PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 99(113) (2016), 295–300 DOI: 10.2298/PIM150203021Z

# PRODUCT OF DIFFERENTIATION AND COMPOSITION OPERATORS ON BLOCH TYPE SPACES

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ABSTRACT. We obtain some simple criteria for the boundedness and compactness of the product of differentiation and composition operator $C_\varphi D^m$ on Bloch type spaces.

## 1. Introduction

Denote by $H(\mathbb{D})$ the space of all analytic functions on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ in the complex plane. A function $f \in H(\mathbb{D})$ is said to belong to Bloch type spaces $\mathcal{B}^\alpha$ (or called $\alpha$-Bloch spaces) if $\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)^\alpha < \infty$, $0 < \alpha < \infty$.

A function $f \in H(\mathbb{D})$ is said to belong to the little Bloch type space $\mathcal{B}_0^\alpha$ (or the little $\alpha$-Bloch space) if $\lim_{|z|\to 1} |f'(z)|(1 - |z|^2)^\alpha = 0, \quad 0 < \alpha < \infty$. The classical Bloch space $\mathcal{B}$ is just $\mathcal{B}^1$. It is clear that $\mathcal{B}^\alpha$ is a Banach space with the norm $\|f\| = |f(0)| + \|f\|_{\mathcal{B}^\alpha}$. See [**24**] for the theory of Bloch type spaces.

Let $\varphi$ be an analytic self-map of $\mathbb{D}$. The composition operator $C_\varphi$ is defined by $C_\varphi(f) = f \circ \varphi$, $f \in H(\mathbb{D})$. The differentiation operator $D$ is defined by $Df = f'$, $f \in H(\mathbb{D})$. For a nonnegative integer $m \in \mathbb{N}$, we define $D^m f = f^{(m)}$, $f \in H(\mathbb{D})$. It is natural to define the product of differentiation and composition operators $C_\varphi D^m$ by $C_\varphi D^m f = f^{(m)} \circ \varphi$, $f \in H(\mathbb{D})$. A basic problem concerning concrete operators on various Banach spaces is to relate their operator theoretic properties to the function theoretic properties of the participating symbols, which attracted a lots of attention recently (the reader can refer to [**1**]–[**27**]).

It is a well-known consequence of the Schwarz–Pick Lemma that the composition operator is bounded on the Bloch space. See [**10**, **11**, **18**–**20**, **23**] for the study of the compactness of composition operator on the Bloch space.

Product-type operators attracted considerable interest recently. The product of differentiation and composition operators has been studied, for example, in [**2**,

2010 *Mathematics Subject Classification:* Primary 47B38; Secondary 30H30.
*Key words and phrases:* Bloch type space, composition operator, differentiation operator.
Partially supported by NSF of China (No. 11471143), NSF of Anhui Province, China (No. 1608085MA01) and NSF of Guangdong Province, China (No. S2013010011978).
Communicated by Stevan Pilipović.

**4**, **7**–**9**, **13**–**17**, **19**, **21**]. For some other product-type operators acting on Bloch-type spaces, see, for example [**3**, **5**, **6**, **12**, **22**, **26**, **27**] and the related references therein.

In [**19**], Wu and Wulan obtained two nice characterizations for the compactness of the product of differentiation and composition operators acting on the Bloch space as follows.

THEOREM 1.1. *Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and $m \in \mathbb{N}$. Then the following statements are equivalent.*

(i) $C_\varphi D^m : \mathcal{B} \to \mathcal{B}$ *is compact.*
(ii) $\lim_{n\to\infty} \left\|C_\varphi D^m(z^n)\right\|_{\mathcal{B}} = 0$.
(iii) $\lim_{|a|\to 1} \left\|C_\varphi D^m\left(\frac{a-z}{1-\bar{a}z}\right)\right\|_{\mathcal{B}} = 0$.

The condition (ii) was extended to the Bloch type spaces by Liang and Zhou in [**9**]. Among other results, they proved the following one.

THEOREM 1.2. *Let $0 < \alpha, \beta < \infty$, $m$ a nonnegative integer, $\varphi$ a self-map of the unit disk $\mathbb{D}$. Suppose that $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. Then $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if $\lim_{n\to\infty} n^{\alpha-1}\left\|C_\varphi D^m(z^n)\right\|_{\mathcal{B}^\beta} = 0$.*

We will extend condition (iii) in Theorem 1.1 to the Bloch type spaces.

Let $X$ and $Y$ be two Banach spaces. Recall that an operator $T : X \to Y$ is said to be bounded if there exists a constant $C > 0$ such that $\|T(f)\|_Y \leqslant C\|f\|_X$. Moreover, $T : X \to Y$ is said to be compact if it takes bounded sets in $X$ to sets in $Y$ which have compact closure.

In addition, we say that $A \lesssim B$ if there exists a constant $C$ such that $A \leqslant CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

## 2. Main results

To prove our main results, we need the following two auxiliary lemmas.

LEMMA 2.1. [**24**] *For $f \in H(\mathbb{D}), m \in \mathbb{N}$ and $\alpha > 0$. Then $f \in \mathcal{B}^\alpha$ if and only if $\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+m-1}|f^{(m)}(z)| < \infty$. Moreover,*

$$\|f\| \approx \sum_{j=0}^{m-1} \left|f^{(j)}(0)\right| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+m-1}\left|f^{(m)}(z)\right|.$$

The following lemma can be proved in a standard way, see e.g. [**1**, Prop. 3.11].

LEMMA 2.2. *Let $m$ be a positive integer and $0 < \alpha, \beta < \infty$. Let $\varphi$ be a holomorphic self-map of $\mathbb{D}$. Then $C_\varphi D^m$ is compact if and only if $C_\varphi D^m : B^\alpha \to \mathcal{B}^\beta$ is bounded and for any bounded sequence $\{f_n\}$ in $\mathcal{B}^\alpha$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, then $\|C_\varphi D^m f\|_{\mathcal{B}^\beta} \to 0$ as $n \to \infty$.*

Since the boundedness of $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ implies that $\varphi \in \mathcal{B}^\beta$, we always assume that $\varphi \in \mathcal{B}^\beta$. We are now ready for the main results in this section.

THEOREM 2.1. *Let $0 < \alpha, \beta < \infty, m \in \mathbb{N}$ and $\varphi$ be a self-map of $\mathbb{D}$ such that $\varphi \in \mathcal{B}^\beta$. Then the following statements are equivalent.*

(i) $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is bounded.*

(ii) $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{B}^\beta$ *is bounded.*
(iii) $\sup_{n\in\mathbb{N}} n^{\alpha-1}\|C_\varphi D^m(z^n)\|_{\mathcal{B}^\beta} < \infty$.
(iv) $\sup_{a\in\mathbb{D}} \big\|C_\varphi D^m\big(\frac{1-|a|^2}{(1-\bar{a}z)^\alpha}\big)\big\|_{\mathcal{B}^\beta} < \infty$.
(v) $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}|\varphi'(z)| < \infty$.

Proof. (i)⇒(ii). This implication is obvious because of $\mathcal{B}_0^\alpha \subset \mathcal{B}^\alpha$.

(ii)⇒(iii). Assume that $\|C_\varphi D^m\|_{\mathcal{B}_0^\alpha\to\mathcal{B}^\beta} < \infty$. Note that the sequence $\{n^{\alpha-1}z^n\}$ is bounded in the little $\alpha$-Bloch space $\mathcal{B}_0^\alpha$ (see, e.g. [**23**]). There exists a constant $C$ such that $n^{\alpha-1}\big\|C_\varphi D^m(z^n)\big\|_{\mathcal{B}^\beta} \leqslant C\big\|C_\varphi D^m\big\|_{\mathcal{B}_0^\alpha\to\mathcal{B}^\beta}\big\|n^{\alpha-1}z^n\big\|_{\mathcal{B}^\alpha} < \infty$, for any $n \in \mathbb{N}$. This implies (iii).

(iii)⇒(iv). Assume that (iii) holds. The MacLaurin expansion of $\frac{1-|a|^2}{(1-\bar{a}z)^\alpha}$ is given by

$$\frac{1-|a|^2}{(1-\bar{a}z)^\alpha} = \big(1-|a|^2\big)\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}\bar{a}^n z^n.$$

By Stirling's formula, we have $\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} \approx n^{\alpha-1}$, as $n \to \infty$. This gives

$$\begin{aligned}\left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\bar{a}z)^\alpha}\right)\right\|_{\mathcal{B}^\beta} &\lesssim (1-|a|^2)\sum_{n=m+1}^{\infty} n^{\alpha-1}|a|^n\|C_\varphi D^m(z^n)\|_{\mathcal{B}^\beta}\\ &\leqslant (1-|a|^2)\Big(\sup_{n\in\mathbb{N}} n^{\alpha-1}\|C_\varphi^n D^m(z^n)\|_{\mathcal{B}^\beta}\Big)\sum_{n=m+1}^{\infty}|a|^n\\ &\leqslant 2\sup_{n\in\mathbb{N}} n^{\alpha-1}\|C_\varphi^n D^m(z^n)\|_{\mathcal{B}^\beta} < \infty,\end{aligned}$$

from which the implication follows.

(iv)⇒(v). If (iv) holds, then

$$\sup_{z\in\mathbb{D}} \alpha(\alpha+1)\cdots(\alpha+m)|a|^{m+1}\frac{1-|a|^2}{|1-\bar{a}\varphi(z)|^{\alpha+m+1}}|\varphi'(z)|(1-|z|^2)^\beta < \infty, \tag{2.1}$$

for any $a \in \mathbb{D}$. If $\varphi(z) = 0$, then (v) automatically holds since $\varphi \in \mathcal{B}^\beta$. When $|\varphi(z)| > 0$, taking $a = \varphi(z)$ in (2.1) we see that (v) holds.

(v)⇒(i). Assume that (v) holds. For any given $f \in \mathcal{B}^\alpha$, by Lemma 2.1 we have

$$\begin{aligned}\|C_\varphi D^m f\|_{\mathcal{B}^\beta} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f^{(m+1)}(\varphi(z))\varphi'(z)|\\ &= \sup_{z\in\mathbb{D}}|f^{(m+1)}(\varphi(z))|(1-|\varphi(z)|^2)^{\alpha+m}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}|\varphi'(z)|\\ &\lesssim \|f\|_{\mathcal{B}^\alpha}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}|\varphi'(z)|.\end{aligned}$$

This implies that $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. □

Theorem 2.2. *Let $0 < \alpha, \beta < \infty, m \in \mathbb{N}$ and $\varphi$ be a self-map of $\mathbb{D}$ such that $\varphi \in \mathcal{B}^\beta$. Then the following statements are equivalent.*

(i) $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is compact.*
(ii) $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{B}^\beta$ *is compact.*
(iii) $\lim_{n\to\infty} n^{\alpha-1}\|C_\varphi D^m(z^n)\|_{\mathcal{B}^\beta} = 0.$
(iv) $\lim_{|a|\to 1^-} \left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\bar{a}z)^\alpha}\right)\right\|_{\mathcal{B}^\beta} = 0.$
(v) $\lim_{|\varphi(z)|\to 1^-} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}|\varphi'(z)| = 0.$

Proof. (i)⇒(ii). This implication is obvious because of $\mathcal{B}_0^\alpha \subset \mathcal{B}^\alpha$.

(ii) ⇒ (iii). Assume that (ii) holds. Note that the sequence $\{n^{\alpha-1}z^n\}$ is bounded in the little $\alpha$-Bloch space $\mathcal{B}_0^\alpha$ and converges to 0 uniformly on a compact subset of $\mathbb{D}$. Lemma 2.2 implies that (iii) holds.

(iii) ⇒ (iv). We now assume that condition (iii) holds. By the argument as in the proof of Theorem 2.1, we have

$$\left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\bar{a}z)^\alpha}\right)\right\|_{\mathcal{B}^\beta} \lesssim (1-|a|^2)\sum_{n=m+1}^{\infty} n^{\alpha-1}|a|^n\|C_\varphi D^m(z^n)\|_{\mathcal{B}^\beta}.$$

If (iii) holds, then for any given $\epsilon > 0$ there exists a positive integer $N(N > m+1)$ such that $n^{\alpha-1}\|C_\varphi D^m(z^n)\|_{\mathcal{B}^\beta} < \epsilon$ for all $n > N$. Combining this with (iii) we obtain

$$\left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\bar{a}z)^\alpha}\right)\right\|_{\mathcal{B}^\beta} \lesssim (1-|a|^2)\sum_{n=m+1}^{N} n^{\alpha-1}|a|^n\|C_\varphi D^m(z^n)\|_{\mathcal{B}^\beta} + 2\epsilon. \tag{2.2}$$

Since $\{n^{\alpha-1}\|C_\varphi^n D^m(z^n)\|_{\mathcal{B}^\beta}\}$ is a bounded sequence, letting $|a| \to 1$ in (2.2) we obtain the following inequality.

$$\left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\bar{a}z)^\alpha}\right)\right\|_{\mathcal{B}^\beta} \lesssim 2\epsilon, \quad \text{as} \quad |a| \to 1.$$

Since $\epsilon$ is arbitrary, we conclude that (iv) holds.

(iv) ⇒ (v). If (iv) holds, then for any given $\epsilon > 0$, there is a $\delta \in (0,1)$ such that

$$\sup_{z\in\mathbb{D}} \alpha(\alpha+1)\cdots(\alpha+m)|a|^{m+1}\frac{1-|a|^2}{|1-\bar{a}\varphi(z)|^{\alpha+m+1}}|\varphi'(z)|(1-|z|^2)^\beta < \epsilon$$

whenever $\delta < |a| < 1$. In particular, if $w \in \mathbb{D}$ satisfies $|\varphi(w)| > \delta$, then we get

$$\sup_{z\in\mathbb{D}} \alpha(\alpha+1)\cdots(\alpha+m)|\varphi(w)|^{m+1}\frac{1-|\varphi(w)|^2}{|1-\overline{\varphi(w)}\varphi(z)|^{\alpha+m+1}}|\varphi'(z)|(1-|z|^2)^\beta < \epsilon,$$

which implies that

$$\sup_{z\in\mathbb{D}} \alpha(\alpha+1)\cdots(\alpha+m)\delta^{m+1}\frac{1-|\varphi(w)|^2}{|1-\overline{\varphi(w)}\varphi(z)|^{\alpha+m+1}}|\varphi'(z)|(1-|z|^2)^\beta < \epsilon.$$

Taking $z = w$ in the above inequality, we get

$$\alpha(\alpha+1)\cdots(\alpha+m)\delta^{m+1}\frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha+m}}|\varphi'(w)| < \epsilon$$

when $|\varphi(w)| > \delta$. This gives that (v) holds.

(v) ⇒ (i). Assume that (v) holds. For any given $\epsilon > 0$ there exists a $\delta$ such that

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}|\varphi'(z)| < \epsilon, \tag{2.3}$$

whenever $\delta < |\varphi(z)| < 1$. This together with the fact that $\varphi \in \mathcal{B}^\beta$ imply (iii). Hence $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded by Theorem 2.1.

Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{B}^\alpha$ that converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then

$$\|C_\varphi D^m f_n\|_{\mathcal{B}^\beta} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |f_n^{(m+1)}(\varphi(z))\varphi'(z)| \leqslant I_1(n) + I_2(n),$$

where

$$I_1(n) = \sup_{|\varphi(z)|\leqslant\delta}(1-|z|^2)^\beta |f_n^{(m+1)}(\varphi(z))\varphi'(z)|,$$
$$I_2(n) = \sup_{\delta<|\varphi(z)|<1}(1-|z|^2)^\beta |f_n^{(m+1)}(\varphi(z))\varphi'(z)|.$$

It follows easily from Cauchy's formula that $f_n^{(m+1)}$ converges to 0 uniformly on a compact subset of $\mathbb{D}$. Since $\varphi \in \mathcal{B}^\beta$, we have $I_1 \to 0$ as $n \to \infty$.

To estimate $I_2(n)$, we note that $\sup_{z\in\mathbb{D}} |f_n^{(m+1)}(\varphi(z))|(1-|\varphi(z)|^2)^{\alpha+m} \lesssim \|f\|_{\mathcal{B}^\alpha}$ by Lemma 2.1. It follows from (2.3) that

$$I_2(n) \lesssim \|f_n\|_{\mathcal{B}^\alpha} \sup_{\delta<|\varphi(z)|<1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}|\varphi'(z)| \lesssim \epsilon\|f_n\|_{\mathcal{B}^\alpha}.$$

Since $\epsilon$ is arbitrary, we obtain $\lim_{n\to\infty} \|C_\varphi D^m f_n\|_{\mathcal{B}^\beta} = 0$. This shows that $C_\varphi D^m$ from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ is compact by Lemma 2.2. □

Remark 2.1. When we take some special value for $m$, $\alpha$ and $\beta$ in Theorems 2.1 and 2.2, we can get many results, which have appeared in [**4**, **9**, **11**, **19**–**21**]. We omit the details.

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(Received 19 08 2014)
(Revised 27 12 2014)



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