PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 99(113) (2016), 217–225 DOI: 10.2298/PIM1613217B

# MAXIMAL TYCHONOFF SPACES AND NORMAL ISOLATOR COVERS

**C. K. Basu and S. S. Mandal**

Abstract. We introduce a new kind of cover called a normal isolator cover to characterize maximal Tychonoff spaces. Such a study is used to provide an alternative proof of an interesting result of Feng and Garcia-Ferreira in 1999 that every maximal Tychonoff space is extremally disconnected. Maximal tychonoffness of subspaces is also discussed.

## 1. Introduction

In the poset $\mathcal{A}(X)$, of all topologies on a given set $X$, having the property $P$, a topological space $(X,\tau)$ is maximal $P$ provided that $\tau$ is a maximal element in $\mathcal{A}(X)$. In [**6**], it had been shown that a topological space $(X,\tau)$ is maximal $P$ if and only if every continuous bijection from a space $(Y,\tau_1)$ with the property $P$ to $(X,\tau)$ is a homeomorphism. In 1943 Hewitt [**15**] and in 1947 Vaidyanathaswamy [**29**] had independently proved that every compact Hausdorff space is maximal compact. Vaidyanathaswamy [**29**] put forward a question if there exists any non-Hausdorff maximal compact space. One year later in 1948 Hing tong [**28**] answered affirmatively Vaidyanathaswamy's question. In the same year Ramanathan [**21**] characterized maximal compact spaces as those whose compact subsets are precisely the closed sets. Levine [**17**] answered affirmatively the question of Vaidyanathaswamy by establishing that a one point compactification of rationals with the usual topology is a non-Hausdorff maximal compact space. In the same paper he exhibited that maximal compact topologies are not productive. On the other hand, Mioduszewski and Rudolf [**18**] demonstrated necessary and sufficient conditions for an absolutely closed (or $H$-closed) space to become maximal absolutely closed.

Thron [**27**] and Aull [**1**] investigated maximal countably compact spaces. Aull, in fact, strengthened the result: a first countable Hausdorff countably compact space is maximal countably compact and minimal first countable Hausdorff of Thron [**27**].

2010 *Mathematics Subject Classification:* 54D05, 54E15, 54E99.

*Key words and phrases:* star refinement, refinement, normally open cover, normal isolator cover, stronger topology, maximal uniformizable, extremally disconnected.

Communicated by Miloš Kurilić.

In 1971, Cameron [**6**] and in 1973 Raha [**20**] investigated exhaustively the various aspects of certain maximal $P$ spaces, where $P =$ Lindeloff, countably compact, sequential compact, pseudocompact, lightly compact or connected. Thomas [**26**] had also discussed maximal connected topologies. Cameron characterized maximal QHC spaces [**7**] and maximal pseudocompact spaces [**5, 9**] and in [**8**] he shown that the maximal topologies of a class of topologies which include lightly compact and QHC spaces are submaximal and $T_1$ spaces. In 1999, Kennedy and McCartan [**16**] investigated spaces which are maximal with respect to a semiregular property and showed new characterizations of maximal QHC spaces and maximal pseudocompact spaces. In 1997, Guthrie, Stone and Wage [**13**] investigated topologies, which are maximal connected Hausdorff and in 1998, Shakhmatov, Tkacenko, Tkachuk, Watson and Wilson [**23**] showed that neither first countable nor Cech-complete spaces are maximal Tychonoff connected also in 2007, Zelenyuk [**31**] investigated almost maximal spaces. In addition, interesting behaviors of some of Maximal topologies and their applications are found in the papers [**10, 11, 14, 19, 22, 24, 25**].

Considering the usefulness and importance of uniformizability (=Tychonoffness in Hausdorff spaces) and the above observations about maximal topologies of various kinds of topological properties, this article is devoted to study maximal uniformizable(=Maximal Tychonoff) spaces. Several characterizations of such spaces have been given in terms of refinement of normally open covers as well as newly introduced normal isolator covers. As a consequence, we provide an alternative proof of the already existing interesting result of Feng, Garcia-Ferreira [**12**] that every maximal Tychonoff space is extremally disconnected. Maximal uniformizability with respect to subspaces has also been discussed.

## 2. Preliminaries

The symbol $X$ or $(X, \tau)$ denotes a topological space without any isolated points which is $T_2$ and the base set $X$ is infinite, unless explicitly stated. For two covers $\mathcal{U}$ and $\mathcal{V}$ of $X$, $\mathcal{U}$ is called a *refinement* of $\mathcal{V}$ denoted by $\mathcal{U} < \mathcal{V}$ if for each $U \in \mathcal{U}$, there exists a $V \in \mathcal{V}$ such that $U \subset V$ and we call $\mathcal{U}$ *star refines* $\mathcal{V}$ or $\mathcal{U}$ is a *star refinement* of $\mathcal{V}$, denoted by $\mathcal{U} \overset{\star}{<} \mathcal{V}$, if for each $U \in \mathcal{U}$, there exists a $V \in \mathcal{V}$ such that $\mathrm{St}(U;\mathcal{U}) \subset V$, where $\mathrm{St}(U;\mathcal{U}) = \bigcup\{W \in \mathcal{U} : W \cap U \neq \emptyset\}$. When $U = \{x\}$, we denote $\mathrm{St}(U;\mathcal{U})$ as $\mathrm{St}(x;\mathcal{U})$. We note that if $\mathcal{U} \overset{\star}{<} \mathcal{V}$, then $\mathcal{U} < \mathcal{V}$.

A *normal sequence* of covers of $X$ is a sequence of covers $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of $X$ such that $\mathcal{U}_{n+1} \overset{\star}{<} \mathcal{U}_n$, for $n = 1, 2, \ldots$; and a *normal cover* is a cover which is $\mathcal{U}_1$ in some normal sequence of covers [**30**, §36.9. p. 247]. An open cover $\mathcal{U}$ of a topological space $X$ is normally open if and only if $\mathcal{U} = \mathcal{U}_1$ in some normal sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ consisting of open covers of $X$ [**30**, §36.14. p. 248].

A collection $\mu'$ of covers of a space $X$ is a *base* for some covering uniformity on $X$ if and only if it satisfies the condition that for $\mathcal{U}_1, \mathcal{U}_2 \in \mu'$ there is a $\mathcal{U}_3 \in \mu'$ such that $\mathcal{U}_3 \overset{\star}{<} \mathcal{U}_1$ and $\mathcal{U}_3 \overset{\star}{<} \mathcal{U}_2$ [**30**, §36.3, p. 245]. It is well known that if $\mu'$ is a base for a covering uniformity $\mu$ on $X$, then $\{\mathrm{St}(x;\mathcal{U}) : \mathcal{U} \in \mu'\}$ is a local base at $x \in X$ in the uniform topology [**30**, §36.6, p. 246]. Also if $X$ is any uniformizable topological space, then there is a finest uniformity on X, compatible with the topology of $X$, called the *fine uniformity* on $X$, denoted by $\mu_F$, which has a base of all normally open covers of X. So a uniformizable space $(X,\tau)$ has at least one normally open cover consisting of proper subsets of $X$.

LEMMA 2.1. *If $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$ are three covers of $X$ such that $\mathcal{U}_1 \overset{\star}{<} \mathcal{U}_2 \overset{\star}{<} \mathcal{U}_3$, then $\mathcal{U}_1 \overset{\star}{<} \mathcal{U}_3$*

PROOF. The proof is obvious. □

LEMMA 2.2. *If $\{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots\}$ is a normal sequence of covers and if $\mathcal{U}_k, \mathcal{U}_m \in \{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots\}$, then for a positive integer $t$ greater than both of $k$ and $m$, $\mathcal{U}_t \overset{\star}{<} \mathcal{U}_k$ and $\mathcal{U}_t \overset{\star}{<} \mathcal{U}_m$.*

PROOF. The proof is obvious. □

THEOREM 2.1 (Hausdorff criterion [**30**]). *For each $x \in X$, let $\mathcal{B}_x^1$ be a neighborhood base at $x$ for the topology $\tau_1$ on $X$ and $\mathcal{B}_x^2$ be a neighborhood base at $x$ for the topology $\tau_2$ on $X$. Then $\tau_1 \subset \tau_2$ if and only if for each $x \in X$ and each $B^1 \in \mathcal{B}_x^1$, there is some $B^2 \in \mathcal{B}_x^2$ such that $B^2 \subset B^1$* [**30**, §4.8. p. 35].

DEFINITION 2.1. For two covers $\mathcal{U}$ and $\mathcal{V}$ of $X$, we denote the intersection of $\mathcal{U}$ and $\mathcal{V}$ as $\mathcal{U} \wedge \mathcal{V}$ and define it as $\mathcal{U} \wedge \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ [**30**, §36.3. p. 245].

## 3. Maximal Tychonoff spaces

Recently, when a space (uniformizable or not) possessing a nontrivial proper uniformizable subtopology is investigated in [**3**] by Basu and Mandal, by the help of normal sequence of covers and star refinement of covers. A useful consequence of that investigation reflects that a sort of converse of A. H. Stone's famous theorem is true when Basu and Mandal [**3**] established that a paracompact $T_2$ space $(X,\tau)$ is either metrizable or $(X,\tau)$ has a nontrivial proper uniformizable subtopology, which is pseudometrizable. In the course of that study, disconnectedness is seen to play a major role, especially when that is of very strong in nature viz. zero-dimensionality, it is shown there that for a paracompact $T_2$ space $(X,\tau)$ containing no isolated points, the cardinality of such nontrivial proper uniformizable subtopologies of $(X,\tau)$ is at least $\aleph_0$. In another paper [**4**], Basu and Mandal characterized minimal Uniformizable spaces in terms of normal sequence of covers and have shown that a minimal uniformizable non-indiscrete space is pseudometrizable. In this section, we investigate maximal uniformizable (=maximal Tychonoff ) spaces in terms of a new kind of cover called a normal isolator cover.

DEFINITION 3.1. A cover $\mathcal{U}$ of $(X,\tau)$ is called an isolator cover of $(X,\tau)$ if $\mathrm{St}(x;\mathcal{U} \wedge \mathcal{V})$ is infinite for every normally open cover $\mathcal{V}$ of $(X,\tau)$ and $\mathcal{U}$ is called a normal isolator cover if it is the first term of a normal sequence of isolator covers.

Clearly every open cover of $(X,\tau)$ is an isolator cover of $(X,\tau)$ and every normally open cover of $(X,\tau)$ is also obviously a normal isolator cover of $(X,\tau)$.

We further note that if $\tau_1$ is a topology on $X$ such that $\tau_1 \supset \tau$ and if $\mathcal{U}$ is a normal isolator cover of $(X, \tau_1)$, then $\mathcal{U}$ is a normal isolator cover of $(X, \tau)$.

Definition 3.2. A Tyconoff (or uniformizable) space $(X, \tau)$ is called maximal Tyconoff [**11**] (or maximal uniformizable) if no topology without any isolated points stronger than $\tau$ is Tyconoff (or uniformizable).

Lemma 3.1. *If* $\mathcal{U}_2 \overset{\star}{<} \mathcal{U}_1$ *and* $\mathcal{V}_2 \overset{\star}{<} \mathcal{V}_1$, *then*
(i) $\mathcal{U}_2 \wedge \mathcal{V}_2 \overset{\star}{<} \mathcal{U}_1 \wedge \mathcal{V}_1$; (ii) $\mathcal{U}_2 \wedge \mathcal{V}_2 \overset{\star}{<} \mathcal{U}_1$ *and* $\mathcal{U}_2 \wedge \mathcal{V}_2 \overset{\star}{<} \mathcal{V}_1$.

Proof. (i) Let $U_2 \cap V_2 \in \mathcal{U}_2 \wedge \mathcal{V}_2$, where $U_2 \in \mathcal{U}_2, V_2 \in \mathcal{V}_2$. Then

$$\begin{aligned}\mathrm{St}(U_2 \cap V_2; \mathcal{U}_2 \wedge \mathcal{V}_2) &\subset \bigcup \{(U_\alpha \cap V_\beta) : U_\alpha \in \mathcal{U}_2 \text{ with } U_\alpha \cap U_2 \neq \phi \text{ and} \\ &\qquad\qquad V_\beta \in \mathcal{V}_2 \text{ with } V_\alpha \cap V_2 \neq \phi\} \\ &\subset \bigcup \{U_\alpha \in \mathcal{U}_2 : U_\alpha \cap U_2 \neq \phi\} \cap \big(\bigcup \{V_\beta \in \mathcal{V}_2 : V_\beta \cap V_2 \neq \phi\}\big) \\ &= \mathrm{St}(U_2; \mathcal{U}_2) \cap \mathrm{St}(V_2; \mathcal{V}) \subset U_1 \cap V_1 \in \mathcal{U}_1 \wedge \mathcal{V}_1\end{aligned}$$

[as $\mathcal{U}_2 \overset{\star}{<} \mathcal{U}_1$ and $\mathcal{V}_2 \overset{\star}{<} \mathcal{V}_1$, so for $U_2 \in \mathcal{U}_2, V_2 \in \mathcal{V}_2$ there exist some $U_1 \in \mathcal{U}_1$ and some $V_1 \in \mathcal{V}_1$ such that $\mathrm{St}(U_2; \mathcal{U}_2) \subset U_1$ and $\mathrm{St}(V_2; \mathcal{V}_2) \subset V_1$]. So for $U_2 \cap V_2 \in \mathcal{U}_2 \wedge \mathcal{V}_2$ there exists $U_1 \cap V_1 \in \mathcal{U}_1 \wedge \mathcal{V}_1$ such that $\mathrm{St}(U_2 \cap V_2; \mathcal{U}_2 \wedge \mathcal{V}_2) \subset U_1 \cap V_1$. Hence $\mathcal{U}_2 \wedge \mathcal{V}_2 \overset{\star}{<} \mathcal{U}_1 \wedge \mathcal{V}_1$.

(ii) We know that $\mathcal{U}_1 \wedge \mathcal{V}_1 < \mathcal{U}_1$ and $\mathcal{U}_1 \wedge \mathcal{V}_1 < \mathcal{V}_1$ and also we know that for three covers $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$ of $X$, if $\mathcal{W}_3 \overset{\star}{<} \mathcal{W}_2 < \mathcal{W}_1$, then $\mathcal{W}_3 \overset{\star}{<} \mathcal{W}_1$. Hence the result follows. □

Lemma 3.2. *For a Tychonoff space* $(X, \tau)$, *if* $\mu$ *is the collection of all normally open covers of* $(X, \tau)$ *and* $\mathcal{U}$ *is a normal isolator cover of* $(X, \tau)$ *with the corresponding normal sequence of isolator covers* $\ldots \mathcal{U}_3 \overset{\star}{<} \mathcal{U}_2 \overset{\star}{<} \mathcal{U}_1 = \mathcal{U}$, *then* $\mu_1' = \mu \cup \{\mathcal{U}_1, \mathcal{U}_2, \ldots\} \cup \{\mathcal{V} \wedge \mathcal{U}_k : k = 1, 2, \ldots; \mathcal{V} \in \mu\}$ *forms a base for some uniformity on* $X$ *which generates a stronger Tychonoff (uniformizable) topology* $\tau_{\mu_1'}$ *on* $X$ *such that* $(X, \tau_{\mu_1'})$ *is* $T_2$ *and contains no isolated points.*

Proof. Here $\mu_1' = \mu \cup \{\mathcal{U}_1, \mathcal{U}_2, \ldots\} \cup \{\mathcal{V} \wedge \mathcal{U}_k : k = 1, 2, \ldots; \mathcal{V} \in \mu\}$. First we shall prove that for $\mathcal{W}_1, \mathcal{W}_2 \in \mu_1'$, there exists a $\mathcal{W}_3 \in \mu_1'$ such that $\mathcal{W}_3 \overset{\star}{<} \mathcal{W}_1$ and $\mathcal{W}_3 \overset{\star}{<} \mathcal{W}_2$.

Now $\mu$ is itself a base for the fine uniformity on $X$ generating the topology $\tau$ on $X$. So for $\mathcal{W}_1, \mathcal{W}_2 \in \mu$, there obviously exists a $\mathcal{W}_3 \in \mu$ such that $\mathcal{W}_3 \overset{\star}{<} \mathcal{W}_1$ and $\mathcal{W}_3 \overset{\star}{<} \mathcal{W}_2$.

Also by Lemma 2.2, for $\mathcal{U}_k, \mathcal{U}_m \in \{\mathcal{U}_1, \mathcal{U}_2, \ldots\}$, there exists an $\mathcal{U}_l \in \{\mathcal{U}_1, \mathcal{U}_2, \ldots\}$ such that $\mathcal{U}_l \overset{\star}{<} \mathcal{U}_k$ and $\mathcal{U}_l \overset{\star}{<} \mathcal{U}_m$.

Now we will have to check it for four possible cases:
(i) $\mathcal{W}_1 \in \mu$ and $\mathcal{U}_k \in \{\mathcal{U}_1, \mathcal{U}_2, \ldots\}$; (ii) $\mathcal{W}_1 \in \mu$ and $\mathcal{W} \in \{\mathcal{V} \wedge \mathcal{U}_k : k = 1, 2, \ldots; \mathcal{V} \in \mu\}$; (iii) $\mathcal{W}_1, \mathcal{W}_2 \in \{\mathcal{V} \wedge \mathcal{U}_k : k = 1, 2, \ldots; \mathcal{V} \in \mu\}$; (iv) for $\mathcal{U}_k \in \{\mathcal{U}_1, \mathcal{U}_2, \ldots\}$ and $\mathcal{W} \in \{\mathcal{V} \wedge \mathcal{U}_k : k = 1, 2, \ldots; \mathcal{V} \in \mu\}$.

Case (i): Suppose $\mathcal{W}_1 \in \mu$ and $\mathcal{U}_k \in \{\mathcal{U}_1, \mathcal{U}_2, \ldots\}$. As $\mu$ is a base for the fine uniformity on $X$, there exists a $\mathcal{W}_2 \in \mu$ such that $\mathcal{W}_2 \overset{\star}{<} \mathcal{W}_1$ and also $\mathcal{U}_t \overset{\star}{<} \mathcal{U}_k$ [where $t$ is a positive integer such that $t > k$]. Hence by Lemma 3.1, $\mathcal{W}_2 \wedge \mathcal{U}_t \overset{\star}{<} \mathcal{W}_1 \wedge \mathcal{U}_k$ and $\mathcal{W}_2 \wedge \mathcal{U}_t \overset{\star}{<} \mathcal{W}_1$; $\mathcal{W}_2 \wedge \mathcal{U}_t \overset{\star}{<} \mathcal{U}_k$, where $\mathcal{W}_2 \wedge \mathcal{U}_t \in \mu_1'$.

Case (ii): Let $\mathcal{U}' \in \mu$, $\mathcal{U} \wedge \mathcal{U}_k \in \{\mathcal{V} \wedge \mathcal{U}_l : l = 1, 2, \ldots; \mathcal{V} \in \mu\}$.

Since $\mathcal{U}', \mathcal{U} \in \mu$ and $\mu$ is being a base for the fine uniformity on $X$, there exists a $\mathcal{U}'' \in \mu$ such that $\mathcal{U}'' \overset{\star}{<} \mathcal{U}'$ and $\mathcal{U}'' \overset{\star}{<} \mathcal{U} \ldots$ (I).

Now as $\mathcal{U}'' \overset{\star}{<} \mathcal{U}$ and $\mathcal{U}_t \overset{\star}{<} \mathcal{U}_k$ (where $t$ is a positive integer greater than $k$), by Lemma 3.1, $\mathcal{U}'' \wedge \mathcal{U}_t \overset{\star}{<} \mathcal{U} \wedge \mathcal{U}_k$.

Again for $U \cap V \in \mathcal{U}'' \wedge \mathcal{U}_t$, we have $\mathrm{St}(U \cap V; \mathcal{U}'' \wedge \mathcal{U}_t) \subset \mathrm{St}(U; \mathcal{U}'') \subset U'$ (for some $U' \in \mathcal{U}'$ as $\mathcal{U}'' \overset{\star}{<} \mathcal{U}'$). So $\mathcal{U}'' \wedge \mathcal{U}_t \overset{\star}{<} \mathcal{U}'$.

Case (iii): Let $\mathcal{U}' \wedge \mathcal{U}_k$, $\mathcal{U}'' \wedge \mathcal{U}_l \in \{\mathcal{V} \wedge \mathcal{U}_t : t = 1, 2, \ldots; \mathcal{V} \in \mu\}$. Now, for $\mathcal{U}', \mathcal{U}'' \in \mu$, there exists a $\mathcal{U}''' \in \mu$, such that $\mathcal{U}''' \overset{\star}{<} \mathcal{U}', \mathcal{U}''' \overset{\star}{<} \mathcal{U}'' \ldots$ (a).

Also Lemma 2.2 ensures that, for a positive integer $t$ greater than both $k$ and $l$, $\mathcal{U}_t \overset{\star}{<} \mathcal{U}_k$ and $\mathcal{U}_t \overset{\star}{<} \mathcal{U}_l \ldots$ (b).

From (a), (b) and Lemma 3.1, we get $\mathcal{U}''' \wedge \mathcal{U}_t \overset{\star}{<} \mathcal{U}' \wedge \mathcal{U}_k$ as well as $\mathcal{U}''' \wedge \mathcal{U}_t \overset{\star}{<} \mathcal{U}'' \wedge \mathcal{U}_l$. Here we note that $\mathcal{U}''' \wedge \mathcal{U}_t \in \{\mathcal{V} \wedge \mathcal{U}_t : t = 1, 2, \ldots; \mathcal{V} \in \mu\} \subset \mu_1'$.

The proof of case (iv) can be done similarly.

So we have for any $\mathcal{W}_1, \mathcal{W}_2 \in \mu_1'$, there exists a $\mathcal{W}_3 \in \mu_1'$ such that $\mathcal{W}_3 \overset{\star}{<} \mathcal{W}_1$, $\mathcal{W}_3 \overset{\star}{<} \mathcal{W}_2$. Hence $\mu_1'$ is a base for some covering uniformity on $X$. Now $\mu_1'$ generates the topology $\tau_{\mu_1'}$ on $X$. Since $\mu \subset \mu_1'$, the topology generated by $\mu$ i.e., the topology $\tau$ is weaker than $\tau_{\mu_1'}$. Now $\{\mathrm{St}(x; \mathcal{U}) : \mathcal{U} \in \mu_1'\}$ forms a local base at $x \in X$ in $(X, \tau_{\mu_1'})$ and also $\mathrm{St}(x; \mathcal{U})$ is infinite for each $x \in X$ and for each $\mathcal{U} \in \mu_1'$. So $(X, \tau_{\mu_1'})$ contains no isolated points. Also $\tau_{\mu_1'}$ is $T_2$ and uniformizable. Hence the Lemma follows. □

THEOREM 3.1. *For a Tychonoff space $(X, \tau)$, the following statements are equivalent:*

(i) *$(X, \tau)$ is maximal Tychonoff.*
(ii) *Every normal isolator cover $\mathcal{U}$ of $(X, \tau)$ has an open refinement $\mathcal{V}$, which is also a cover of $(X, \tau)$.*
(iii) *Every normal isolator cover $\mathcal{U}$ of $(X, \tau)$ has an open star refinement $\mathcal{V}$, which is also a cover of $(X, \tau)$.*

PROOF. We shall proceed to prove in the following manner: (i)⇔(ii), (ii)⇔(iii).

(i)⇒(ii): Let $(X, \tau)$ be a maximal Tychonoff space and also let $\mathcal{U}$ be a normal isolator cover of $(X, \tau)$ and $\mathcal{U} = \mathcal{U}_1, \mathcal{U}_2, \ldots$ be the corresponding normal sequence of isolator covers of $(X, \tau)$.

Now we shall consider the collection $\mu_1'$ of covers consisting of all normally open covers of $(X, \tau)$, the covers $\{\mathcal{U}_1, \mathcal{U}_2, \ldots\}$ and the covers $\{\mathcal{V} \wedge \mathcal{U}_k : k = 1, 2, \ldots; \mathcal{V} \in \mu\}$, where $\mu$ is the collection of all normally open covers of $(X, \tau)$.

So by Lemma 3.2, $\mu_1'$ is a base for some uniformity on $X$, which generates a stronger Tychonoff(or uniformizable) topology $\tau_{\mu_1'}$ on $X$ (i.e. $\tau_{\mu_1'} \supset \tau$) such that $(X, \tau_{\mu_1'})$ is $T_2$ and contains no isolated points. Then $\tau = \tau_{\mu_1'}$, by the maximality of $\tau$ as a Tychonoff (or uniformizable) topology. So we can write $\tau_{\mu_1'} \subset \tau$.

Now $\{\mathrm{St}(x;\mathcal{W}) : \mathcal{W} \in \mu_1'\}$ is a local base at $x \in X$ in $(X, \tau_{\mu_1'})$ and $\{\mathrm{St}(x;\mathcal{W}') : \mathcal{W}' \in \mu\}$ is a local base at $x \in X$ in $(X, \tau)$. By the *Hausdorff criterion*, for $\mathcal{U}_2 \in \mu_1'$, there exists a $\mathcal{V} \in \mu$ such that $\mathrm{St}(x;\mathcal{V}) \subset \mathrm{St}(x;\mathcal{U}_2)$. Take a $V_x \in \mathcal{V}$ containing $x$. Then $V_x \in \mathrm{St}(x;\mathcal{V}) \subset \mathrm{St}(x;\mathcal{U}_2) \subset \mathrm{St}(U_2;\mathcal{U}_2)$ [for some $U_2 \in \mathcal{U}_2$ containing $x$]. As $\mathcal{U}_2 \overset{\star}{<} \mathcal{U}_1 = \mathcal{U}$, there exists some $U_x \in \mathcal{U}_1 = \mathcal{U}$ such that $\mathrm{St}(U_2;\mathcal{U}_2) \subset U_x$. So $V_x \subset U_x$. Therefore the cover $\mathcal{W} = \{V_x : x \in X\}$ is the required open cover of $(X, \tau)$, which is a refinement of $\mathcal{U}$.

(ii) $\Rightarrow$ (i): Let the condition holds and if possible let $\tau_1$ be a Tychonoff (or uniformizable) topology that contains no isolated points satisfying $\tau \subset \tau_1$. It is sufficient to prove that $\tau_1 \subset \tau$.

Let $U \in \tau_1$. Consider the collection $\mu_1$ of all normally open covers of $(X, \tau_1)$. Then for $x \in U \in \tau_1$, there exists a $\mathcal{U}' \in \mu_1$ such that $x \in \mathrm{St}(x;\mathcal{U}') \subset U$. But as $\mathcal{U}'$ is a normal isolator cover of $(X, \tau_1)$, it is therefore so in $(X, \tau)$, as well. Hence the assumption shows the existence of an open cover $\mathcal{W}$ of $(X, \tau)$ satisfying $\mathcal{W} < \mathcal{U}'$. Obviously $x \in \mathrm{St}(x;\mathcal{W}) \subset \mathrm{St}(x;\mathcal{U}') \subset U$ and hence $U \in \tau$ as $U$ is a neighborhood of $x$ in $(X, \tau)$. So $\tau_1 \subset \tau$ and hence (i) is followed.

(ii)$\Rightarrow$(iii): Let (ii) holds i.e. every normal isolator cover of $(X, \tau)$ has a $\tau$-open refinement. Let $\mathcal{U}$ be a normal isolator cover of $(X, \tau)$ with $\ldots, \mathcal{U}_3, \mathcal{U}_2, \mathcal{U}_1 = \mathcal{U}$ be the corresponding normal sequence of isolator covers. Since $\mathcal{U}_2$ is also a normal isolator cover, by (ii), it has a $\tau$-open refinement $\mathcal{V}$. Now for $V \in \mathcal{V}$, there exists a $U_2 \in \mathcal{U}_2$, such that $V \subset U_2$. So $\mathrm{St}(V;\mathcal{V}) \subset \mathrm{St}(U_2;\mathcal{U}_2)$, as $\mathcal{V} < \mathcal{U}_2$ and every member of $\mathcal{V}$ which intersects $V$ must be contained in a member of $\mathcal{U}_2$ intersecting $U_2$. Since $\mathcal{U}_2 \overset{\star}{<} \mathcal{U}$, $\mathrm{St}(V;\mathcal{V}) \subset \mathrm{St}(U_2;\mathcal{U}_2) \subset U$ for some $U \in \mathcal{U}$. That is for $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ such that $\mathrm{St}(V;\mathcal{V}) \subset U$ and hence $\mathcal{V}$ is a star refinement of $\mathcal{U}$.

(iii)$\Rightarrow$(ii): As every star refinement of a cover $\mathcal{W}$ is obviously a refinement of $\mathcal{W}$, (iii)$\Rightarrow$(ii) follows obviously. □

Corollary 3.1. *In a maximal Tychonoff space $(X, \tau)$, a subset $G$ of $X$ is open if and only if for every $x \in G$ there exists a normal isolator cover $\mathcal{U}$ of $(X, \tau)$ such that $x \in \mathrm{St}(x;\mathcal{U}) \subset G$*

Proof. Let $G$ be an open subset of $(X, \tau)$ and $x \in G$. Now we know that the collection of all normally open covers $\mu$ (say) of $(X, \tau)$ forms a base for a uniformity on $X$, which generates $\tau$ on $X$ where $\beta_x = \{\mathrm{St}(x;\mathcal{V}) : \mathcal{V} \in \mu\}$ forms a local base at $x \in X$ in $(X, \tau)$. Hence the necessary part is followed as every normally open cover is also a normal isolator cover.

Conversely, let $G$ be a subset of $X$ such that for every $x \in G$ there exists a normal isolator cover $\mathcal{U}$ of $(X, \tau)$ with $x \in \mathrm{St}(x;\mathcal{U}) \subset G$. Since $(X, \tau)$ is maximal Tychonoff, then by Theorem 3.1 the normal isolator cover $\mathcal{U}$ has an $\tau$-open refinement $\mathcal{V}$ (say). Now if $U$ is a member of $\mathcal{V}$ containing $x$, then obviously

$x \in U \subset \mathrm{St}(x;\mathcal{U})$ and so $x \in U \subset G$. Hence $G$ is a $\tau$-neighborhood of $x$ and this is true for every $x \in G$. Thus $G$ is an open set in $(X, \tau)$. □

Basu and Mandal [**2**] recently characterized various disconnectedness in terms of star refinement of covers. Here using the tool of normal isolator covers, we have shown that a maximal Tychonoff (or maximal uniformizable) space is extremally disconnected.

THEOREM 3.2. *Let $(X, \tau)$ be a maximal Tychonoff space. Then for every nonempty proper subset $G$ of $X$ containing no isolated points in $G$ (with the induced subspace topology),* $\mathrm{cl}(G)$ *is open.*

PROOF. Let $G$ be a nonempty proper subset of $X$ containing no isolated points in $(X, \tau)$. If $\mathrm{cl}(G) = X$, then $\mathrm{cl}(G)$ is obviously open; so let $\mathrm{cl}(G) \subsetneq X$. Then $X - \mathrm{cl}(G)$ is an open set and obviously contains no isolated points in $(X, \tau)$. Also $\mathrm{cl}(G)$ obviously contains no isolated points in $(X, \tau)$.

Now we consider the cover $\mathcal{U} = \{\mathrm{cl}(G), X - \mathrm{cl}(G)\}$ of $X$. Here we see that for any normally open cover $\mathcal{V}$ of $(X, \tau)$ if $x$ belongs to $X - \mathrm{cl}(G)$, then $\mathrm{St}(x;\mathcal{U} \wedge \mathcal{V})$ being a union of open sets is itself an open set and hence is infinite. Let $x \in \mathrm{cl}(G)$. Now if $x \in G$, then $x$ is not an isolated point of $G$. So when an open set containing $x$ intersects $G$, it intersects at infinite number of points. Again if $x$ is a limit point of $G$, then as $(X, \tau)$ is $T_1$, every open set containing $x$ intersects $G$ at infinite number of points. So in both cases $\mathrm{St}(x;\mathcal{U} \wedge \mathcal{V})$ is infinite. Hence $\mathcal{U}$ is an isolator cover of $(X, \tau)$. Also $\ldots, \mathcal{U}, \mathcal{U}, \mathcal{U}$ is a normal sequence and so $\mathcal{U}$ is a normal isolator cover of $(X, \tau)$. Thus by Theorem 3.1, $\mathcal{U}$ has a $\tau$-open refinement. So $\mathrm{cl}(G)$ is obviously $\tau$-open. Hence the result follows. □

L. Feng and S. Garcia-Ferreira, in their paper [**12**], have established that every maximal Tychonoff space is extremally disconnected [**12**, Lemma 1.6]. Here an alternative proof of this result is established.

COROLLARY 3.2. *A maximal Tychonoff space is extremally disconnected.*

PROOF. As in a maximal Tychonoff space, every nonempty proper open set contains no isolated points in that space, so by Theorem 3.2, the closure of every open set is open and hence a maximal Tychonoff space is extremally disconnected. □

COROLLARY 3.3. *A maximal Tychonoff space is zero dimensional space.*

PROOF. The proof follows immediately, since an extremally disconnected regular space is zero-dimensional. □

THEOREM 3.3. *In a maximal Tychonoff space $(X, \tau)$, for any proper nonempty subset $G$ containing no isolated points in $(X, \tau)$,* $\mathrm{cl}(G)$ *is maximal Tychonoff.*

PROOF. Let $(X, \tau)$ be maximal Tychonoff and $G$ be a proper nonempty subset of $X$ such that $G$ contains no isolated points and also let $\mathrm{cl}(G) \neq X$. Then obviously $(\mathrm{cl}(G), \tau_{\mathrm{cl}(G)})$ is $T_2$, uniformizable and also contains no isolated points.

Now if $\mathcal{U}$ is any normal isolator cover of $(\mathrm{cl}(G), \tau_{\mathrm{cl}(G)})$, then $\mathcal{U} \cup \{X - \mathrm{cl}(G)\}$ is also a normal isolator cover of $(X, \tau)$.

As $(X, \tau)$ is maximal Tychonoff, by Theorem 3.1, $\mathcal{U} \cup \{X - \mathrm{cl}(G)\}$ has a $\tau$-open refinement $\mathcal{V}'$, which is also a cover of $X$. Now $\mathcal{V} = \mathcal{V}'_{\mathrm{cl}(G)} = \{V \cap \mathrm{cl}(G) : V \in \mathcal{V}'\}$ is an open cover of $\mathrm{cl}(G)$ and it refines $\mathcal{U}$. So $\mathcal{V}$ is a $\tau_{\mathrm{cl}(G)}$-open refinement of $\mathcal{U}$ i.e., every normal isolator cover of $(\mathrm{cl}(G), \tau_{\mathrm{cl}(G)})$ has a $\tau_{\mathrm{cl}(G)}$-open refinement. Hence by Theorem 3.1, $(\mathrm{cl}(G), \tau_{\mathrm{cl}(G)})$ is a maximal Tychonoff space. □

COROLLARY 3.4. *In a maximal Tychonoff space $(X, \tau)$, for every proper nonempty open set $A$, $\mathrm{cl}(A)$ is maximal Tychonoff.*

PROOF. As every proper nonempty open set contains no isolated points, the proof follows from Theorem 3.3. □

COROLLARY 3.5. *A topological space $(X, \tau)$ is maximal Tychonoff if and only if every nonempty open subset is maximal Tychonoff.*

PROOF. The proof follows from Corollary 3.4. □

**Acknowledgement.** The authors would like to thank the learned referee for her valuable comments and suggestions.

## References

1. C. E. Aull, *A certain class of topological spaces*, Prace Mat. **11** (1967), 49–53.
2. C. K. Basu, S. S. Mandal, *A note on discoonectedness*, Chaos Solitons Fractals **42** (2009), 3242–3246.
3. ———, *Uniformizable Subtopology and a sort of converse of A. H. Stone's Theorem*, Real Anal. Exch. (summer symposium) 2010, 67–75.
4. ———, *Minimal uniformizablity revisited in terms of normal sequence of covers*, Lobachevskii J. Math. **36**(2) (2015), 139–143.
5. D. E. Cameron, *Maximal pseudocompactness*, Proc. Gen. Topol. Conf., Emory University, 1970, 26–31.
6. ———, *Maximal and minimal topologies*, Trans. Am. Math. Soc. **160** (1971), 229–248.
7. ———, *Maximal QHC-spaces*, Rocky Mt. J. Math. **7**(2) (1977), 313–322.
8. ———, *A class maximal topologies*, Pac. J. Math. **70**(1) (1977), 101–104.
9. ———, *A note on maximal pseudocompactness*, Bull. Malays. Math. Soc., II. Ser. **2** (1979), 45–46.
10. B. Clark, V. Schneider, *A characterization of maximal connected spaces and maximal arcwise connected spaces*, Proc. Am. Math. Soc. **104**(4) (1998), 1256–1260.
11. E. van Douwen, *Applications of maximal topologies*, Topology Appl. **51** (1993), 125–139.
12. L. Feng, S. Garcia-Ferreira, *Some Examples of MI-Spaces and SI-Spaces*, Topol. Proc. **24** (1999), 153–164.
13. J. A. Guthrie, H. E. Stone, M. L. Wage, *Maximal connected hausdorff topologies*, Topol. Proc. **2** (1977), 349–353.
14. ———, *Maximal connected expansions of reals*, Proc. Am. Math. Soc. **69** (1978), 159–165.
15. E. Hewitt, *A problem of set theoretic topology*, Duke Math. J. **10** (1943), 309–333.
16. G. J. Kennedy, S. D. McCartan, *Singular sets and maximal topologies*, Proc. Am. Math. Soc. **127** (1999), 3375–3382.
17. N. Levine, *When are compact and closed equivalent?*, Am. Math. Mon. **72** (1965), 41–44.
18. J. Mioduszewski, L. Rudolf, *H-closed and extremally disconnected Hausdorff spaces*, Diss. Math. **66** (1969), p. 55.
19. I. Protasov, *Maximal topologies on groups*, Sib. Math. J. **39** (1998), 1184–1194.

20. A. B. Raha, *Maximal topologies*, J. Aust. Math. Soc. **15** (1973), 279–290.
21. A. Ramanathan, *A characterization of maximal Hausdorff spaces*, J. Indian Math. Soc. **11** (1947), 73–80.
22. ———, *Maximal Hausdorff spaces*, Proc. Indian Acad. Sci., Sect. A **26** (1947), 31–42.
23. D. B. Shakhmatov, M. G. Tkacenko, V. V. Tkachuk, S. Watson, R. G. Wilson, *Neither first countable nor Cech-complete spaces are maximal Tychonoff connected*, Proc. Am. Math. Soc. **126** (1998), 279–287.
24. P. Simon, *An example of a maximal connected Hausdorff space*, Fundam. Math. **100** (1978), 157–163.
25. N. Smythe, C. A. Wilkins, *Minimal Hausdorff and maximal compact spaces*, J. Aust. Math. Soc. **3** (1963), 167–171.
26. J. P. Thomas, *Maximal connected topologies*, J. Aust. Math. Soc. **8** (1968), 700–705.
27. W. J. Thron, *Topological Stuctures*, Holt, Rinehart and Winston, New York, 1966.
28. Hing Tong, *Note on Minimal bicompact spaces (preliminary report)*, Bull. Am. Math. Soc. **54** (1948), 478–479.
29. R. Vaidyanathaswamy, *Set Topology*, Chelsea, New York, 1947.
30. S. Willard, *General Topology*, Addision-Wesley, Reading, Mass, 1970.
31. Yevhen Zelenyuk, *Almost maximal spaces*, Topology Appl. **154** (2007), 339–357.

Department of Mathematics
West Bengal State University
Berunanpukuria, Kolkata, West Bengal
India
ckbasu1962@yahoo.com

(Received 02 05 2013)
(Revised 21 05 2015 and 25 09 2015)

Department of Mathematics
Krishnagar Women's College
Nadia, West Bengal
India
snehadrisekharmandal@gmail.com