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DEGENERATE MULTI-TERM FRACTIONAL DIFFERENTIAL EQUATIONS IN LOCALLY CONVEX SPACES

Marko Kostić

Abstract. We investigate, in the setting of sequentially complete locally convex spaces, degenerate multi-term fractional differential equations with Caputo derivatives. The obtained theoretical results are illustrated with some examples.

1. Introduction and preliminaries

Let $n \in \mathbb{N} \smallsetminus \{1\}$, $0 \leqslant \alpha_1 < \cdots < \alpha_n$, let $A_1, \dots, A_{n-1}$ be closed linear operators on a Hausdorff sequentially complete locally convex space E, and let $f\colon [0,\infty) \to E$ be a continuous function. The well-posedness of the following multi-term fractional differential equation has been analyzed in a series of recent papers (cf. [**20**, Section 2.10] for an extensive survey of results on abstract multi-term fractional differential equations with Caputo fractional derivatives)

$$\mathbf{D}_t^{\alpha_n} u(t) + \sum_{i=1}^{n-1} A_i \mathbf{D}_t^{\alpha_i} u(t) = f(t), \quad t \geqslant 0; \quad u^{(j)}(0) = u_j,\ j = 0, \dots, \lceil \alpha_n \rceil - 1.$$

Define $m_i := \lceil \alpha_i \rceil$, $i \in \mathbb{N}_{n-1}$, $T_{i,L}u(t) := A_i \mathbf{D}_t^{\alpha_i} u(t)$, if $t \geqslant 0$, $i \in \mathbb{N}_{n-1}$ and $\alpha_i > 0$, and $T_{i,R}u(t) := \mathbf{D}_t^{\alpha_i} A_i u(t)$, if $t \geqslant 0$ and $i \in \mathbb{N}_{n-1}$. Henceforth it will be assumed that, for every $t \geqslant 0$ and $i \in \mathbb{N}_{n-1}$, $T_i u(t)$ denotes either $T_{i,L}u(t)$ or $T_{i,R}u(t)$. In this paper, we will consider the following degenerate abstract multi-term problem:

$$\sum_{i=1}^{n-1} T_i u(t) = f(t), \quad t \geqslant 0. \tag{1.1}$$

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In such a way, we will continue our previous research studies [**22**, **27**], where we have looked at generation of degenerate fractional resolvent operator families and hypercyclic properties of degenerate multi-term fractional differential equations.

In order to subject initial conditions to equation (1.1), we shall follow the approach from [**27**]. First of all, assume that $\alpha > 0$ and $m = \lceil \alpha \rceil$. Let us recall that the Caputo fractional derivative $\mathbf{D}_t^{\alpha} u$ [**6**, **20**] is defined for those functions $u \in C^{m-1}([0,\infty) : E)$ for which $g_{m-\alpha} * (u - \sum_{j=0}^{m-1} u^{(j)}(0) g_{j+1}) \in C^m([0,\infty) : E)$, by

$$\mathbf{D}_t^{\alpha} u(\cdot) = \frac{d^m}{dt^m}\left[g_{m-\alpha} * \left(u - \sum_{j=0}^{m-1} u^{(j)}(0) g_{j+1}\right)\right].$$

If $\alpha \in \mathbb{N}$, then the Caputo fractional derivative $\mathbf{D}_t^{\alpha} u(\cdot)$ is defined iff $u \in C^m([0,\infty) : E)$; in this case, $\mathbf{D}_t^{\alpha} u(\cdot) = (d^{\alpha}/dt^{\alpha}) u(\cdot) \in C([0,\infty) : E)$. The following facts can be proved by using the equality [**6**, (1.21)], induction and closedness of A:

(a) Suppose that $l \in \mathbb{N}$ and $u,\ Au \in C^l([0,\infty) : E)$. Then $u^{(j)}(t) \in D(A)$, $t \geqslant 0$ and $Au^{(j)}(t) = (Au)^{(j)}(t)$, $t \geqslant 0$ $(0 \leqslant j \leqslant l)$.

(b) Suppose that the Caputo fractional derivatives $\mathbf{D}_t^{\alpha} u$ and $\mathbf{D}_t^{\alpha} Au$ are defined. Then $u^{(j)}(t) \in D(A)$, $t \geqslant 0$, $Au^{(j)}(t) = (Au)^{(j)}(t)$, $t \geqslant 0$ $(0 \leqslant j \leqslant m-1)$, $\mathbf{D}_t^{\alpha} u(t) \in D(A)$, $t \geqslant 0$, $A\mathbf{D}_t^{\alpha} u(t) = \mathbf{D}_t^{\alpha} Au(t)$, $t \geqslant 0$, and

$$J_t^{\alpha} A \mathbf{D}_t^{\alpha} u(t) = J_t^{\alpha} \mathbf{D}_t^{\alpha} Au(t) = Au(t) - \sum_{j=0}^{m-1} Au^{(j)}(0) g_{j+1}(t), \quad t \geqslant 0,$$

where $J_t^{\alpha} u(t) := (g_{\alpha} * u)(t)$, $t \geqslant 0$.

Set $P_{\lambda} := \sum_{i=1}^{n-1} \lambda^{\alpha_i - \alpha_{n-1}} A_i$, $\lambda \in \mathbb{C} \smallsetminus \{0\}$, $\mathcal{I} := \{i \in \mathbb{N}_{n-1} : \alpha_i > 0$ and $T_{i,L}u(t)$ appears on the left hand side of (1.1)$\}$, $Q := \max \mathcal{I}$, if $\mathcal{I} \neq \emptyset$ and $Q := m_Q := 0$, if $\mathcal{I} = \emptyset$. We will subject the following initial conditions to the equation (1.1), cf. (a)–(b):

(1.2) $$u^{(j)}(0) = u_j, \ \ 0 \leqslant j \leqslant m_Q - 1 \ \text{ and } \ (A_i u)^{(j)}(0) = u_{i,j} \ \text{ if } \ m_i - 1 \geqslant j \geqslant m_Q.$$

If $T_{n-1}u(t) = T_{n-1,L}u(t)$, then (1.2) reads $u^{(j)}(0) = u_j$, $0 \leqslant j \leqslant m_{n-1} - 1$. If this is not the case, then choice (1.2) may be nonoptimal, the index $i \in \mathbb{N}_{n-1}$ has to satisfy the inequality $m_i - 1 \geqslant m_Q$ in the second equality appearing in (1.2), and we cannot expect the existence of solutions of problem (1.1)–(1.2), in general (consider, for example, the case $n = 3$, $A_2 = A_1$ and $u_{2,0} \neq u_{1,0}$); furthermore, for any index $i \in \mathbb{N}_{n-1}$ satisfying the inequality $m_i - 1 \geqslant m_Q$ and for every nonnegative integer $k \in [m_Q, m_i - 1]$, we need to introduce exactly one initial value $u_{i,k}$.

The most important subcases of problem (1.1))–(1.2) are the following fractional Sobolev degenerate equations:

$$(\mathrm{DFP})_R : \begin{cases} \mathbf{D}_t^{\alpha} Bu(t) = Au(t) + f(t), & t \geqslant 0, \\ Bu(0) = Bx; \quad (Bu)^{(j)}(0) = 0, & 1 \leqslant j \leqslant \lceil \alpha \rceil - 1, \end{cases}$$

and

$$(\mathrm{DFP})_L : \begin{cases} B\mathbf{D}_t^{\alpha} u(t) = Au(t) + f(t), & t \geqslant 0, \\ u(0) = x; \quad u^{(j)}(0) = 0, & 1 \leqslant j \leqslant \lceil \alpha \rceil - 1, \end{cases}$$

where $\alpha > 0$. For further information concerning the wellposedness of Sobolev first order degenerate equations, the reader may consult the monographs by Favini, Yagi [**9**], Krein [**31**], Carroll, Showalter [**7**], Melnikova, Filinkov [**34**] and Sviridyuk, Fedorov [**45**], as well as the papers [**1**, **10**–**14**, **35**, **50**, **51**, **55**]. The well-posedness of various types of degenerate Sobolev equations of second order have been analyzed in [**2**, **4**, **7**, **9**, **15**, **22**, **36**, **44**, **52**, **56**]. The corresponding results on degenerate Sobolev equations with integer higher-order derivatives can be found in [**3**], [**9**, Section 5.7], [**45**–**49**, **56**].

For the purpose of study of abstract multi-term problem (1.1)–(1.2), we introduce the classes of exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation families (Subsection 2.1) and exponentially equicontinuous $(k;C)$-regularized resolvent (i,j)-propagation families (Subsection 2.2). We investigate subordination principles, regularity properties, existence and uniqueness of solutions of problem (1.1)–(1.2) and its integral analogons; besides this, we clarify some results on the C-wellposedness of the equation $(\mathrm{DFP})_L$ in Subsection 2.2 (cf. Abdelaziz, Neubrander [**1**] for the case $\alpha = 1$). In a follow-up research [**25**], we are going to analyze various types of degenerate Volterra integro-differential equations by using results from the theory of multivalued linear operators [**9**, **34**].

Before we explain the notation used throughout the paper, it should be noticed that we take under consideration some equations that are valuable only from the mathematical point of view and do not have any physical significance, for now at least.

Unless specifed otherwise, we shall always assume that E is a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short; the abbreviation $\circledast$ stands for the fundamental system of seminorms which defines the topology of E. If X is also an SCLCS over the field of complex numbers, then we denote by $L(E,X)$ the space of all continuous linear mappings from E into X; $L(E) \equiv L(E,E)$. By E^* we denote the dual space of E; if E is a Banach space, then we denote by $\|x\|$ the norm of an element $x \in E$. If A is a linear operator acting on E, then the domain and range of A will be denoted by $D(A)$ and $R(A)$, respectively. Since no confusion seems likely, we will identify A with its graph. Given $s \in \mathbb{R}$ in advance, set $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leqslant l\}$. Define $\Sigma_\alpha := \{z \in \mathbb{C} \smallsetminus \{0\} : |\arg(z)| < \alpha\}$ $(\alpha \in (0,\pi])$. By $AC_{\mathrm{loc}}([0,\infty))$ we denote the space consisting of all complex valued functions that are absolutely continuous on any finite interval $[0,T]$ $(T > 0)$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers; the convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s)\,ds$. Set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$, $0^\zeta := 0$ $(\zeta > 0,\ t > 0)$, $\mathbb{N}_l := \{1,\dots,l\}$, $\mathbb{N}_l^0 := \{0,1,\dots,l\}$ $(l \in \mathbb{N})$ and $g_0(t) :=$ the Dirac δ-distribution.

Fairly complete information about fractional calculus and fractional differential equations can be obtained by consulting [**6**, **8**, **16**–**22**, **27**, **28**, **32**, **39**, **40**]. In the sequel, we shall use the following fact about Caputo fractional derivatives: Assume that $\alpha > 0$, $m = \lceil \alpha \rceil$, $\beta \in (0,\alpha)$ and the Caputo fractional derivative $\mathbf{D}_t^\alpha u(\cdot)$ is defined. Then we know that the Caputo fractional derivative $\mathbf{D}_t^\beta u(\cdot)$ is also defined and the following equality holds:

$$\mathbf{D}_t^{\beta} u(t) = \big(g_{\alpha-\beta} * \mathbf{D}_t^{\alpha} u(\cdot)\big)(t) + \sum_{j=\lceil \beta \rceil}^{m-1} u^{(j)}(0) g_{j+1-\beta}(t), \quad t \geqslant 0. \tag{1.3}$$

The Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C}.$$

In this place, we assume that $1/\Gamma(\alpha n + \beta) = 0$ if $\alpha n + \beta \in -\mathbb{N}_0$. Set, for short, $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$. The asymptotic behaviour of the entire function $E_{\alpha,\beta}(z)$ is given in the following auxiliary lemma; cf. also [**32**, **42**] and the formulae [**6**, (1.27)–(1.28)] for the case $0 < \alpha < 2$, which is the most important in our analysis.

LEMMA 1.1. [**53**] *Let $0 < \sigma < \frac{1}{2}\pi$. Then, for every $z \in \mathbb{C} \smallsetminus \{0\}$ and $m \in \mathbb{N} \smallsetminus \{1\}$,*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} \sum_s Z_s^{1-\beta} e^{Z_s} - \sum_{j=1}^{m-1} \frac{z^{-j}}{\Gamma(\beta - \alpha j)} + O\big(|z|^{-m}\big), \quad |z| \to \infty,$$

where Z_s is defined by $Z_s := z^{1/\alpha} e^{2\pi i s/\alpha}$ and the first summation is taken over all those integers s satisfying $|\arg(z) + 2\pi s| < \alpha(\frac{\pi}{2} + \sigma)$.

Throughout the paper, we shall always assume that the function $k(t)$ is a scalar-valued continuous kernel on $[0, \infty)$. The following conditions on function $k(t)$ will be used occasionally:

(P1): $k(t)$ is Laplace transformable, i.e., it is locally integrable on $[0, \infty)$ and there exists $\beta \in \mathbb{R}$ such that $\tilde{k}(\lambda) := \mathcal{L}(k)(\lambda) := \lim_{b \to \infty} \int_0^b e^{-\lambda t} k(t)\,dt := \int_0^\infty e^{-\lambda t} k(t)\,dt$ exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \beta$. Put $abs(k) := \inf\{\operatorname{Re}\lambda : \tilde{k}(\lambda) \text{ exists}\}$, $\tilde{\delta}(\lambda) := 1$ and denote by $\mathcal{L}^{-1}$ the inverse Laplace transform.

(P2): $k(t)$ satisfies (P1) and $\tilde{k}(\lambda) \neq 0$, $\operatorname{Re}\lambda > \beta$ for some $\beta \geqslant \operatorname{abs}(k)$.

Let $\gamma \in (0,1)$ and $\omega \in \mathbb{R}$. Recall that the Wright function $\Phi_\gamma(\cdot)$ is defined by $\Phi_\gamma(t) := \mathcal{L}^{-1}(E_\gamma(-\lambda))(t)$, $t \geqslant 0$. Following [**54**, Definition 1.1.3], we say that a function $h : (\omega, \infty) \to E$ belongs to the class $LT - E$ iff there exists a function $f \in C([0,\infty) : E)$ such that for each $p \in \circledast$ there exists $M_p > 0$ satisfying $p(f(t)) \leqslant M_p e^{\omega t}$, $t \geqslant 0$ and $h(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\,dt$, $\lambda > \omega$; if this is the case, then we know that the function $\lambda \mapsto h(\lambda)$, $\lambda > \omega$ can be analytically extended to the right half plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\}$.

The reader may consult [**54**] and [**20**] for further information concerning the Laplace transform of functions with values in sequentially complete locally convex spaces (cf. [**5**] for the Banach space case). In the sequel, we shall use the following uniqueness type theorem for the Laplace transform.

LEMMA 1.2. *Let $\omega \geqslant 0$, and let f_1, $f_2 \in C([0,\infty) : E)$ satisfy that for each $p \in \circledast$ there exists $M_p' > 0$ such that $p(f_1(t)) + p(f_2(t)) \leqslant M_p' e^{\omega t}$, $t \geqslant 0$. Suppose* *that a continuous function* $g\colon [0,\infty) \to \mathbb{C}$ *satisfies* (P1) *and that* A *is a closed linear operator on* E *satisfying that for* $\lambda > a$,

$$\int_0^\infty e^{-\lambda t} f_1(t)dt \in D(A),$$
$$A\int_0^\infty e^{-\lambda t} f_1(t)dt = \int_0^\infty e^{-\lambda t} f_2(t)dt + \int_0^\infty e^{-\lambda t} g(t)x\,dt, \quad \lambda > a,$$

for some $x \in E$. *Then, for every* $t \geqslant 0$, *one has* $f_1(t) \in D(A)$ *and* $Af_1(t) = f_2(t) + g(t)x$.

In [**22**], we have recently considered the C-wellposedness of the following degenerate abstract Volterra equation:

$$Bu(t) = f(t) + \int_0^t a(t-s)Au(s)ds, \quad t \geqslant 0, \tag{1.4}$$

where $t \mapsto f(t)$, $t \geqslant 0$ is a continuous E-valued mapping, $a \in L^1_{\mathrm{loc}}([0,\infty))$ and $A,\ B$ are closed linear operators with domain and range contained in E (cf. also [**11**–**13**] and [**18**]). Following Xiao and Liang [**55**, **56**], we have introduced in [**22**] the class of exponentially equicontinuous (a,k)-regularized C-resolvent families for (1.4) as follows.

Definition 1.1. [**22**] Suppose that the functions $a(t)$ and $k(t)$ satisfy (P1), as well as that $R(t)\colon D(B) \to E$ is a linear mapping ($t \geqslant 0$). Let $C \in L(E)$ be injective. Then the operator family $(R(t))_{t\geqslant 0}$ is said to be an exponentially equicontinuous (a,k)-regularized C-resolvent family for (1.4) iff there exists $\omega \geqslant \max(0, \mathrm{abs}(a), \mathrm{abs}(k))$ such that the following holds:

(i) The mapping $t \mapsto R(t)x$, $t \geqslant 0$ is continuous for every fixed element $x \in D(B)$.
(ii) The family $\{e^{-\omega t}R(t) : t \geqslant 0\}$ is equicontinuous, i.e., for every $p \in \circledast$, there exist $c > 0$ and $q \in \circledast$ such that $p(e^{-\omega t}R(t)x) \leqslant cq(x)$, $x \in D(B)$, $t \geqslant 0$.
(iii) For every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, the operator $B - \tilde{a}(\lambda)A$ is injective, $C(R(B)) \subseteq R(B - \tilde{a}(\lambda)A)$ and
$$\tilde{k}(\lambda)\big(B - \tilde{a}(\lambda)A\big)^{-1}CBx = \int_0^\infty e^{-\lambda t}R(t)x\,dt, \quad x \in D(B).$$

If $k(t) = g_{r+1}(t)$ for some $r \geqslant 0$, then it is also said that $(R(t))_{t\geqslant 0}$ is an exponentially equicontinuous r-times integrated (a,C)-regularized resolvent family for (1.4); an exponentially equicontinuous 0-times integrated (a,C)-regularized resolvent family for (1.4) is also said to be an exponentially equicontinuous (a,C)-regularized resolvent family for (1.4).

For further information concerning the applications of (a,k)-regularized C-resolvent families to non-degenerate Volterra integro-differential equations (cf. Definition 1.1 with $B = I$), the reader may consult the monograph [**20**]. For abstract non-scalar Volterra equations, we refer the reader to [**25**, **29**, **30**, **40**].

2. Degenerate k-regularized C-resolvent propagation families for problem (1.1)–(1.2)

Let us recall that $n \in \mathbb{N} \setminus \{1\}$, $0 \leqslant \alpha_1 < \cdots < \alpha_{n-1}$, as well as that $A_1, \ldots, A_{n-1}$ are closed linear operators on E and that $f\colon [0,\infty) \to E$ is a continuous function. Let the set $\mathcal{I}$ and number Q be defined as above.

Suppose, for the time being, that the initial values $u_j \in E$ $(0 \leqslant j \leqslant m_Q - 1)$ satisfy $u_j \in D(A_i)$, provided $i \in \mathbb{N}_{n-1}$, $T_i u(t) = T_{i,R}u(t)$ and $0 \leqslant j \leqslant m_i - 1$ (put $u_{i,j} := A_i u_j$ in this case), and $u_{i,j} \in E$, provided $i \in \mathbb{N}_{n-1}$, $T_i u(t) = T_{i,R}u(t)$ and $m_i - 1 \geqslant j \geqslant m_Q$. We start this section by introducing the notion of a strong solution of problem (1.1)–(1.2).

Definition 2.1. A function $u \in C([0,\infty) : E)$ is said to be a strong solution of problem (1.1)–(1.2) iff the term $T_i u(t)$ is well defined and continuous for any $t \geqslant 0$, $i \in \mathbb{N}_{n-1}$, and (1.1)–(1.2) holds identically on $[0,\infty)$.

Now we would like to observe the following fact. If $Q > 0$, then we can consider the problem obtained from problem (1.1) by replacing some of the terms $T_{i,R}(t)$, for $1 \leqslant i \leqslant Q$, with the corresponding terms of form $T_{i,L}(t)$. By (1.3) and (b), it readily follows that a strong solution of problem (1.1)–(1.2)) is also a strong solution of the problem described above, when endowed with initial conditions (1.2). It is also worth noting that we have considered in [**27**, Remark 11(i)] problem (1.1) endowed with slightly different initial conditions, and that the existence of strong solutions of certain classes of multi-term problems with hypercyclic behaviour has been investigated in [**27**, Theorem 10] (cf. [**27**, Example 13] for an interesting application of the above-mentioned theorem involving the Ornstein-Uhlenbeck operators on L^2-type spaces).

Define now, for every $i \in \mathbb{N}_{n-1}$ and $t \geqslant 0$,

$$\mathcal{T}_{i,L}u(t) := g_{\alpha_{n-1}-\alpha_i} * A_i\Bigg[u(\cdot) - \sum_{j=0}^{m_i-1} u_j g_{j+1}(\cdot)\Bigg](t), \text{ if } T_i u(t) = T_{i,L}u(t), \tag{2.1}$$

$$\mathcal{T}_{i,R}u(t) := g_{\alpha_{n-1}-\alpha_i} * \Bigg[A_i u(\cdot) - \sum_{j=0}^{m_i-1} u_{i,j} g_{j+1}(\cdot)\Bigg](t), \text{ if } T_i u(t) = T_{i,R}u(t). \tag{2.2}$$

Let $\mathcal{T}_i u(t)$ denote, as before, exactly one of the terms $\mathcal{T}_{i,L}u(t)$ or $\mathcal{T}_{i,R}u(t)$. Integrating equation (1.1) α_{n-1} times, the foregoing arguments imply that any strong solution $t \mapsto u(t)$, $t \geqslant 0$ of problem 1.1–(1.2) satisfies the following integral equation

$$\sum_{i=1}^{n-1} \mathcal{T}_i u(t) = \big(g_{\alpha_{n-1}} * f\big)(t), \quad t \geqslant 0. \tag{2.3}$$

This motivates us to introduce the following definition.

Definition 2.2. Let $u_j \in E$ $(0 \leqslant j \leqslant m_Q - 1)$, let $u_{i,j} \in E$, provided $i \in \mathbb{N}_{n-1}$, $T_i u(t) = T_{i,R}u(t)$ and $0 \leqslant j \leqslant m_i - 1$, and let $\mathcal{V} \subseteq \mathbb{N}_{n-1}$. Then a continuous E-valued function $t \mapsto u(t)$, $t \geqslant 0$ is said to be a $\mathcal{V}$-mild solution of (2.3) iff (i)–(v) hold where

(i) $g_{\alpha_{n-1}-\alpha_i} * [u(\cdot) - \sum_{j=0}^{m_i-1} u_j g_{j+1}(\cdot)](t) \in D(A_i)$ for all $t \geqslant 0$ and $i \in \mathcal{I} \cap \mathcal{V}$, the mapping $t \mapsto A_i(g_{\alpha_{n-1}-\alpha_i} * [u - \sum_{j=0}^{m_i-1} u_j g_{j+1}])(t)$, $t \geqslant 0$ is well-defined and continuous for all $i \in \mathcal{I} \cap \mathcal{V}$,

(ii) the mapping $t \mapsto (g_{\alpha_{n-1}-\alpha_i} * A_i[u - \sum_{j=0}^{m_i-1} u_j g_{j+1}])(t)$, $t \geqslant 0$ is continuous for all $i \in \mathcal{I} \smallsetminus \mathcal{V}$,

(iii) $(g_{\alpha_{n-1}-\alpha_i} * u)(t) \in D(A_i)$ for all $t \geqslant 0$ and $i \in (\mathbb{N}_{n-1} \smallsetminus \mathcal{I}) \cap \mathcal{V}$, the mapping $t \mapsto A_i(g_{\alpha_{n-1}-\alpha_i} * u)(t)$, $t \geqslant 0$ is continuous for all $i \in (\mathbb{N}_{n-1} \smallsetminus \mathcal{I}) \cap \mathcal{V}$,

(iv) the mapping $t \mapsto (g_{\alpha_{n-1}-\alpha_i} * A_i u)(t)$, $t \geqslant 0$ is well-defined and continuous for all $i \in (\mathbb{N}_{n-1} \smallsetminus \mathcal{I}) \smallsetminus \mathcal{V}$,

(v) for every $t \geqslant 0$, the following holds:

$$\begin{aligned}
&\sum_{i \in \mathcal{I} \cap \mathcal{V}} A_i\biggl(g_{\alpha_{n-1}-\alpha_i} * \biggl[u(\cdot) - \sum_{j=0}^{m_i-1} u_j g_{j+1}(\cdot)\biggr]\biggr)(t) \\
&\quad + \sum_{i \in \mathcal{I} \smallsetminus \mathcal{V}} \biggl(g_{\alpha_{n-1}-\alpha_i} * A_i\biggl[u(\cdot) - \sum_{j=0}^{m_i-1} u_j g_{j+1}(\cdot)\biggr]\biggr)(t) \\
&\quad + \sum_{i \in (\mathbb{N}_{n-1} \smallsetminus \mathcal{I}) \smallsetminus \mathcal{V}} (g_{\alpha_{n-1}-\alpha_i} * A_i u)(t) + \sum_{i \in (\mathbb{N}_{n-1} \smallsetminus \mathcal{I}) \cap \mathcal{V}} A_i(g_{\alpha_{n-1}-\alpha_i} * u)(t) \\
&= \sum_{i \in \mathbb{N}_{n-1} \smallsetminus \mathcal{I}} \sum_{j \in \mathbb{N}^0_{m_i-1}} g_{\alpha_{n-1}-\alpha_i+1+j}(t) u_{i,j} + (g_{\alpha_{n-1}} * f)(t), \quad t \geqslant 0.
\end{aligned}$$

If $\mathcal{V} = \emptyset$ ($\mathcal{V} = \mathcal{N}_{n-1}$), then we also say that $u(t)$ is a strong (mild) solution of (2.3).

Any strong solution of problem (1.1)–(1.2) is also a strong solution of problem (2.3), and any $\mathcal{V}$-mild solution of problem (2.3) is also a $\mathcal{V}'$-mild solution of (2.3) provided that $\mathcal{V}, \mathcal{V}' \subseteq \mathbb{N}_{n-1}$ and $\mathcal{V} \subseteq \mathcal{V}'$. As already observed in [**27**] for the problem $(\mathrm{DFP})_L$, a sufficiently smooth strong solution of the problem (2.3) need not be a strong solution of problem (1.1)–(1.2) in the case that $\mathcal{I} \neq \emptyset$. The situation is quite complicated even in the case that $\mathcal{I} = \emptyset$ because then we can only prove that a strong solution of problem (2.3) satisfies the equation

$$\sum_{i \in \mathbb{N}_{n-1}} \biggl(g_{m_{n-1}-m_i} * g_{m_i-\alpha_i} * \biggl[A_i u(\cdot) - \sum_{j=0}^{m_i-1} u_{i,j} g_{1+j}(\cdot)\biggr]\biggr)(t) = (g_{\alpha_{n-1}} * f)(t), \quad t \geqslant 0,$$

which does not imply, in general, that the function

$$t \mapsto g_{m_i-\alpha_i} * \biggl[A_i u - \sum_{j=0}^{m_i-1} u_{i,j} g_{1+j}\biggr](t), \quad t \geqslant 0$$

is m_i-times continuously differentiable for $i \in \mathbb{N}_{n-1}$ (the problem $(\mathrm{DFP})_R$ is an exception, cf. [**27**]). Because of that, we shall primarily consider degenerate integral equation (2.3) in the sequel.

REMARK 2.1. Before dividing our further research into two separate subsections, it should be observed that we can further generalize the abstract form of problem (1.1) by assuming that some of the terms $T_i u(t)$ can be expressed as sums of terms like $A_i'\mathbf{D}_t^{\alpha_i}(B_i'\mathbf{D}_t^{\beta_i}u(t))$ and $\mathbf{D}_t^{\alpha_i}A_i''(\mathbf{D}_t^{\beta_i}B_i''u(t))$, with A_i', B_i', A_i'', B_i'' being closed linear operators on E and $\beta_i \geqslant 0$ (cf. [**44**, Chapter VI] for corresponding examples). It would take too long to go into further details concerning this topic here.

2.1. Exponentially equicontinuous *k*-regularized *C*-resolvent (i,j)-propagation families for (1.1)–(1.2). Following the method employed in the papers [**55**,**56**] and [**22**], we introduce the notion of an exponentially equicontinuous k-regularized C-resolvent propagation family for problem (1.1)–(1.2) as follows (cf. the problem (2.3) with $\mathcal{I} = \emptyset$, $x = u_{i,j}$, the other initial values being zeroes, and then apply the formula [**6**, (1.23)] for the Laplace transform of Caputo derivatives of the α^{th} order).

Definition 2.3. Suppose that the function $k(t)$ satisfies (P1), as well as that $1 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m_i - 1$ and $R_{i,j}(t)\colon D(A_i) \to E$ is a linear mapping ($t \geqslant 0$). Let the operator $C \in L(E)$ be injective. Then the operator family $(R_{i,j}(t))_{t \geqslant 0}$ is said to be an exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation family for problem (1.1)–(1.2) iff there exists $\omega \geqslant \max(0, \operatorname{abs}(k))$ such that the following holds:

- (i) The mapping $t \mapsto R_{i,j}(t)x$, $t \geqslant 0$ is continuous for every fixed element $x \in D(A_i)$.
- (ii) The family $\{e^{-\omega t}R_{i,j}(t) : t \geqslant 0\}$ is equicontinuous, i.e., for every $p \in \circledast$, there exist $c > 0$ and $q \in \circledast$ such that $p(e^{-\omega t}R_{i,j}(t)x) \leqslant cq(x)$, $x \in D(A_i)$, $t \geqslant 0$.
- (iii) For every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, the operator P_λ is injective, $C(R(A_i)) \subseteq R(P_\lambda)$ and

$$\lambda^{\alpha_i - \alpha_{n-1} - j}\tilde{k}(\lambda)P_\lambda^{-1}CA_i x = \int_0^\infty e^{-\lambda t}R_{i,j}(t)x\,dt, \quad x \in D\big(A_i\big). \tag{2.4}$$

If $k(t) = g_{r+1}(t)$ for some $r \geqslant 0$, then it is also said that $(R_{i,j}(t))_{t \geqslant 0}$ is an exponentially equicontinuous r-times integrated C-regularized resolvent (i,j)-propagation family for (1.1)–(1.2); an exponentially equicontinuous 0-times integrated C-regularized resolvent (i,j)-propagation family for (1.1)–(1.2) is also said to be an exponentially equicontinuous C-regularized resolvent (i,j)-propagation family for (1.1)–(1.2).

Before we state the following important extension of [**56**, Theorem 3.1], it is worth noting that we do not use here the condition $CA_i \subseteq A_iC$, in contrast to the corresponding definitions from [**55**, **56**] and [**22**], and that the existence of an exponentially equicontinuous k-regularized C-resolvent $(i,0)$-propagation family for problem (1.1)–(1.2) implies the existence of an exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation family for problem (1.1)–(1.2) ($j \in \mathbb{N}^0_{m_i-1}$); if this is the case, we have $R_{i,j}(t)x = (g_j * R_{i,0}(\cdot)x)(t)$, $t \geqslant 0$, $j \in \mathbb{N}^0_{m_i-1}$, $x \in D(A_i)$. Observe also that the uniqueness theorem for Laplace transform implies that there exists at most one exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation family for problem (1.1)–(1.2) and that the assertions of [**22**, Remark 2.3(iv), Proposition 2.4, Theorem 2.5] can be reformulated in our context.

THEOREM 2.1. *Suppose that* $1 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m_i - 1$ *and there exists an exponentially equicontinuous* k*-regularized* C*-resolvent* (i,j)*-propagation family* $(R_{i,j}(t))_{t \geqslant 0}$ *for problem* (1.1)–(1.2).

(i) *Assume that there exists* $l \in \mathbb{N}_{n-1}$ *such that the following condition:*

(C.1) *For every* $v \in \mathbb{N}_{n-1} \smallsetminus \{l\}$ *and* $x \in D(A_i)$, *there exist a number* $\omega_0 > \omega$ *and a continuous* E*-valued function* $t \mapsto f_{i,j,v}(t;x)$, $t \geqslant 0$ *such that, for every* $p \in \circledast$, *there exists* $M_p > 0$ *with* $p(f_{i,j,v}(t;x)) \leqslant M_p e^{\omega t}$, $t \geqslant 0$ $(v \in \mathbb{N}_{n-1} \smallsetminus \{l\})$ *and that, for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega_0$ *and* $\tilde{k}(\lambda) \neq 0$,
$$\int_0^\infty e^{-\lambda t} f_{i,j,v}(t;x)\,dt = \lambda^{\alpha_i - \alpha_{n-1} - j + \alpha_v - \alpha_{n-1}} \tilde{k}(\lambda) A_v P_\lambda^{-1} C A_i x$$

holds. Then for each $v_0 \in D(A_i)$ *the function* $u(t) := R_{i,j}(t)v_0$, $t \geqslant 0$ *is a mild solution of the integral equation*
$$\sum_{v=1}^{n-1} A_v \big(g_{\alpha_{n-1}-\alpha_v} * u\big)(t) = \big(g_{\alpha_{n-1}-\alpha_i+j} * k\big)(t) C A_i v_0, \quad t \geqslant 0, \tag{2.5}$$

defined in the same way as in Definition 2.2(ii).

(ii) *Let* $\emptyset \neq \mathcal{K} \subseteq \mathbb{N}_{n-1}$. *If the following conditions hold,*

(C.2) *For every* $l \in \mathcal{K}$ *and* $x \in D(A_i)$, *and for every* $v \in \mathbb{N}_{n-1} \smallsetminus \{l\}$, *there exist a number* $\omega_{l,v} > \omega$ *and a continuous* E*-valued function* $t \mapsto g_{i,j,l,v}(t;x)$, $t \geqslant 0$ *such that, for every* $p \in \circledast$, *there exists* $M_{p,l,v} > 0$ *with* $p(g_{i,j,l,v}(t;x)) \leqslant M_{p,l,v} e^{\omega_{l,v} t}$, $t \geqslant 0$ $(l \in \mathcal{K},\ v \in \mathbb{N}_{n-1} \smallsetminus \{l\})$ *and that, for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega_{l,v}$ *and* $\tilde{k}(\lambda) \neq 0$,
$$\int_0^\infty e^{-\lambda t} g_{i,j,l,v}(t;x) dt = \lambda^{\alpha_i - \alpha_{n-1} - j + \alpha_v - \alpha_l} \tilde{k}(\lambda) A_v P_\lambda^{-1} C A_i x$$

(C.3) *For every* $l \in \mathcal{K}$, *there exist a number* $\omega_l > \omega$ *and a continuous function* $h_l \colon [0,\infty) \to \mathbb{C}$ *satisfying* (P1) *and*
$$\widetilde{h_l}(\lambda) = \tilde{k}(\lambda)\lambda^{\alpha_i - \alpha_l - j}, \quad \operatorname{Re}\lambda > \omega_l,$$

then for each $v_0 \in D(A_i)$ *the function* $u(t) = R_{i,j}(t)v_0$, $t \geqslant 0$ *satisfies that the mappings* $t \mapsto A_l u(t)$, $t \geqslant 0$ *are well-defined, continuous and that for each* $p \in \circledast$ *there exist* $M_p > 0$ *and* $\omega_0 > \omega$ *with* $p(A_l u(t) - h_l(t) C A_i v_0) \leqslant M_p e^{\omega_0 t}$, $t \geqslant 0$ $(l \in \mathcal{K})$. *Furthermore, for every* $t \geqslant 0$,
$$\begin{aligned}\sum_{l \in \mathcal{K}} (g_{\alpha_{n-1}-\alpha_l} * A_l u)(t) + \sum_{l \in \mathbb{N}_{n-1} \smallsetminus \mathcal{K}} A_l (g_{\alpha_{n-1}-\alpha_l} * u)(t) \\ = \big(g_{\alpha_{n-1}-\alpha_i+j} * k\big)(t) C A_i v_0.\end{aligned} \tag{2.6}$$

(iii) *Suppose that* (C.1) *holds. Let* $v_0 \in \bigcap_{i=1}^{n-1} D(A_i)$, *let* $\emptyset \neq \mathcal{K} \subseteq \mathbb{N}_{n-1}$, *and let* $CA_p \subseteq A_p C$ *for all* $p \in \mathbb{N}_{n-1}$. *If the following condition holds,*

(C.4) *For every* $l \in \mathcal{K}$ *and for every* $v \in \mathbb{N}_{n-1} \smallsetminus \{i\}$, *there exist a number* $\omega_{l,v} > \omega$ *and a continuous function* $h_{l,v} \colon [0,\infty) \to E$ *satisfying that, for every* $p \in \circledast$, *there exists* $M_{p,l,v} > 0$ *with* $p(h_{l,v}(t)) \leqslant M_{p,l,v} e^{\omega_{l,v} t}$, $t \geqslant 0$ *and* $\widetilde{h_{l,v}}(\lambda) = \tilde{k}(\lambda)\lambda^{\alpha_v - \alpha_{n-1} - j} A_l P_\lambda^{-1} C A_v v_0$, $\operatorname{Re}\lambda > \omega_{l,v}$, $\tilde{k}(\lambda) \neq 0$, *then the function $u(t)=R_{i,j}(t)v_0$, $t\geqslant 0$ satisfies that the mappings $t\mapsto A_l u(t)$, $t\geqslant 0$ are well-defined, continuous and that for each $p\in\circledast$ there exist $M_p>0$ and $\omega_0>\omega$ with $p(A_l u(t)-(g_j*k)(t)CA_l v_0)\leqslant M_p e^{\omega_0 t}$, $t\geqslant 0$ ($l\in\mathcal{K}$). Furthermore, for every $t\geqslant 0$,* (2.6) *holds.*

(iv) *Suppose that $CA_p\subseteq A_pC$, $p\in\mathbb{N}_{n-1}$ and $k(t)$ satisfies* (P2)*, as well as that $n=3$ or that $n\geqslant 4$ and the following condition holds:*

(C.5) *For every $p\in\circledast$ and $l\in\mathbb{N}_{n-1}\smallsetminus\{i\}$, there exist numbers $\lambda_{p,l},\sigma_{p,l}>0$, a seminorm $q_{p,l}\in\circledast$ and a function $h_{p,l}\colon(\lambda_{p,l},\infty)\to(0,\infty)$ such that $p(P_\lambda^{-1}CA_l x)\leqslant[q_{p,l}(x)+q_{p,l}(A_l x)]h_{p,l}(\lambda)$, $\lambda>\lambda_{p,l}$, $x\in D(A_l)$, and $\lim_{\lambda\to+\infty}e^{-\lambda\sigma_{p,l}}h_{p,l}(\lambda)=0$.*

Then the function $u(t)=R_{i,j}(t)v_0$, $t\geqslant 0$ is a unique mild solution of integral equation (2.5)*, provided that $v_0\in D(A_i)$ and the assumptions of* (i) *hold. Furthermore, the function $u(t)=R_{i,j}(t)v_0$, $t\geqslant 0$ is a unique function satisfying that the mapping $t\mapsto A_l u(t)$, $t\geqslant 0$ is well-defined, continuous ($l\in\mathcal{K}$) and that* (2.6) *holds, provided that $v_0\in D(A_i)$ and the assumptions of* (ii) *hold, resp. $v_0\in\bigcap_{i=1}^{n-1}D(A_i)$ and the assumptions of* (iii) *hold.*

Proof. Let $v_0\in D(A_i)$. Due to the condition (C.1) and Lemma 1.2, we have that the function $t\mapsto A_v(g_{\alpha_{n-1}-\alpha_v}*R_{i,j}(\cdot)v_0)(t)$, $t\geqslant 0$ is well-defined, continuous and that for each $p\in\circledast$ there exist $M_p'>0$ and $\omega'>\omega$ with $p(A_v(g_{\alpha_{n-1}-\alpha_v}*R_{i,j}(\cdot)v_0)(t))\leqslant M_p'e^{\omega' t}$, $t\geqslant 0$ ($v\in\mathbb{N}_{n-1}\smallsetminus\{l\}$); furthermore,

$$\int_0^\infty e^{-\lambda t}A_v(g_{\alpha_{n-1}-\alpha_v}*R_{i,j}(\cdot)v_0)(t)\,dt=\tilde{k}(\lambda)\lambda^{\alpha_i-\alpha_{n-1}-j+\alpha_v-\alpha_{n-1}}A_vP_\lambda^{-1}CA_iv_0,$$

for any $v\in\mathbb{N}_{n-1}\smallsetminus\{l\}$ and for any $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\omega'$ and $\tilde{k}(\lambda)\neq 0$. Using the identity

$$\begin{aligned}(2.7)\quad &\tilde{k}(\lambda)\lambda^{\alpha_i-\alpha_{n-1}-j+\alpha_l-\alpha_{n-1}}A_lP_\lambda^{-1}CA_iv_0\\&=\tilde{k}(\lambda)\lambda^{\alpha_i-\alpha_{n-1}-j}\Big[CA_iv_0-\sum_{v\in\mathbb{N}_{n-1}\smallsetminus\{l\}}\lambda^{\alpha_v-\alpha_{n-1}}A_vP_\lambda^{-1}CA_iv_0\Big],\end{aligned}$$

for any $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\omega'$ and $\tilde{k}(\lambda)\neq 0$, and Lemma 1.2, it readily follows that the function $t\mapsto A_l(g_{\alpha_{n-1}-\alpha_l}*R_{i,j}(\cdot)v_0)(t)$, $t\geqslant 0$ is well-defined, continuous and that

$$\begin{aligned}A_l(g_{\alpha_{n-1}-\alpha_l}*R_{i,j}(\cdot)v_0)(t)&=(g_{\alpha_{n-1}-\alpha_i+j}*k)(t)CA_iv_0\\&\quad-\sum_{v\in\mathbb{N}_{n-1}\smallsetminus\{l\}}A_l(g_{\alpha_{n-1}-\alpha_v}*R_{i,j}(\cdot)v_0)(t),\quad t\geqslant 0,\end{aligned}$$

proving that the function $u(t)=R_{i,j}(t)v_0$, $t\geqslant 0$ is a mild solution of integral equation (2.5). Suppose now that conditions (C.2)–(C.3) hold, as well as that $v_0\in D(A_i)$ and $\emptyset\neq\mathcal{K}\subseteq\mathbb{N}_{n-1}$. Clearly, (C.2) implies (C.1) with any $l\in\mathcal{K}$. Similarly as in the proof of (i), the conditions (C.2)–(C.3) in combination with equation (2.7), multiplied by $\lambda^{\alpha_{n-1}-\alpha_l}$, imply that there exists a sufficiently large number $\omega_l' > \omega$ such that

$$A_l \int_0^\infty e^{-\lambda t} R_{i,j}(t) v_0 \, dt = \widetilde{h}_l(\lambda) C A_i v_0 - \int_0^\infty e^{-\lambda t} \sum_{v \in \mathbb{N}_{n-1} \smallsetminus \mathcal{K}} g_{i,j,l,v}(t; v_0) dt,$$

for any $l \in \mathcal{K}$ and for any $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \omega_l'$. Then we can use the assertion (i) and Lemma 1.2 to complete the proof of (ii). In order to prove (iii), observe first that the assumptions $v_0 \in \bigcap_{i=1}^{n-1} D(A_i)$ and $CA_p \subseteq A_p C$, $p \in \mathbb{N}_{n-1}$ imply that

$$P_\lambda^{-1} C(\lambda^{\alpha_i - \alpha_{n-1}} A_i v_0) + \sum_{v \in \mathbb{N}_{n-1} \smallsetminus \{i\}} P_\lambda^{-1} C(\lambda^{\alpha_v - \alpha_{n-1}} A_v v_0) = P_\lambda^{-1} C P_\lambda v_0 = C v_0, \tag{2.8}$$

provided $\operatorname{Re} \lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$. Making use of (2.4) and (2.8), we obtain that, for any such value of complex parameter λ, the following holds:

$$\begin{aligned} A_l \int_0^\infty e^{-\lambda t} R_{i,j}(t) v_0 \, dt &= \lambda^{-j} \tilde{k}(\lambda) A_l P_\lambda^{-1} C(\lambda^{\alpha_i - \alpha_{n-1}} A_i v_0) \\ &= \lambda^{-j} \tilde{k}(\lambda) A_l \Big[C v_0 - \sum_{v \in \mathbb{N}_{n-1} \smallsetminus \{i\}} \lambda^{\alpha_v - \alpha_{n-1}} P_\lambda^{-1} C A_v v_0 \Big]. \end{aligned}$$

Keeping in mind the last equation, as well as condition (C.4) and Lemma 1.2, the proof of (iii) follows instantly. We will prove the uniqueness of solutions in (iv) only in the case that $v_0 \in D(A_i)$ and the assumptions of (i) hold. Let $t \mapsto u(t)$, $t \geqslant 0$ be a mild solution of integral equation (2.5) with $v_0 = 0$. Convoluting the function $u(\cdot)$ with $g_\xi(\cdot)$, for a sufficiently large number $\xi > 0$, we may assume without of generality that, for every $v \in \mathbb{N}_{n-1}$, the mapping $t \mapsto A_v u(t)$, $t \geqslant 0$ is well-defined and continuous. Set, for every $t \geqslant 0$ and $\zeta > 0$, $v_{t,\zeta}(\lambda) := (g_\zeta * e^{\lambda \cdot})(t) - \lambda^{-\zeta} e^{t\lambda}$, $\lambda > 0$; $v_{t,0}(\lambda) := 0$ ($t \geqslant 0$, $\lambda > 0$). Then the mapping $t \mapsto v_{t,\zeta}(\lambda)$ is continuous in $t \geqslant 0$, for any fixed numbers $\zeta \geqslant 0$ and $\lambda > 0$, and by [**54**, Lemma 1.5.5, p. 23], there exists $M \geqslant 1$ such that the mapping $\lambda \mapsto v_{t,\zeta}(\lambda)$, $\lambda > 0$ satisfies

$$|v_{t,\zeta}(\lambda)| \leqslant M\big[(1+t)^{\zeta-1} \lambda^{-1} (1 + \lambda^{1-\zeta}) + t^{\zeta-1} \lambda^{-1}\big], \quad \lambda > 0,\ t > 0,\ \zeta > 0. \tag{2.9}$$

Keeping in mind that $CA_p \subseteq A_p C$, $p \in \mathbb{N}_{n-1}$, we have that, for every $t \geqslant 0$ and $\lambda > 0$,

$$\begin{aligned} &\lambda^{\alpha_i - \alpha_{n-1}} \int_0^t e^{\lambda(t-s)} A_i C u(s) ds + \int_0^t v_{t-s, \alpha_{n-1} - \alpha_i}(\lambda) A_i C u(s) ds \\ &= C \int_0^t e^{\lambda(t-s)} (g_{\alpha_{n-1} - \alpha_i} * A_i u)(s) ds \\ &= (-C) \sum_{v \in \mathbb{N}_{n-1} \smallsetminus \{i\}} \int_0^t e^{\lambda(t-s)} (g_{\alpha_{n-1} - \alpha_v} * A_v u)(s) ds \\ &= - \sum_{v \in \mathbb{N}_{n-1} \smallsetminus \{i\}} \Big[\lambda^{\alpha_v - \alpha_{n-1}} \int_0^t e^{\lambda(t-s)} A_v C u(s) ds + \int_0^t v_{t-s, \alpha_{n-1} - \alpha_v}(\lambda) A_v C u(s) ds \Big], \end{aligned}$$

which clearly implies that, for every $\lambda > \omega$, $\sigma > 0$ and $t \geqslant 0$, the following holds:
(2.10)

$$\begin{aligned}
&\lambda^{\alpha_i-\alpha_{n-1}-j}\tilde{k}(\lambda)e^{-\lambda\sigma}\int_0^t e^{\lambda(t-s)}Cu(s)ds \\
&\quad = -\lambda^{\alpha_i-\alpha_{n-1}-j}\tilde{k}(\lambda)e^{-\lambda\sigma}P_\lambda^{-1}C\int_0^t v_{t-s,\alpha_{n-1}-\alpha_i}(\lambda)A_iu(s)ds \\
&\qquad - \lambda^{\alpha_i-\alpha_{n-1}-j}\tilde{k}(\lambda)e^{-\lambda\sigma}\sum_{v\in\mathbb{N}_{n-1}\smallsetminus\{i\}}P_\lambda^{-1}C\int_0^t v_{t-s,\alpha_{n-1}-\alpha_v}(\lambda)A_vu(s)ds.
\end{aligned}$$

By (2.4) and (2.10), we obtain that, for every $\lambda > \omega$, $\sigma > 0$ and $t \geqslant 0$, the following holds:

$$\begin{aligned}
&e^{-\lambda\sigma}\int_0^t e^{\lambda(t-s)}Cu(s)ds \\
&\quad = -\frac{\lambda^{\alpha_{n-1}+j-\alpha_i}e^{-\lambda\sigma}}{\tilde{k}(\lambda)}\int_0^\infty e^{-\lambda s}R_{i,j}(s)\Big(\int_0^t v_{t-r,\alpha_{n-1}-\alpha_i}(\lambda)u(r)dr\Big)ds \\
&\qquad - e^{-\lambda\sigma}\sum_{v\in\mathbb{N}_{n-1}\smallsetminus\{i\}}P_\lambda^{-1}CA_v\int_0^t v_{t-s,\alpha_{n-1}-\alpha_v}(\lambda)u(s)ds.
\end{aligned}$$

For the estimation of the first addend on the right-hand side of the above equality, we can use the fact that there exist numbers $\sigma_0 > 0$ and $M' \geqslant 1$ such that

$$\frac{e^{-\lambda\sigma_0}}{|\tilde{k}(\lambda)|} \leqslant M', \quad \lambda > \omega + 1; \tag{2.11}$$

cf. the proof of [**22**, Theorem 2.8]. Keeping in mind (2.9) and (2.11), it can be simply proved that, for every $\sigma > \sigma_0$ and for every $p \in \circledast$, we have
(2.12)

$$\lim_{\lambda\to+\infty} p\Big(\frac{\lambda^{\alpha_{n-1}+j-\alpha_i}e^{-\lambda\sigma}}{\tilde{k}(\lambda)}\int_0^\infty e^{-\lambda s}R_{i,j}(s)\Big(\int_0^t v_{t-r,\alpha_{n-1}-\alpha_i}(\lambda)u(r)dr\Big)ds\Big) = 0.$$

If $n \geqslant 4$, then condition (C.5) in combination with the previous equality and (2.9) shows that, for every $p \in \circledast$, there exists a sufficiently large number $\sigma_p > 0$ such that $\lim_{\lambda\to+\infty} e^{-\lambda\sigma_p}p((e^{\lambda\cdot} * Cu)(t)) = 0$, $t \geqslant 0$; the same holds in the case that $n = 3$ because then we can use, instead of condition (C.5), equation (2.8) and the arguments already seen in proving equation (2.12), to conclude that

$$\lim_{\lambda\to+\infty} p\Big(e^{-\lambda\sigma}\sum_{v\in\mathbb{N}_{n-1}\smallsetminus\{i\}}P_\lambda^{-1}CA_v\int_0^t v_{t-s,\alpha_{n-1}-\alpha_v}(\lambda)v(s)ds\Big) = 0,$$

for any $\sigma > \sigma_0$ and $t \geqslant 0$. In such a way, we obtain that for each $p \in \circledast$ the following holds: $\lim_{\lambda\to+\infty}\int_0^t e^{\lambda(t-s-\sigma)}Cu(s)ds = 0$, $t \geqslant 0$, $\sigma > \sigma_p$. By the dominated convergence theorem, it readily follows that for each $p \in \circledast$ we have: $\lim_{\lambda\to+\infty} p(\int_0^{t-\sigma} e^{\lambda(t-s-\sigma)}Cu(s)ds) = 0$, $t \geqslant \sigma > \sigma_p$. Therefore,

$$\lim_{\lambda\to+\infty}\int_0^t e^{\lambda(t-s)}Cu(s)ds = 0, \quad t \geqslant 0.$$

Since C is injective, we can apply [**20**, Lemma 2.1.33(iii)] (cf. [**38**, Lemma 1.4.4, p. 100] for the Banach space case) to complete the proof. $\square$

The uniqueness of solutions of integral equation (2.5), resp. (2.6), can be proved even in the case of non-existence of an exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation family for problem (1.1)–(1.2). Strictly speaking, the proof of Theorem 2.1 implies the following uniqueness type theorem for degenerate multi-term problems (cf. [**10**, Theorem 3.1] for a pioneering result on the uniqueness of degenerate first order equations):

Theorem 2.2. *Suppose that $CA_p \subseteq A_pC$ for all $p \in \mathbb{N}_{n-1}$, $\mathcal{V} \subseteq \mathbb{N}_{n-1}$ and the requirements in* (C.5) *hold for every seminorm $p \in \circledast$ and for every number $l \in \mathbb{N}_{n-1}$. Then there exists at most one mild solution $t \mapsto u(t)$, $t \geqslant 0$ of integral equation* (2.5) *with $v_0 = 0$, resp. there exists at most one continuous E-valued function $t \mapsto u(t)$, $t \geqslant 0$ satisfying that the mapping $t \mapsto A_l u(t)$, $t \geqslant 0$ is well-defined, continuous ($l \in \mathcal{K}$) and that* (2.6) *holds with $v_0 = 0$. In particular, there exists at most one $\mathcal{V}$-mild solution of problem* (2.3) *and there exists at most one strong solution of problem* (1.1)–(1.2).

Remark 2.2. Suppose again that the general assumptions of Theorem 2.1 hold, i.e., that $1 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m_i - 1$ and there exists an exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation family $(R_{i,j}(t))_{t\geqslant 0}$ for problem (1.1)–(1.2).

(i) Suppose that $k(t)$ satisfies (P2) and that, for every $l \in \mathbb{N}_{n-1} \smallsetminus \{i\}$, there exists $j_l \in \mathbb{N}^0_{m_l-1}$ such that there exists an exponentially equicontinuous k-regularized C-resovent (l, j_l)-propagation family for problem (1.1)–1.2. By the proof of Theorem 2.1(iv), we have that condition (C.5) automatically holds.

(ii) The uniqueness of solutions of non-degenerate integral equations has been recently considered in [**30**]. It ought to be observed that we must impose the additional condition $CA_p \subseteq A_pC$, $p \in \mathbb{N}^0_{n-1}$ in the formulation of Theorem 3.2 in [**30**] in order for its proof to work.

(iii) Let $\emptyset \neq \mathcal{K} \subseteq \mathbb{N}_{n-1}$. Suppose that $v_0 \in D(A_i)$ and conditions (C.2)–(C.3) hold, or that $v_0 \in \bigcap_{i=1}^{n-1} D(A_i)$, $CA_p \subseteq A_pC$ for all $p \in \mathbb{N}_{n-1}$, and the conditions (C.1) and (C.4) hold. Let $u(t)$ be the solution of (2.6), satisfying the properties stated above. Consider now equation (2.3) and the notion introduced in Definition 2.2 with indexes i,j replaced by i',j'. Then the following holds:
   (a) If $i \in \mathbb{N}_{n-1} \smallsetminus \mathcal{I}$, $k(t) = 1$, $u_{j'} = 0$ $(0 \leqslant j' \leqslant m_{i'} - 1)$, $u_{i',j'} = CA_i v_0$, provided $i' = i$ and $j' = j$, and $u_{i',j'} = 0$, otherwise, then $u(t)$ is a $(\mathbb{N}_{n-1} \smallsetminus \mathcal{K})$-mild solution of (2.3) with $f(t) = 0$.
   (b) If $i \in \mathcal{I}$, $k(t) = 1$, $u_{i',j'} = 0$ $(i' \in \mathbb{N}_{n-1}$, $j' \in \mathbb{N}^0_{m_{i'}-1})$, $u_{j'} = Cv_0$, provided $j' = j$, $u_{j'} = 0$, otherwise, and $CA_i \subseteq A_iC$, then $u(t)$ is a $(\mathbb{N}_{n-1} \smallsetminus \mathcal{K})$-mild solution of (2.3) with $f(t) = 0$, provided that for each $i' \in \{s \in \mathcal{I} \smallsetminus \{i\} : m_s - 1 \geqslant j\}$ one has $A_{i'}Cv_0 = 0$.

(iv) Making use of [**54**, Theorem 1.1.9], Lemma 1.2 and the formula [**6**, (1.23)], we can clarify some sufficent conditions for the existence of terms $A_p\mathbf{D}_t^{\alpha_p}u(t)$ and $\mathbf{D}_t^{\alpha_p}A_pu(t)$ $(p \in \mathbb{N}_{n-1})$. Unfortunately, it is very hard to verify these conditions in practical situations because we do not know the precise values of elements $R_{i,j}(0)x$, $R'_{i,j}(0)x, \dots$ $(x \in D(A_i))$.

The notion of an exponentially equicontinuous (equicontinuous), analytic k-regularized C-resolvent (i,j)-propagation family $(R_{i,j}(t))_{t\geqslant 0}$ for problem (1.1)–(1.2) is introduced in the following definition.

Definition 2.4. Suppose that $1 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m_i - 1$, $0 < \alpha \leqslant \pi$ and there exists an exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation family $(R_{i,j}(t))_{t\geqslant 0}$ for problem (1.1)–(1.2). Suppose, further, that the function $k(t)$ satisfies (P1), as well as that $C \in L(E)$ is an injective mapping. Then it is said that $(R_{i,j}(t))_{t\geqslant 0}$ is an exponentially equicontinuous (equicontinuous), analytic k-regularized C-resolvent (i,j)-propagation family $(R_{i,j}(t))_{t\geqslant 0}$ for problem (1.1)–(1.2), of angle α, iff the following holds:

(i) For every $x \in D(A_i)$, the mapping $t \mapsto R_{i,j}(t)x$, $t > 0$ can be analytically extended to the sector Σ_α; since no confusion seems likely, we shall denote the extension by the same symbol.
(ii) For every $x \in D(A_i)$ and $\beta \in (0,\alpha)$, one has $\lim_{z\to 0, z\in\Sigma_\beta} R_{i,j}(z)x = R_{i,j}(0)x$.
(iii) For every $\beta \in (0,\alpha)$, there exists $\omega_\beta \geqslant \max(0, \operatorname{abs}(k))$ $(\omega_\beta = 0)$ such that the family $\{e^{-\omega_\beta z} R_{i,j}(z) : z \in \Sigma_\beta\}$ is equicontinuous, i.e., for every $p \in \circledast$, there exist $c > 0$ and $q \in \circledast$ such that $p(e^{-\omega_\beta z} R_{i,j}(z)x) \leqslant cq(x)$, $x \in D(A_i)$, $z \in \Sigma_\beta$.

The proof of following theorem can be given by using the arguments given in that of [**24**, Theorem 3.7].

Theorem 2.3. *Assume that the function $k(t)$ satisfies* (P1)*, $1 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m_i - 1$, $\omega \geqslant \max(0, \operatorname{abs}(k))$, $\alpha \in (0, \pi/2]$ and the operator $C \in L(E)$ is injective. Assume, further, that for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, we have that the operator P_λ is injective and $C(R(A_i)) \subseteq R(P_\lambda)$. Let for each $x \in D(A_i)$ there is an analytic function $q_x \colon \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to E$ such that*

$$q_x(\lambda) = \lambda^{\alpha_i - \alpha_{n-1} - j} \tilde{k}(\lambda) P_\lambda^{-1} C A_i x, \quad \operatorname{Re}\lambda > \omega,\ \tilde{k}(\lambda) \neq 0.$$

Suppose that, for every $\beta \in (0,\alpha)$ and $p \in \circledast$, there exist $c_{p,\beta} > 0$ and $r_{p,\beta} \in \circledast$ such that $p((\lambda - \omega) q_x(\lambda)) \leqslant c_{p,\beta} r_{p,\beta}(x)$, $x \in D(A_i)$, $\lambda \in \omega + \Sigma_{\beta + (\pi/2)}$ and that, for every $x \in D(A_i)$, there exists the limit $\lim_{\lambda\to+\infty} \lambda q_x(\lambda)$ in E. Then there exists an exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation family $(R_{i,j}(t))_{t\geqslant 0}$ for problem (1.1)–(1.2)*, of angle α, and for each $\beta \in (0,\alpha)$ the family $\{e^{-\omega z} R_{i,j}(z) : z \in \Sigma_\beta\}$ is equicontinuous.*

Differential properties of (a,k)-regularized C-resolvent families have been investigated in [**22**, Theorems 3.4 and 3.5]; these assertions can be simply reformulated for exponentially equicontinuous (analytic) k-regularized C-resolvent (i,j)-propagation families in locally convex spaces. As the following theorem shows, this is also the case with the assertion of [**22**, Theorem 3.3].

Theorem 2.4. *Suppose that $1 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m_i - 1$, $0 < \gamma < 1$, $0 \leqslant j' \leqslant m_i - 1$, and there exists an exponentially equicontinuous k-regularized C-resolvent (i,j)-propagation family $(R_{i,j}(t))_{t\geqslant 0}$ for problem* (1.1)–(1.2) *satisfying* *that the family* $\{e^{-\omega t}R_{i,j}(t) : t \geqslant 0\}$ *is equicontinuous for some* $\omega \geqslant \max(0, \operatorname{abs}(k))$. *Assume that* $k(t)$ *satisfies* (P1) *and there exists a scalar-valued continuous kernel* $k_\gamma(t)$ *on* $[0,\infty)$, *satisfying* (P1), *and a positive real number* $\eta > 0$ *such that*

$$\widetilde{k_\gamma}(\lambda) = \lambda^{\gamma-1+j'-\gamma j}\tilde{k}\big(\lambda^\gamma\big), \quad \lambda > \eta.$$

Then there exists an exponentially equicontinuous k*-regularized* C*-resolvent* (i,j')*-propagation family* $(R_{i,j',\gamma}(t))_{t\geqslant 0}$ *for problem* (1.1)–(1.2), *with* α_i *replaced by* $\gamma\alpha_i$ *(*$i \in \mathbb{N}_{n-1}$*)* , *and* $(R_{i,j',\gamma}(t))_{t\geqslant 0}$ *is given by* $R_{i,j',\gamma}(0) := R_{i,j}(0)$,

$$R_{i,j',\gamma}(t)x := \int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma})R_{i,j}(s)x\,ds, \quad x \in D(A_i),\ t > 0.$$

Furthermore, the family $\{e^{-\omega^{1/\gamma}t}R_{i,j',\gamma}(t) : t \geqslant 0\}$ *is equicontinuous and, for every* $\zeta \geqslant 0$, *the equicontinuity of the family* $\{e^{-\omega t}(1+t^\zeta)^{-1}R_{i,j}(t) : t \geqslant 0\}$, *resp.* $\{e^{-\omega t}t^{-\zeta}R_{i,j}(t) : t \geqslant 0\}$, *implies the equicontinuity of the family*

$$\begin{gathered}\{e^{-\omega^{1/\gamma}t}(1+t^{\gamma\zeta})^{-1}(1+\omega t^{\zeta(1-\gamma)})^{-1}R_{i,j',\gamma}(t) : t \geqslant 0\},\\ \textit{resp. } \{e^{-\omega^{1/\gamma}t}t^{-\gamma\zeta}(1+\omega t^{\zeta(1-\gamma)})^{-1}R_{i,j',\gamma}(t) : t \geqslant 0\},\end{gathered}$$

and the following holds:

(i) *The mapping* $t \mapsto R_{i,j',\gamma}(t)x$, $t > 0$ *admits an analytic extension to the sector* $\Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)}$ *for all* $x \in D(A_i)$.

(ii) *If* $\omega = 0$ *and* $\varepsilon \in \big(0, \min\big(\big(\frac{1}{\gamma}-1\big)\frac{\pi}{2},\pi\big)\big)$, *then the family* $\{R_{i,j',\gamma}(z) : z \in \Sigma_{\min((\frac{2}{\gamma}-1)\frac{\pi}{2},\pi)-\varepsilon}\}$ *is equicontinuous and* $\lim_{z\to 0, z\in\Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)-\varepsilon}} R_{i,j',\gamma}(z)x = R_{i,j',\gamma}(0)x$ *for all* $x \in D(A_i)$.

(iii) *If* $\omega > 0$ *and* $\varepsilon \in (0, \min((\frac{1}{\gamma}-1)\frac{\pi}{2}, \frac{\pi}{2}))$, *then there exists* $\delta_{\gamma,\varepsilon} > 0$ *such that the family* $\{e^{-\delta_{\gamma,\varepsilon}\operatorname{Re} z}R_{i,j',\gamma}(z) : z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\frac{\pi}{2})-\varepsilon}\}$ *is equicontinuous. Moreover,* $\lim_{z\to 0, z\in\Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\frac{\pi}{2})-\varepsilon}} R_{i,j',\gamma}(z)x = R_{i,j',\gamma}x$ *for all* $x \in D(A_i)$.

Remark 2.3. Using the proof of [**6**, Theorem 3.1] and an elementary argumentation, it can be simply verified that any of conditions (C.1)–(C.5) is invariant under the action of subordination principle described in Theorem 2.4.

Before illustrating our abstract results from this subsection by some examples, we would like to observe that the analysis carried out in [**40**, Theorem 4.1, p. 101], [**22**, Theorem 2.6(i)] and Theorem 2.1 can be used in the study of the following degenerate integral equation:

$$\sum_{j\in\mathcal{K}}(a_j * A_j u)(t) + \sum_{j\in\mathbb{N}_{n-1}\smallsetminus\mathcal{K}} A_j(a_j * u)(t) = f(t), \quad t \geqslant 0,$$

where $\emptyset \neq \mathcal{K} \subseteq \mathbb{N}_{n-1}$, $f \in C([0,\infty) : E)$, and the functions $a_1(t),\dots,a_{n-1}(t)$ satisfy certain properties. For the sake of brevity and better exposition, we shall only refer the reader to [**29**, Theorem 4.4] for the corresponding result in the case of non-degenerate equations.

EXAMPLE 2.1. (cf. also [**29**, Example 5.1(i)]) Suppose that $c_l \in \mathbb{C} \smallsetminus 0$ $(1 \leqslant l \leqslant n-1)$, as well as that A and B are closed linear operators on E, and $A_l = c_l B$ for $1 \leqslant l \leqslant n-1$. We consider the following degenerate multi-term problem:

$$\mathbf{D}_t^{\alpha_n} Bu(t) + \sum_{l=1}^{n-1} c_l \mathbf{D}_t^{\alpha_l} Bu(t) = \mathbf{D}_t^{\alpha} Au(t), \quad t \geqslant 0, \tag{2.13}$$

equipped with the initial conditions of the form (1.2). Here $0 \leqslant \alpha_1 < \cdots < \alpha_n$, $0 \leqslant \alpha < \alpha_n$, and

$$P_\lambda = \sum_{l=1}^{n-1} c_l \lambda^{\alpha_l - \alpha_n} B - \lambda^{\alpha - \alpha_n} A + B, \quad \lambda \in \mathbb{C} \smallsetminus \{0\}.$$

(i) (a) Suppose $0 < \delta \leqslant 2$, $\sigma \geqslant 1$, $\frac{\pi\delta}{2(\alpha_n - \alpha)} - \frac{\pi}{2} > 0$, $0 \leqslant j \leqslant \lceil \alpha_n \rceil - 1$, and there exists an exponentially equicontinuous $(\sigma - 1)$-times integrated C-resolvent propagation family $(R(t))_{t \geqslant 0}$ for problem (1.4), with $a(t) = g_\delta(t)$. Put $\sigma' := \max(1, 1 - j + (\alpha_n - \alpha)(\sigma - 1)\delta^{-1})$ and $\theta := \min\left(\frac{\pi}{2}, \frac{\pi\delta}{2(\alpha_n - \alpha)} - \frac{\pi}{2}\right)$. Then, for every sufficiently small number $\varepsilon > 0$, there exists $\omega_\varepsilon > 0$ such that $C(R(B)) \subseteq R(P_\lambda)$ for all $\lambda \in \omega_\varepsilon + \Sigma_{\frac{\pi}{2}\delta - \varepsilon}$ and that the family $\{|\lambda|^{\frac{\delta - \sigma}{\delta}}(1 + |\lambda|^{\frac{1}{\delta}})(\lambda B - A)^{-1} CBx : \lambda \in \omega_\varepsilon + \Sigma_{\frac{\pi}{2}\delta - \varepsilon},\ x \in D(B)\}$ is equicontinuous. Noting also that

$$\arg\left(\lambda^{\alpha_n - \alpha} + \sum_{l=1}^{n-1} c_l \lambda^{\alpha_l - \alpha}\right) \approx (\alpha_n - \alpha)\arg(\lambda), \quad \lambda \to \infty,\ \arg(\lambda) < \frac{\pi}{\alpha_n - \alpha},$$

our choice of θ implies that, for every sufficiently small number $\varepsilon > 0$, there exists $\omega'_\varepsilon > 0$ such that, for every $\lambda \in \omega'_\varepsilon + \Sigma_{\frac{\pi}{2} + \theta - \varepsilon}$, one has

$$\lambda^{\alpha_n - \alpha} + \sum_{l=1}^{n-1} c_l \lambda^{\alpha_l - \alpha} \in \omega_\varepsilon + \Sigma_{\frac{\pi}{2}\delta - \varepsilon}.$$

Put now, for every $x \in D(B)$ and $\lambda \in \omega'_\varepsilon + \Sigma_{\frac{\pi}{2} + \theta - \varepsilon}$,

$$q_x(\lambda) := \lambda^{-j - \sigma'} P_\lambda^{-1} CBx.$$

Then $q_x : \omega'_\varepsilon + \Sigma_{\frac{\pi}{2} + \theta - \varepsilon} \to E$ is an analytic function and, for every $\beta \in (0, \theta)$ and $p \in ⊛$, there exist $c_{p,\beta} > 0$ and $r_{p,\beta} \in ⊛$ such that $p((\lambda - \omega'_\varepsilon) q_x(\lambda)) \leqslant c_{p,\beta} r_{p,\beta}(x)$, $x \in D(B)$, $\lambda \in \omega'_\varepsilon + \Sigma_{\frac{\pi}{2} + \theta - \varepsilon}$. By the proof of [**5**, Proposition 4.1.3, p. 248], we have that $\lim_{\operatorname{Re}\lambda \to +\infty} \lambda^{\delta - \sigma + 1}(\lambda^\delta B - A)^{-1} CBx = R_{i,j}(0)x$, $x \in D(B)$, which simply implies that, for every $x \in D(B)$, there exists the limit $\lim_{\lambda \to +\infty} \lambda q_x(\lambda)$ in E. Therefore, Theorem 2.3 implies that there exists an exponentially equicontinuous, analytic $(\sigma' - 1)$-times integrated C-resolvent (n, j)-propagation family $(R_{n,j}(t))_{t \geqslant 0}$ for problem (2.13), of angle θ (with the clear meaning).

(b) Suppose $0 < \delta \leqslant 2$, $\sigma \geqslant 1$, $0 \leqslant j \leqslant \lceil \alpha_n \rceil - 1$, $\gamma \in (0, \frac{\pi}{2}]$ and $\frac{\delta(\frac{\pi}{2} + \gamma)}{(\alpha_n - \alpha)} - \frac{\pi}{2} > 0$. Put $\sigma_1 := \sigma'$ and $\theta_1 := \min(\frac{\pi}{2}, \frac{\delta(\frac{\pi}{2} + \gamma)}{(\alpha_n - \alpha)} - \frac{\pi}{2})$. Arguing similarly as in (a), one can prove the following: Suppose that for each $\varepsilon \in (0, \frac{\pi}{2} + \gamma)$ there exists $\omega_\varepsilon > 0$ such that for each $x \in D(B)$ there exists an analytic function $q_x : \omega_\varepsilon + \Sigma_{\frac{\pi}{2}+\gamma-\varepsilon} \to E$ satisfying that

$$q_x(\lambda) = \lambda^{\delta-\sigma}(\lambda^\delta B - A)^{-1}CBx, \quad \lambda \in \omega_\varepsilon + \Sigma_{\frac{\pi}{2}+\gamma-\varepsilon},\ x \in D(B),$$

and that for each $p \in ⊛$ there exist $c_p > 0$ and $q_p \in ⊛$ so that

$$p(q_x(\lambda)) \leqslant c_p \frac{q_p(x)}{1+|\lambda|}, \quad \lambda \in \omega_\varepsilon + \Sigma_{\frac{\pi}{2}+\gamma-\varepsilon},\ x \in D(B).$$

Then the existence of limit $\lim_{\mathrm{Re}\,\lambda\to+\infty} \lambda^{\delta-\sigma+1}(\lambda^\delta B - A)^{-1}CBx$ in E, for all $x \in D(B)$, implies that there exists an exponentially equicontinuous, analytic $(\sigma_1 - 1)$-times integrated C-resolvent (n, j)-propagation family $(R_{n,j}(t))_{t\geqslant 0}$ for problem (2.13), of angle θ_1; if there is an element $x \in D(B)$ such that the limit $\lim_{\mathrm{Re}\,\lambda\to+\infty} \lambda^{\delta-\sigma+1}(\lambda^\delta B - A)^{-1}CBx$ does not exist in E, then the above holds with any number $\sigma_2 > \sigma_1$. For the purpose of illustration of obtained results, assume now that $n \in \mathbb{N}$ and iA'_l, $1 \leqslant l \leqslant n$ are commuting generators of bounded C_0-groups on a Banach space E. Put $A' := (A'_1, \dots, A'_n)$; we refer the reader to [**57**] and [**22**, Section 4] for the definition of a closable operator $P(A')$, where $P(x)$ is a complex polynomial in n variables, and for more details about functional calculus for commuting generators of bounded C_0-groups (if $E = L^p(\mathbb{R}^n)$ for some $p \in [1, \infty)$, then the obvious choice is $A'_l := -i\partial/\partial x_l$, with maximal distibutional domain). Suppose $0 < \delta < 2$, $\omega \geqslant 0$, $P_1(x)$ and $P_2(x)$ are non-zero complex polynomials, $N_1 = dg(P_1(x))$, $N_2 = dg(P_2(x))$, $\beta > \frac{n}{2}\frac{(N_1+N_2)}{\min(1,\delta)}$ (resp. $\beta \geqslant n|\frac{1}{p} - \frac{1}{2}|\frac{(N_1+N_2)}{\min(1,\delta)}$, if $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$), $P_2(x) \neq 0$, $x \in \mathbb{R}^n$ and

$$\sup_{x\in\mathbb{R}^n} \mathrm{Re}\Big(\Big(\frac{P_1(x)}{P_2(x)}\Big)^{1/\delta}\Big) \leqslant \omega. \tag{2.14}$$

Set

$$R_\delta(t) := \Big(E_\delta\Big(t^\delta \frac{P_1(x)}{P_2(x)}\Big)(1 + |x|^2)^{-\beta/2}\Big)(A'), \quad t \geqslant 0. \tag{2.15}$$

By [**22**, Theorem 4.3], we know that $(R_\delta(t))_{t\geqslant 0} \subseteq L(E)$ is a global exponentially bounded $(g_\delta, R_\delta(0))$-regularized resolvent family for problem (1.4) with $B = \overline{P_2(A')}$, $A = \overline{P_1(A')}$ and $a(t) = g_\delta(t)$. By the conclusion in (a), it readily follows that there exists an exponentially equicontinuous, analytic C-resolvent (n, j)-propagation family $(R_{n,j}(t))_{t\geqslant 0}$ for problem (2.13), of angle $\theta = \min(\frac{\pi}{2}, \frac{\pi\delta}{2(\alpha_n-\alpha)} - \frac{\pi}{2})$. Since condition (ii.1) given in the formulation of [**22**, Theorem 2.8] holds, with $a(t) = g_\delta(t)$ and $k(t) = 1$, it is not diificult to prove, with the help of our previous consideration and the results concerning the Laplace transform of analytic vector-valued functions (see e.g. [**24**, Section 3]) that conditions (C.1) and (C.5) hold for (2.13), as well as that condition (C.4) holds for (2.13) provided that $\alpha_{n-1} \leqslant \alpha$; we need the last condition because the inclusion $\lambda^{\alpha_{n-1}-\alpha_n-1}AP_\lambda^{-1}Cx \in LT - E$ has to be satisfied $(x \in E)$, it is also worth noting that we do not need the condition $\alpha_{n-1} \leqslant \alpha$ for the existence of solutions of the integral equation

$$Bu(t)+\sum_{l=1}^{n-1}c_l(g_{\alpha_n-\alpha_l}*Bu)(t)=A(g_{\alpha_n-\alpha}*u)(t)+CBv_0,\quad t\geqslant 0,$$

cf. (2.6). It is also worth noting that we can refine the results on C-wellposednes of equation (2.13) by using the estimates quoted in [**22**, Remark 4.4(ii)] and that we can similarly consider equation (2.13) in E_l-type spaces (cf. [**22**, Remark 4.5]).

(ii) (cf. [**29**, Example 5.1(i)-(b)] and [**22**, Example 3.8] for more details) Let $s>1$, $0\leqslant j\leqslant \lceil\alpha_n\rceil-1$, $k_{a,b}(t):=\mathcal{L}^{-1}(\exp(-a\lambda^b))(t)$, $t\geqslant 0$ ($a>0$, $b\in(0,1)$),

$$E:=\Big\{f\in C^{\infty}[0,1];\ \|f\|:=\sup_{p\geqslant 0}\frac{\|f^{(p)}\|_{\infty}}{p!^s}<\infty\Big\}$$
$$A':=-d/ds,\quad D(A'):=\{f\in E;\ f'\in E,\ f(0)=0\}.$$

Let $P_1(z)=\sum_{l=0}^{N_1}a_{l,1}z^l$, $z\in\mathbb{C}$, $a_{N_1,1}\neq 0$ be a complex non-zero polynomial, and let $P_2(z)=\sum_{l=0}^{N_2}a_{l,2}z^l$, $z\in\mathbb{C}$, $a_{N_2,2}\neq 0$ be a complex non-zero polynomial so that $N_1=dg(P_1)\geqslant 1+dg(P_2)=1+N_2$ (we leave to the interested reader the analysis of the case $N_1\leqslant N_2$, in which we always have the existence of integrated solution resolvent (n,j)-propagation families for problem (2.13)). Set $A:=P_1(A')$ and $B:=P_2(A')$. Using the consideration given in [**22**, Example 3.8], we can prove that there exist numbers $b>0$ and $c>0$ such that

$$\|(\lambda B-A)^{-1}\|=O\big(e^{b|\lambda|^{1/(N_1-N_2)s}+c|\lambda|^{1/(N_1-N_2)}}\big),\quad \lambda\in\mathbb{C},\tag{2.16}$$

and that, for every complex non-zero polynomial $P(z)$ with $dg(P)\leqslant N_1$, there exists $\zeta>0$ such that

$$\|(\lambda B-A)^{-1}P(A')f\|\leqslant \zeta\|f\|e^{b|\lambda|^{1/(N_1-N_2)s}+c|\lambda|^{1/(N_1-N_2)}},\tag{2.17}$$

for all $\lambda\in\mathbb{C}$ and $f\in D(P(A'))$ (observe that the above estimates can be used in proving the existence of convoluted solutions of fractional analogs of equation $(\lambda-\Delta)u_t=\alpha\Delta-\beta\Delta^2$ ($\alpha,\beta>0$, $\lambda\in\mathbb{R}$), in contrast with the assertions of [**22**, Theorems 4.2 and 4.3] which can be applied only in the case that $\lambda>0$; as observed by G. A. Sviridyuk, this equation is important in evolution modeling of some problems appearing in the theory of liquid filtration, see e.g. [**9**, p. 6]). Let $\theta\in(0,\pi/2]$, $b'=(\alpha_n-\alpha)/(N_1-N_2)$ and let $b'\leqslant\pi/(\pi+2\theta)$. Owing to (2.16), (2.17) and Theorem 2.3, we obtain that there is a sufficiently large number $a'>0$ such that there exists an exponentially equicontinuous, analytic $k_{a',b'}$-regularized I-resolvent (n,j)-propagation family $(R_{n,j}(t))_{t\geqslant 0}$ for problem (2.13), of angle θ, satisfying conditions (C.1)–(C.5). Denote, as before, $T_{l,L}u(t)=B\mathbf{D}_t^{\alpha_l}u(t)$, $t\geqslant 0$ if $l\in\mathbb{N}_n$ and $\alpha_l>0$, $T_{l,R}u(t)=\mathbf{D}_t^{\alpha_l}Bu(t)$, $t\geqslant 0$ if $l\in\mathbb{N}_n$, $T_{0,L}u(t)=A\mathbf{D}_t^{\alpha}u(t)$, $t\geqslant 0$ if $\alpha>0$, and $T_{0,R}u(t)=\mathbf{D}_t^{\alpha}Au(t)$, $t\geqslant 0$. Let $T_lu(t)$ be either $T_{l,L}u(t)$ or $T_{l,R}u(t)$ ($l\in\mathbb{N}_n^0$). Then it can be easily seen that for each $x\in D(B)$ the function $u(t)=R_{n,j}(t)x$, $t\geqslant 0$ is a unique strong solution of the problem

$$T_nu(t)+\sum_{l=1}^{n-1}c_lT_lu(t)=T_0u(t)+\big(k_{a',b'}^{(m_n)}*g_{j+m_n-\alpha_n}\big)(t),\quad t\geqslant 0,$$

with all initial values chosen to be zeroes. Observe, finally, that the analysis contained in [**29**, Example 5.4] can be used for the construction of hypoanalytic exponentially equicontinuous k-regularized I-resolvent (n, j)-propagation families for the problem

$$\sum_{l\in\mathcal{K}} P_l(A')\mathbf{D}_t^{\alpha_l}u(t) + \sum_{l\in\mathbb{N}_{n-1}\smallsetminus\mathcal{K}} \mathbf{D}_t^{\alpha_l}P_l(A')u(t) = 0, \quad t \geqslant 0,$$

where $P_1(z), \ldots, P_{n-1}(z)$ are complex non-zero polynomials satysfying certain properties, and $\mathcal{K} \subseteq \mathbb{N}_{n-1}$.

2.2. Exponentially equicontinuous (a, k)-regularized C-resolvent families generated by A, B; exponentially equicontinuous $(k; C)$-regularized resolvent (i, j)-propagation families for (1.1)–(1.2). In this subsection, we shall mainly consider the C-wellposedness of problem $(\mathrm{DFP})_L$ with A and B being closed linear operators on E. Set $p_B(x) := p(x) + p(Bx)$, $x \in D(B)$, $p \in ⊛$. Then the calibration $(p_B)_{p\in ⊛}$ induces the Hausdorff sequentially complete locally convex topology on $D(B)$. We shall denote this space simply by $[D(B)]$.

Following the consideration given in [**1**, Section 2], we introduce the following definition.

Definition 2.5. Suppose that the functions $a(t)$ and $k(t)$ satisfy (P1), as well as that $R(t) \in L(E, [D(B)])$ for all $t \geqslant 0$. Let $C \in L(E)$ be injective, and let $CA \subseteq AC$ and $CB \subseteq BC$. Then the operator family $(R(t))_{t\geqslant 0}$ is said to be an exponentially equicontinuous (a, k)-regularized C-resolvent family generated by A, B iff there exists $\omega \geqslant \max(0, \mathrm{abs}(a), \mathrm{abs}(k))$ such that the following holds:

(i) The mappings $t \mapsto R(t)x$, $t \geqslant 0$ and $t \mapsto BR(t)x$, $t \geqslant 0$ are continuous for every fixed element $x \in E$.

(ii) The family $\{e^{-\omega t}R(t) : t \geqslant 0\} \subseteq L(E, [D(B)])$ is equicontinuous, i.e., for every $p \in ⊛$, there exist $c > 0$ and $q \in ⊛$ such that

$$p(e^{-\omega t}R(t)x) + p(e^{-\omega t}BR(t)x) \leqslant cq(x), \quad x \in E,\ t \geqslant 0.$$

(iii) For every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, the operator $B - \tilde{a}(\lambda)A$ is injective, $R(C) \subseteq R(B - \tilde{a}(\lambda)A)$ and

$$\tilde{k}(\lambda)(B - \tilde{a}(\lambda)A)^{-1}Cx = \int_0^\infty e^{-\lambda t}R(t)x\,dt, \quad x \in E.$$

If $k(t) = g_{r+1}(t)$ for some $r \geqslant 0$, then it is also said that $(R(t))_{t\geqslant 0}$ is an exponentially equicontinuous r-times integrated (a, C)-regularized resolvent family generated by A, B; an exponentially equicontinuous 0-times integrated (a, C)-regularized resolvent family generated by A, B is also said to be an exponentially equicontinuous (a, C)-regularized resolvent family generated by A, B.

Before going any further, it should be noticed that we have already constructed some examples of (g_α, k)-regularized C-resolvent families generated by A, B in Example 2.1(ii).

Remark 2.4. Suppose that the functions $a(t)$ and $k(t)$ satisfy (P1), as well as that $CA \subseteq AC$ and $CB \subseteq BC$.

(i) It is clear that an exponentially equicontinuous (a,k)-regularized C-resolvent family generated by $A,\ B$, if exists, must be unique.

(ii) If for each $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$ the operator B commutes with $(B - \tilde{a}(\lambda)A)^{-1}C$, then the operator family $(BR(t))_{t\geqslant 0}$ is an exponentially equicontinuous (a,k)-regularized C-resolvent family for (1.4), and the condition (ii.1) stated in the formulation of [**22**, Theorem 2.8] holds. Furthermore, for each $t \geqslant 0$ the operator $BR(t)$ can be continuously extended from $D(B)$ to the whole space E.

(iii) Assume that $(R(t))_{t\geqslant 0}$ is an exponentially equicontinuous (a,k)-regularized C-resolvent family for (1.4) and that there exists a strongly continuous operator family $(\hat{R}(t))_{t\geqslant 0} \subseteq L(E)$ such that $\hat{R}(t)x = R(t)x$, $t \geqslant 0$, $x \in D(B)$ (the last condition automatically follows from the previous one if E is complete and B is densely defined; cf. [**22**, Remark 2.3(iv)]). If $B^{-1} \in L(E)$ and $BR(t) \subseteq R(t)B$, $t \geqslant 0$, then $(R(t)B^{-1})_{t\geqslant 0}$ is an exponentially equicontinuous (a,k)-regularized C-resolvent family generated by $A,\ B$.

The proof of the following theorem can be deduced by using slight modifications of the proofs of [**1**, Proposition 2.1, Lemma 2.2] and the fact that the assertion of [**21**, Lemma 2.4] continues to hold in SCLCSs.

Theorem 2.5. *Let $(R(t))_{t\geqslant 0}$ be an exponentially equicontinuous (a,k)-regularized C-resolvent family generated by $A,\ B$, and let $abs(|a|) < \infty$. Then the following holds:*

(i) *For every $x \in E$ and for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, we have*

$$\tilde{k}(\lambda)B(B - \tilde{a}(\lambda)A)^{-1}Cx = \int_0^\infty e^{-\lambda t}BR(t)x\,dt.$$

(ii) $R(t)Bx = k(t)Cx + \int_0^t a(t-s)R(s)Ax\,ds,\ t \geqslant 0,\ x \in D(A) \cap D(B).$

(iii) $\int_0^t a(t-s)R(s)x\,ds \in D(A) \cap D(B),\ t \geqslant 0,\ x \in E.$

(iv) $BR(t)x = k(t)Cx + A\int_0^t a(t-s)R(s)x\,ds,\ t \geqslant 0,\ x \in E.$

(v) $R(t)B(D(A) \cap D(B)) \subseteq D(A) \cap D(B),\ t \geqslant 0.$

(vi) $B(B-\tilde{a}(\lambda)A)^{-1}CAx = A(B-\tilde{a}(\lambda)A)^{-1}CBx$ *for every* $x \in D(A)\cap D(B)$ *and for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega$ *and* $\tilde{k}(\lambda) \neq 0$; $AR(t)Bx = BR(t)Ax,\ t \geqslant 0,\ x \in D(A) \cap D(B).$

(vii) *Suppose that the function $k(t)$ is differentiable at a point $t_0 \geqslant 0$ and that $a \in AC_{\mathrm{loc}}([0,\infty))$. If $\lambda \in \mathbb{C}$ satisfies $\operatorname{Re}\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, then for every $j \in \mathbb{N}_0$, $z \in \mathbb{C}$ and for every complex polynomial $P(\cdot)$, we have*

$$\begin{aligned}&\Big(\frac{d}{dt}\big[(z(B - \tilde{a}(\lambda)A)^{-1}C - P(C))^j R(t)Bx\big]\Big)_{t=t_0}\\&\qquad = (z\big(B - \tilde{a}(\lambda)A)^{-1}C - P(C))^j \Big(\frac{d}{dt}R(t)Bx\Big)_{t=t_0}.\end{aligned}$$

(viii) *Let $x \in D(A) \cap D(B)$. Then the function $t \mapsto u(t)$, $t \geqslant 0$, defined by $u(t) := R(t)Bx$, $t \geqslant 0$ satisfies $u \in C([0,\infty) : [D(A)]) \cap C([0,\infty) : [D(B)])$ and*

$$Bu(t) = k(t)CBx + \int_0^t a(t-s)Au(s)ds, \quad t \geqslant 0.$$

REMARK 2.5. (i) Suppose that $x \in D(A) \cap D(B)$, $\alpha > 0$ and there exists an exponentially equicontinuous (g_α, C)-regularized resolvent family $(R(t))_{t\geqslant 0}$ generated by A, B. Using the identity $R(t)Bx = Cx + \int_0^t g_\alpha(t-s)R(s)Ax\,ds$, $t \geqslant 0$, it readily follows that the mapping $t \mapsto R(t)Bx$, $t \geqslant 0$ is $(m-1)$-times continuously differentiable on $[0,\infty)$, where $m = \lceil \alpha \rceil$. Furthermore, it can be easily verified that the Caputo derivative $\mathbf{D}_t^\alpha R(t)Bx$ is well defined as well as that $\mathbf{D}_t^\alpha R(t)Bx = R(t)Ax$, $t \geqslant 0$. Keeping in mind Remark 2.2(ii) and Proposition 2.5(vi), we get that the function $u(t) := R(t)Bx$, $t \geqslant 0$ is a unique solution of the following Cauchy problem:

$$\begin{aligned}
&u \in C([0,\infty) : [D(A)]) \cap C([0,\infty) : [D(B)]) \cap C^{m-1}([0,\infty) : E),\\
&B\mathbf{D}_t^\alpha u(t) = Au(t), \quad t \geqslant 0,\\
&u(0) = Cx; \quad u^{(j)}(0) = 0, \quad 1 \leqslant j \leqslant m-1.
\end{aligned}$$

In Theorem 2.6 below, we extend this result to the class of exponentially equicontinuous $(g_\alpha, g_{\alpha l+1})$-regularized C-resolvent families generated by A, B $(l \in \mathbb{N})$.

(ii) Now we would like to illustrate the conclusion deduced in the first part of this remark to degenerate fractional equations associated with the abstract differential operators [**20**,**57**]. For the sake of simplicity, we shall only consider the equations of order $\alpha \in (0,2)$; cf. [**22**, Subsection 4.1] for further information concerning the case $\alpha = 2$. Assume that $n \in \mathbb{N}$ and iA_j, $1 \leqslant j \leqslant n$ are commuting generators of bounded C_0-groups on a Banach space E. Suppose again that $0 < \alpha < 2$, $\omega \geqslant 0$, $P_1(x)$ and $P_2(x)$ are non-zero complex polynomials, $N_1 = dg(P_1(x))$, $N_2 = dg(P_2(x))$, $\beta > \frac{n}{2}\frac{(N_1+N_2)}{\min(1,\alpha)}$ (resp. $\beta \geqslant n|\frac{1}{p} - \frac{1}{2}|\frac{(N_1+N_2)}{\min(1,\alpha)}$, if $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$), $P_2(x) \neq 0$, $x \in \mathbb{R}^n$ and that (2.14) holds with δ replaced by α. Define $(R_\alpha(t))_{t\geqslant 0}$ as in (2.15), with δ replaced by α; $C \equiv R_\alpha(0)$. Then we know that $(R_\alpha(t))_{t\geqslant 0} \subseteq L(E)$ is a global exponentially bounded $(g_\alpha, R_\alpha(0))$-regularized resolvent family for the problem

$$(P)_R : \begin{cases} \mathbf{D}_t^\alpha \overline{P_2(A)}u(t) = \overline{P_1(A)}u(t), \quad t \geqslant 0,\\ u(0) = Cx; \quad u^{(j)}(0) = 0, \quad 1 \leqslant j \leqslant \lceil \alpha \rceil - 1; \end{cases}$$

cf. Definition 1.1 with $a(t) = g_\alpha(t)$ and $k(t) = 1$. Furthermore, the analysis contained in [**22**, Remark 4.4(i)] implies that there exists an exponentially bounded, strongly continuous operator family $(G_\alpha(t))_{t\geqslant 0}$ such that $G_\alpha(t)x = \overline{P_2(A)}^{-1}R_\alpha(t)x$, $t \geqslant 0$, $x \in E$ and $\lambda^{\alpha-1}(\lambda^\alpha B - A)^{-1}Cx = \int_0^\infty e^{-\lambda t}G_\alpha(t)x\,dt$ for any $x \in E$ and $\lambda > 0$ sufficiently large. Hence, $(G_\alpha(t))_{t\geqslant 0}$ is an exponentially equicontinuous (g_α, C)-regularized resolvent family generated by $\overline{P_1(A)}$, $\overline{P_2(A)}$, so that for each $f \in D(\overline{P_1(A)}) \cap D(\overline{P_2(A)})$, the function $u(t) := R_\alpha(t)x$, $t \geqslant 0$ is a unique solution of the following Cauchy problem:

$$(P)_L : \begin{cases} u \in C([0,\infty) : [D(\overline{P_1(A)})]) \cap C([0,\infty) : [D(\overline{P_2(A)})]), \\ \overline{P_2(A)}\mathbf{D}_t^{\alpha} u(t) = \overline{P_1(A)} u(t), \quad t \geqslant 0, \\ u(0) = Cx; \quad u^{(j)}(0) = 0, \quad 1 \leqslant j \leqslant \lceil \alpha \rceil - 1. \end{cases}$$

The consideration is quite similar in the case that the requirements of [**22**, Theorem 4.3] hold (cf. also [**22**, Remark 4.4(ii)] and [**22**, Remark 4.5], which enables us to consider the wellposedness of problem $(P)_L$ in E_l-type spaces [**54**]).

We shall employ the following auxiliary lemma in the proof of Theorem 2.6 mentioned above.

Lemma 2.1. (cf. [**20**, Corollary 2.1.20]) *Suppose $\alpha > 0$, $l \in \mathbb{N}$, $z \in \mathbb{C}$, A is a subgenerator of an exponentially equicontinuous $(g_\alpha, g_{l\alpha+1})$-regularized C-resolvent family $(S_{l,\alpha}(t))_{t\geqslant 0}$ on E, $z - A$ is injective, $R(C) \subseteq R((z-A)^l)$ and $(z-A)^{-1}C \in L(E)$, ..., $(z-A)^{-l}C \in L(E)$. Set, for every $x \in E$ and $t \geqslant 0$,*

$$S_\alpha(t)x := (-1)^l S_{l,\alpha}(t)x + \sum_{j=0}^{l-1} (-1)^{j+1} \binom{l}{j} z^{l-j} \Big[\mathcal{L}^{-1}\Big(\frac{r^{\alpha j}}{(r^\alpha - z)^l} \Big) * S_{l,\alpha}(\cdot)x \Big](t)$$
$$+ \sum_{j=1}^{l} (-1)^{l-j} \mathcal{L}^{-1}\Big(\frac{r^{\alpha-1}}{(r^\alpha - z)^{l+1-j}} \Big)(t)(z-A)^{-j}Cx.$$

Then $(S_\alpha(t))_{t\geqslant 0}$ is an exponentially equicontinuous $(g_\alpha, (z-A)^{-l}C)$-regularized resolvent family with a subgenerator A.

Now we state the following important extension of [**1**, Theorem 2.2] (cf. the forthcoming monograph [**26**] for more details about applications of Theorem 2.6 in the study of analytical solutions to fractional Barenblatt–Zheltov–Kochina equation in finite domains).

Theorem 2.6. *Suppose that $\alpha > 0$, $l \in \mathbb{N}$, $z \in \mathbb{C}$, there exists an exponentially equicontinuous $(g_\alpha, g_{l\alpha+1})$-regularized C-resolvent family $(S_{l,\alpha}(t))_{t\geqslant 0}$ generated by A, B, the operator $zB - A$ is injective and $x \in D(A) \cap D(B) \cap D(((zB-A)^{-1}B)^l C)$. Define*

$$u(t) := (-1)^l S_{l,\alpha}(t)Bx + \sum_{j=0}^{l-1} (-1)^{j+1} \binom{l}{j} z^{l-j} \Big[\mathcal{L}^{-1}\Big(\frac{r^{\alpha j}}{(r^\alpha - z)^l} \Big) * S_{l,\alpha}(\cdot)Bx \Big](t)$$
$$+ \sum_{j=1}^{l} (-1)^{l-j} \mathcal{L}^{-1}\Big(\frac{r^{\alpha-1}}{(r^\alpha - z)^{l+1-j}} \Big)(t)((zB-A)^{-1}B)^j Cx, \quad t \geqslant 0.$$

Then the function $u(t)$ is a unique solution of the problem $(\mathrm{DFP})_L$ with $f(t) \equiv 0$ and the initial value x replaced by $((zB-A)^{-1}B)^l Cx$ (we will designate this problem by $(\mathrm{DFP})_{L,l}$ in the sequel).

Proof. The uniqueness of solutions follows similarly as in Remark 2.5(i) and we shall only prove that the function $u(t)$ is a solution of the problem $(\mathrm{DFP})_{L,l}$.

Denote $x_j := ((zB - A)^{-1}B)^j Cx$ $(j \in \mathbb{N}_l^0)$, $F_{j,l}(t) := \mathcal{L}^{-1}(\frac{r^{\alpha j}}{(r^\alpha - z)^l})(t)$, $t > 0$ $(0 \leqslant j \leqslant l-1)$ and $G_{j,l}(t) := \mathcal{L}^{-1}(\frac{r^{\alpha-1}}{(r^\alpha - z)^{l+1-j}})(t)$, $t \geqslant 0$ $(1 \leqslant j \leqslant l)$. Then the function $F_{j,l}(t)$ is continuous on $(0,\infty)$, locally integrable on $[0,\infty)$ and exponentially bounded on $[1,\infty)$ $(0 \leqslant j \leqslant l-1)$, while the function $G_{j,l}(t)$ is continuous and exponentially bounded on $[0,\infty)$ $(1 \leqslant j \leqslant l)$; cf. [**20**]. Set $m := \lceil \alpha \rceil$. By Theorem 2.5(ii), we have that the mapping $t \mapsto S_{l,\alpha}(t)Bx$, $t \geqslant 0$ is $(m-1)$-times continuously differentiable and

$$\frac{d^{m-1}}{dt^{m-1}} S_{l,\alpha}(t)Bx = g_{\alpha l+2-m}(t)Cx + \int_0^t g_{\alpha+1-m}(t-s)S_{l,\alpha}(s)Ax\,ds, \quad t \geqslant 0;$$

hence $(\frac{d^j}{dt^j} S_{l,\alpha}(t)Bx)_{t=0} = 0$, $0 \leqslant j \leqslant m-1$. This simply implies that the mapping $t \mapsto [F_{j,l} * S_{l,\alpha}(\cdot)Bx](t)$, $t \geqslant 0$ is $(m-1)$-times continuously differentiable as well as that

$$\frac{d^{m-1}}{dt^{m-1}}[F_{j,l} * S_{l,\alpha}(\cdot)Bx](t) = \Big[F_{j,l} * \frac{d^{m-1}}{dt^{m-1}} S_{l,\alpha}(\cdot)Bx\Big](t), \quad t \geqslant 0,$$

provided $0 \leqslant j \leqslant l-1$; hence, $(\frac{d^p}{dt^p}[F_{j,l} * S_{l,\alpha}(\cdot)Bx](t))_{t=0} = 0$, $0 \leqslant p \leqslant m-1$ $(0 \leqslant j \leqslant l-1)$. Now it is not difficult to prove that

$$\mathbf{D}_t^\alpha S_{l,\alpha}(t)Bx = g_{l\alpha+1-\alpha}(t)Cx + S_{l,\alpha}(t)Ax, \quad t \geqslant 0,$$

$$\mathbf{D}_t^\alpha \big[F_{j,l} * S_{l,\alpha}(\cdot)Bx\big](t) = g_{l\alpha+1-\alpha}(t)Cx + S_{l,\alpha}(t)Ax,\ t \geqslant 0 \quad (0 \leqslant j \leqslant l-1).$$

Suppose, for the time being, that the assumptions of Lemma 2.1 hold. Since for each $x \in D(A)$ the function $v(t) := S_\alpha(t)x$, $t \geqslant 0$ is a unique solution of the problem

$$\begin{cases} \mathbf{D}_t^\alpha v(t) = Av(t), \quad t \geqslant 0, \\ v(0) = Cx; \quad v^{(j)}(0) = 0, \quad 1 \leqslant j \leqslant m-1, \end{cases}$$

we may conclude from the above (by plugging $l = 1, 2, \ldots$ successively in Lemma 2.1) that for each $j \in \mathbb{N}_l$ the function $G_{j,l}(t)$ is $(m-1)$-times continuously differentiable on $[0,\infty)$ as well as that $(\frac{d^p}{dt^p} G_{j,l}(t))_{t=0} = 0$, $1 \leqslant p \leqslant m-1$ $(1 \leqslant j \leqslant l)$ and that the Caputo derivative $\mathbf{D}_t^\alpha G_{j,l}(t)$ is well defined $(1 \leqslant j \leqslant l)$. Since $G_{l,l}(t) = E_\alpha(zt^\alpha)$, $t \geqslant 0$, it readily follows that the function $u(t)$ satisfies $u(0) = x_l$ and $u^{(j)}(0) = 0$, $1 \leqslant j \leqslant m-1$. It remains to be proved that $B\mathbf{D}_t^\alpha = Au(t)$, $t \geqslant 0$. Carrying out a straightforward computation, it can be easily seen that this equality holds iff

$$(-1)^l g_{\alpha l+1-\alpha}(t)BCx + \sum_{j=0}^{l-1}(-1)^{j+1}\binom{l}{j} z^{l-j}[F_{j,l} * g_{\alpha l+1-\alpha}](t)BCx$$
$$+ \sum_{j=1}^{l}(-1)^{l-j}\mathbf{D}_t^\alpha G_{j,l}(t)Bx_j = \sum_{j=1}^{l}(-1)^{l-j}G_{j,l}(t)Ax_j, \quad t \geqslant 0$$

iff

$$(-1)^l g_{\alpha l+1-\alpha}(t)BCx + \sum_{j=0}^{l-1}(-1)^{j+1}\binom{l}{j} z^{l-j}[F_{j,l} * g_{\alpha l+1-\alpha}](t)BCx$$

$$+\sum_{j=1}^{l}(-1)^{l-j}\mathbf{D}_t^{\alpha}G_{j,l}(t)Bx_j=\sum_{j=1}^{l}(-1)^{l-j}G_{j,l}(t)[zBx_j-Bx_{j-1}],\quad t\geqslant 0.$$

This is true because the coefficients of Bx_j, for every fixed number $j\in\mathbb{N}_l^0$, on both sides of previous equality are equal (cf. also the proof of [**20**, Theorem 2.1.19]). □

Suppose that the operator B is injective, $x\in D(AB^{-1})$, $\alpha>0$ and there exists an exponentially equicontinuous (g_α, C)-regularized resolvent family $(R(t))_{t\geqslant 0}$ generated by A, B. Then it can be easily checked that the function $u(t):=R(t)x$, $t\geqslant 0$ is a unique solution of problem $(\mathrm{DFP})_L$ with $f(t)\equiv 0$ and the initial value x replaced by $CB^{-1}x$. We leave to the interested reader the problem of transferring this conclusion, as well as the other ones from [**1**, Remark 2.4], to degenerate fractional equations whose solutions are goverened by $(g_\alpha, g_{\alpha l+1})$-regularized C-resolvent families generated by A, B $(l\in\mathbb{N})$.

Assume now that $n\in\mathbb{N}\smallsetminus\{1\}$, $0\leqslant\alpha_1<\cdots<\alpha_{n-1}$, and $A_1,\dots,A_{n-1}$ are closed linear operators on E. In the analysis of existence and uniqueness of integral equations associated with the problem (1.1)–(1.2), we can also use the notion of an exponentially equicontinuous (analytic) $(k;C)$-regularized resolvent (i,j)-propagation family.

Definition 2.6. (cf. Definition 2.3 and Definition 2.4) Suppose that the function $k(t)$ satisfies (P1), as well as that $1\leqslant i\leqslant n-1$, $0\leqslant j\leqslant m_i-1$ and $R_{i,j}(t)\in L(E,[D(A_i)])$ for all $t\geqslant 0$. Let the operator $C\in L(E)$ be injective.

(i) Then the operator family $(R_{i,j}(t))_{t\geqslant 0}$ is said to be an exponentially equicontinuous $(k;C)$-regularized resolvent (i,j)-propagation family for problem (1.1)–(1.2) iff there exists $\omega\geqslant\max(0,\mathrm{abs}(k))$ such that the following holds:

(a) The mappings $t\mapsto R_{i,j}(t)x$, $t\geqslant 0$ and $t\mapsto A_iR_{i,j}(t)x$, $t\geqslant 0$ are continuous for every fixed element $x\in E$.

(b) The family $\{e^{-\omega t}R_{i,j}(t):t\geqslant 0\}\subseteq L(E,[D(A_i)])$ is equicontinuous, i.e., for every $p\in\circledast$, there exist $c>0$ and $q\in\circledast$ such that

$$p(e^{-\omega t}R_{i,j}(t)x)+p(e^{-\omega t}A_iR_{i,j}(t)x)\leqslant cq(x),\quad x\in E,\quad t\geqslant 0.$$

(c) For every $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\omega$ and $\tilde{k}(\lambda)\neq 0$, the operator P_λ is injective, $R(C)\subseteq R(P_\lambda)$ and

$$\lambda^{\alpha_i-\alpha_{n-1}-j}\tilde{k}(\lambda)P_\lambda^{-1}Cx=\int_0^\infty e^{-\lambda t}R_{i,j}(t)x\,dt,\quad x\in E.$$

(ii) Let $(R_{i,j}(t))_{t\geqslant 0}$ be an exponentially equicontinuous $(k;C)$-regularized resolvent (i,j)-propagation family for problem (1.1)–(1.2). Then it is said that $(R_{i,j}(t))_{t\geqslant 0}$ is an exponentially equicontinuous (equicontinuous), analytic $(k;C)$-regularized resolvent (i,j)-propagation family for problem (1.1)–(1.2), of angle α, iff the following holds:

(a) For every $x\in E$, the mappings $t\mapsto R_{i,j}(t)x$, $t>0$ and $t\mapsto A_iR_{i,j}(t)x$, $t>0$ can be analytically extended to the sector Σ_α; since no confusion seems likely, we shall denote these extensions by the same symbols.

(b) For every $x \in E$ and $\beta \in (0, \alpha)$, one has $\lim_{z \to 0, z \in \Sigma_\beta} R_{i,j}(z)x = R_{i,j}(0)x$ and $\lim_{z \to 0, z \in \Sigma_\beta} A_i R_{i,j}(z)x = A_i R_{i,j}(0)x$.

(c) For every $\beta \in (0, \alpha)$, there exists $\omega_\beta \geqslant \max(0, \mathrm{abs}(k))$ ($\omega_\beta = 0$) such that the family $\{e^{-\omega_\beta z} R_{i,j}(z) : z \in \Sigma_\beta\} \subseteq L(E, [D(A_i)])$ is equicontinuous, i.e., for every $p \in \circledast$, there exist $c > 0$ and $q \in \circledast$ such that

$$p(e^{-\omega_\beta z} R_{i,j}(z)x) + p(e^{-\omega_\beta z} A_i R_{i,j}(z)x) \leqslant cq(x), \quad x \in E, \; z \in \Sigma_\beta.$$

Exponentially equicontinuous (analytic) $(k; C)$-regularized resolvent (i, j)-propagation families yield results very similar to those obtained by k-regularized C-resolvent (i, j)-propagation families. Without going into a deeper analysis, we shall only observe that the assertions of Theorem 2.1(i)–(iii), Remark 2.2(i), (iii), Theorem 2.3 and Theorem 2.4 can be restated for exponentially equicontinuous $(k; C)$-regularized resolvent (i, j)-propagation families. Details can be left to the interested reader.

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Faculty of Technical Sciences
University of Novi Sad, Novi Sad, Serbia
marco.s@verat.net

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