}} |\tau(k+1) - \tau(k)| \leqslant C,$$

where C is a positive constant, which is independent of n, $\nu(n) = \sup_{k \geqslant n} \psi(k)$ and

$$\tau(k) := \begin{cases} 0, & 1 \leqslant k \leqslant n-1, \\ \frac{\nu(n)}{\psi(k)}, & n \leqslant k \leqslant 2n. \end{cases} \tag{3.11}$$

Theorem 3.3. *Let $p, s \in \mathcal{P}^\gamma$, $p(x) < s(x)$, $x \in [0;2\pi]$ and $(\psi;\beta) \in \mathfrak{B}$. Then, for all $n \in \mathbb{N}$ we have $\mathcal{E}_n(L^\psi_{\beta,p(\cdot)}; \hat{Z}_n)_{s(\cdot)} \geqslant C_{p,s}\nu(n) n^{1/\overline{p} - 1/\underline{s}}$, where $C_{p,s}$ is a positive constant depending on $p = p(x)$ and $s = s(x)$ only.*

Proof. For obtaining a lower estimate, let us show that for any positive integer n in class $L^\psi_{\beta,p(\cdot)}$ there exists a function f_n^*, for which the inequality is true

$$\|f_n^* - \hat{Z}_n\|_{s(\cdot)} \geqslant C_{p,g} \nu(n) n^{1/\overline{p} - 1/\underline{s}}.$$

For this we fix $n \in \mathbb{N}$ and consider the function $f_n^*(x) = \sum_{k=n}^{2n} \psi(k) \cos\left(kx - \frac{\beta\pi}{2}\right)$. Since

$$(f_n^*(x))_\beta^\psi = \sum_{k=n}^{2n} \cos kx = \frac{\sin nx/2 \cos(3n+1)x/2}{\sin x/2},$$

then using relation (3.7) and also the well-known inequality

$$\frac{x}{\pi} \leqslant \sin x/2, \quad \sin x \leqslant x, \quad x \in [0;\pi], \tag{3.12}$$

we obtain

$$\|(f_n^*)_\beta^\psi\|_{p(\cdot)} = \Big\|\sum_{k=n}^{2n}\cos kx\Big\|_{p(\cdot)} \leqslant K_p\Big\|\sum_{k=n}^{2n}\cos kx\Big\|_{\overline{p}}$$
$$= \bigg(2\int_0^\pi \Big|\sum_{k=n}^{2n}\cos kx\Big|^{\overline{p}}dx\bigg)^{1/\overline{p}} \leqslant \bigg(2\int_0^\pi \Big|\frac{\sin nx/2}{\sin x/2}\Big|^{\overline{p}}dx\bigg)^{1/\overline{p}}$$
$$\leqslant C_p^* n^{1-1/\overline{p}}.$$

This implies that the function

$$g_n^*(x) = \frac{n^{1/\overline{p}-1}}{C_p^*}f_n^*(x) = \frac{n^{1/\overline{p}-1}}{C_p^*}\sum_{k=n}^{2n}\psi(k)\cos\Big(kx-\frac{\beta\pi}{2}\Big)$$

belongs to the class $L^\psi_{\beta,p(\cdot)}$.

Again using the inequalities (3.7) and (3.12), we find

$$\Big\|\sum_{k=n}^{2n}\cos kx\Big\|_{s(\cdot)} \geqslant \bigg(2\int_0^\pi \Big|\frac{\sin nx/2\cos(3n+1)x/2}{\sin x/2}\Big|^{\underline{s}}dx\bigg)^{1/\underline{s}} \tag{3.13}$$
$$\geqslant C_s n^{1-1/\underline{s}}\bigg(\int_0^{\pi/2}(\cos x)^{\underline{s}}\,dx\bigg)^{1/\underline{s}} \geqslant K_s n^{1-1/\underline{s}}.$$

If now by T_ψ we denote the operator-multiplier that generates sequence (3.11), then by applaying Lemma 2.1 to the condition $(\psi;\beta)\in\mathfrak{B}$ we will have

$$\Big\|\sum_{k=n}^{2n}\cos\Big(kx-\frac{\beta\pi}{2}\Big)\Big\|_{s(\cdot)} = \Big\|T_\psi\Big(\sum_{k=n}^{2n}\frac{\psi(k)}{\nu(n)}\cos\Big(kx-\frac{\beta\pi}{2}\Big)\Big)\Big\|_{s(\cdot)}$$
$$\leqslant C\Big\|\sum_{k=n}^{2n}\frac{\psi(k)}{\nu(n)}\cos\Big(kx-\frac{\beta\pi}{2}\Big)\Big\|_{s(\cdot)}.$$

Hence, considering inequality (3.13) we find

$$\Big\|\sum_{k=n}^{2n}\frac{\psi(k)}{\nu(n)}\cos\Big(kx-\frac{\beta\pi}{2}\Big)\Big\|_{s(\cdot)} \geqslant K\Big\|\sum_{k=n}^{2n}\cos\Big(kx-\frac{\beta\pi}{2}\Big)\Big\|_{s(\cdot)} \tag{3.14}$$
$$\geqslant C_s\Big\|\sum_{k=n}^{2n}\cos kx\Big\|_{\underline{s}} \geqslant K_s n^{1-1/\underline{s}}.$$

Using relation (3.14), we obtain

$$\mathcal{E}_n(L^\psi_{\beta,p(\cdot)};\hat{Z}_n)_{s(\cdot)} \geqslant \|g_n^*-\hat{Z}_n(g_n^*)\|_{s(\cdot)} = \Big\|\frac{n^{1/\overline{p}-1}}{C_p^*}\sum_{k=n}^{2n}\psi(k)\cos\Big(kx-\frac{\beta\pi}{2}\Big)\Big\|_{s(\cdot)}$$
$$\geqslant \frac{n^{1/\overline{p}-1}}{C_p^*}\nu(n)\Big\|\sum_{k=n}^{2n}\frac{\psi(k)}{\nu(n)}\cos\Big(kx-\frac{\beta\pi}{2}\Big)\Big\|_{s(\cdot)}$$
$$\geqslant C_{p,s}n^{1/\overline{p}-1}\nu(n)n^{1-1/\underline{s}} = C_{p,s}\nu(n)n^{1/\overline{p}-1/\underline{s}}. \qquad \square$$

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