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ABOUT A CONJECTURE ON DIFFERENCE EQUATIONS IN QUASIANALYTIC CARLEMAN CLASSES

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ABSTRACT. We consider the difference equation $\sum_{j=1}^{q} a_j(x)\varphi(x+\alpha_j) = \chi(x)$ where $\alpha_1 < \cdots < \alpha_q$ ($q \geqslant 3$) are given real constants, a_j ($j = 1, \dots, q$) are given holomorphic functions on a strip $\mathbb{R}_\delta$ ($\delta > 0$) such that a_1 and a_q vanish nowhere on it, and χ is a function belonging to a quasianalytic Carleman class $C_M\{\mathbb{R}\}$. We prove, under a growth condition on the functions a_j, that the difference equation above is solvable in $C_M\{\mathbb{R}\}$.

1. Introduction

Belitskii, Dyn'kin and Tkachenko in [**1**] formulated the following conjecture.

CONJECTURE. *Let χ, a_j, $j = 1, \dots, q$, be functions in a Carleman class $C_M\{\mathbb{R}\}$ such that a_1 and a_q nowhere vanish on $\mathbb{R}$, and $\alpha_1 < \cdots < \alpha_q$ some real numbers. Then the difference equation*

$$\sum_{j=1}^{q} a_j(x)\varphi(x+\alpha_j) = \chi(x) \tag{1.1}$$

is solvable in the Carleman class $C_M\{\mathbb{R}\}$.

In that paper, the authors, relying on a result of decomposition in Carleman classes, proved the conjecture in the particular cases where the coefficients a_j are constants or when the coefficients are variables with $q = 2$. They also suggested that the same method could be used to show the solvability of equation (1.1) in a quasianalytic Carleman class $C_M\{\mathbb{R}\}$, if we assume that the functions $\frac{1}{a_1}$, $\frac{1}{a_q}$, $\frac{a_2}{a_1}, \dots, \frac{a_q}{a_1}$, $\frac{a_1}{a_q}, \dots, \frac{a_{q-1}}{a_q}$ ($q \geqslant 3$) can be continued in a strip $\mathbb{R}_\delta := \{z \in \mathbb{C} : |\operatorname{Im}(z)| < \delta\}$ as analytic functions increasing on $\mathbb{R}_\delta$, not too rapidly in infinity. As an example of such coefficients, they mentioned the class of rational functions. Our aim here is to give a precise meaning to this assertion, by proving that the result is

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true even if the functions $\frac{1}{a_1}, \frac{1}{a_q}, \frac{a_2}{a_1}, \dots, \frac{a_q}{a_1}, \frac{a_1}{a_q}, \dots, \frac{a_{q-1}}{a_q}$ have more rapid increase in infinity, provided that it is of the form $\exp(e^{C|\operatorname{Re}(z)|})$ where $C > 0$ is a constant.

2. Notations, definitions and statement of the main result

We set for every $\rho > 0$, $a \geqslant 0$

$$\begin{aligned} \mathbb{R}_\rho &:= \{z \in \mathbb{C} : |\operatorname{Im}(z)| < \rho\}, \quad \mathbb{R}_\rho^{\pm} := \{z \in \mathbb{R}_\rho : \pm\operatorname{Re}(z) > \rho\} \\ \mathbb{R}_{\rho,a} &:= \{z \in \mathbb{R}_\rho : |\operatorname{Re}(z)| \leqslant a\} \\ \Delta_\rho &:= \{z \in \mathbb{C} : |z| < \rho\}, \quad \Delta_\rho^{\pm} := \{z \in \Delta_\rho : \pm\operatorname{Re}(z) \leqslant 0\} \\ \Gamma_\rho &:= \{z \in \mathbb{C} : |z| = \rho\}, \quad \Gamma_\rho^{\pm} := \{z \in \Gamma_\rho : \pm\operatorname{Re}(z) \leqslant 0\} \end{aligned}$$

For every nonempty subset V of $\mathbb{C}$ and every $z \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we set

$$\begin{gathered} V^{(0)} := V, \quad V^{(n)} := \{u_1 + \dots + u_n : u_j \in V, \ j = 1, \dots, n\}, \quad n \geqslant 1 \\ z + V := \{z + u : u \in V\}, \quad z - V := \{z - u : u \in V\} \end{gathered}$$

Denote by $dm(\zeta)$ the Lebesgue measure on $\mathbb{C}$. Let S be a nonempty subset of $\mathbb{C}$. By $O(S)$ we denote the set of holomorphic functions on some neighborhood of S. Let $F : U \subset \mathbb{C} \to \mathbb{C}$ be a function of class C^1 on an open subset U of $\mathbb{C}$. For all $z \in U$ we set

$$\overline{\partial} F(z) := \frac{1}{2}\Big[\frac{\partial F}{\partial x}(z) + i\,\frac{\partial F}{\partial y}(z)\Big];$$

$\overline{\partial}$ is called the operator of Cauchy–Riemann.

Let $M := (M_n)_{n \geqslant 0}$ be a sequence of strictly positive real numbers. The Carleman class $C_M\{\mathbb{R}\}$ is the set of all functions $f : \mathbb{R} \to \mathbb{C}$ of class C^∞ such that

$$\|f^{(n)}\|_{\infty,I} \leqslant C_I \rho_I^n M_n, \quad n \in \mathbb{N}$$

for every compact interval I of $\mathbb{R}$ with some constants $C_I, \rho_I > 0$. The Carleman class $C_M\{\mathbb{R}\}$ is said to be quasinalytic if every function $f \in C_M\{\mathbb{R}\}$ such that $f^{(n)}(u) = 0$ for some $u \in \mathbb{R}$ and every $n \in \mathbb{N}$ is identically equal to 0. The Carleman class $C_M\{\mathbb{R}\}$ is called regular when the following conditions hold

$$\begin{gathered} \Big(\frac{M_{n+1}}{(n+1)!}\Big)^2 \leqslant \frac{M_n}{n!}\frac{M_{n+2}}{(n+2)!}, \quad n \in \mathbb{N} \\ \sup_{n \in \mathbb{N}}\Big(\frac{M_{n+1}}{(n+1)M_n}\Big)^{\frac{1}{n}} < +\infty, \\ \lim_{n \to +\infty} M_n^{\frac{1}{n}} = +\infty \end{gathered}$$

To the Carleman $C_M\{\mathbb{R}\}$ we associate its weight H_M defined by the following relation

$$H_M(x) := \lim_{n \in \mathbb{N}}\Big[\frac{M_n}{n!}x^n\Big], \quad x > 0$$

In this paper, the following result will play a crucial role.

THEOREM 2.1. [**3**] *We assume that the Carleman class $C_M\{\mathbb{R}\}$ is regular. A function $f:\mathbb{R}\to\mathbb{C}$ belongs to $C_M\{\mathbb{R}\}$ if and only if there exists for every compact interval I of $\mathbb{R}$ a compactly supported function $F:\mathbb{C}\to\mathbb{C}$ of class C^1 such that F is an extension to $\mathbb{C}$ of the restriction to I of the function f and satisfies the following estimate*

$$|\overline{\partial}F(z)| \leqslant A_I H_M(B_I|\operatorname{Im}(z)|), \quad z\in\mathbb{C}$$

where $A_I, B_I > 0$ are constants.

Throughout the paper, we assume that the Carleman class $C_M\{\mathbb{R}\}$ is regular and quasianalytic. Our main result is the following.

THEOREM 2.2. *Let $q\in\mathbb{N}^*\smallsetminus\{1,2\}$, $\delta>0$, $\chi\in C_M\{\mathbb{R}\}$, and $a_j\in O(\mathbb{R}_\delta)$ $(j=1,\dots,q)$ such that a_1 and a_q nowhere vanish on $\mathbb{R}_\delta$. We assume that the following growth condition holds*

$$\sup_{z\in\mathbb{R}_\delta}\Big(\sum_{j=2}^{q}\Big|\frac{a_j(z)}{a_1(z)}\Big| + \sum_{j=1}^{q-1}\Big|\frac{a_j(z)}{a_q(z)}\Big| + \frac{1}{|a_1(z)|} + \frac{1}{|a_q(z)|}\Big)e^{-e^{C|\operatorname{Re} z|}} < +\infty \tag{2.1}$$

for a constant $C>0$. Then difference equation (1.1) *is solvable in the class $C_M\{\mathbb{R}\}$.*

3. Proof of the main result

Let us first prove the following lemma.

LEMMA 3.1. *Given $f\in C_M\{\mathbb{R}\}$, $C_0>0$ and $\rho\in\left]0;\frac{\pi}{2C_0}\right[$, there exist two functions $f_\pm : \big(\mathbb{C}\smallsetminus\Delta_\rho^\pm\big)\cup\mathbb{R}\to\mathbb{C}$ which are holomorphic on $\mathbb{C}\smallsetminus\big(\Gamma_\rho^\pm\cup\Delta_\rho^\pm\big)$, whose restrictions to $\mathbb{R}$ belong to $C_M\{\mathbb{R}\}$, and such that the following conditions hold*

$$f(x) = f_+(x) + f_-(x), \quad x\in[-\rho,\rho]$$
$$|f_\pm(z)| \leqslant D_0\exp\big(-\cos(\rho C_0)\,e^{C_0|\operatorname{Re}(z)|}\big), \quad z\in\mathbb{R}_\rho^\pm$$

where $D_0>0$ is a constant.

PROOF. Since f belongs to $C_M\{\mathbb{R}\}$, there exists, according to Dyn'kin's theorem [**3**], a compactly supported function $F:\mathbb{C}\to\mathbb{C}$ of class C^1 such that F is an extension of the restriction of f to the interval $[-\rho,\rho]$ and satisfies the following estimate

$$|\overline{\partial}F(z)| \leqslant AH_M(B|\operatorname{Im}(z)|), \quad z\in\mathbb{C}$$

where $A, B>0$ are constants. Following the same approach as that of [**1**, pp. 34,35], [**2**, pp. 148–150], but using the Cauchy–Pompeiu formula on the disk Δ_ρ, for the function $\exp(e^{C_0z}+e^{-C_0z})f(z)$, we show that the functions

$$\begin{aligned} f_\pm(z) = {}& \frac{1}{2i\pi}\exp(-e^{C_0z}-e^{-C_0z})\int_{\Gamma_\rho^\pm}\frac{\exp(e^{C_0\zeta}+e^{-C_0\zeta})F(\zeta)}{\zeta-z}\,d\zeta \\ & -\frac{1}{\pi}\exp(-e^{C_0z}-e^{-C_0z})\iint_{\Delta_\rho^\pm}\frac{\exp(e^{C_0\zeta}+e^{-C_0\zeta})\overline{\partial}F(\zeta)}{\zeta-z}\,dm(\zeta)\end{aligned}$$

satisfy the required conditions. □

Now we set

$$\beta_j := \alpha_j - \alpha_1, \ \ j = 2, \dots, q, \qquad b_j(z) := -\frac{a_j(z)}{a_1(z)}, \ \ z \in \mathbb{R}_\delta, \ \ j = 2, \dots, q$$
$$\gamma_j := \alpha_q - \alpha_j, \ \ j = 1, \dots, q-1, \quad c_j(z) := -\frac{a_j(z)}{a_q(z)}, \ \ z \in \mathbb{R}_\delta, \ \ j = 1, \dots, q-1$$

Let $C_1 > C\big(\frac{\beta_q}{\beta_2} + \frac{\gamma_1}{\gamma_{q-1}}\big)$ and $\delta_0 \in \big]0, \min(\delta, \frac{\pi}{2C_1})\big[$. Then according to the lemma above, there exists a constant $D_1 > 0$ and two functions $\chi_\pm : (\mathbb{C} \smallsetminus \Delta^\pm_{\delta_0}) \cup \mathbb{R} \to \mathbb{C}$ which are holomorphic on $\mathbb{C} \smallsetminus (\Gamma^\pm_{\delta_0} \cup \Delta^\pm_{\delta_0})$, whose restrictions to $\mathbb{R}$ belong to $C_M\{\mathbb{R}\}$, and such that the following conditions hold

$$\begin{aligned} &\chi(x) = \chi_+(x) + \chi_-(x), \ \ x \in [-\delta_0, \delta_0], \\ &|\chi_\pm(z)| \leqslant D_1 \exp\!\big(-\cos(C_1\delta_0) e^{C_1|\operatorname{Re}(z)|}\big), \ \ z \in \mathbb{R}^\pm_{\delta_0}. \end{aligned} \tag{3.1}$$

Let $(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ be the sequences of complex valued functions defined on the strip $\mathbb{R}_{\delta_0}$ by

$$g_0(z) := \frac{\chi_+(z)}{a_1(z)}, \quad g_{n+1}(z) := \sum_{j=2}^{q} b_j(z) g_n(z + \beta_j),$$
$$h_0(z) := \frac{\chi_-(z)}{a_q(z)}, \quad h_{n+1}(z) := \sum_{j=1}^{q-1} c_j(z) h_n(z - \gamma_j).$$

It is clear that all the functions $g_{n|\mathbb{R}}$ and $h_{n|\mathbb{R}}$ belong to $C_M\{\mathbb{R}\}$.

Let us set

$$K_1 := \{\beta_j : j = 2, \dots, q\}, \quad K_2 := \{\gamma_j : j = 1, \dots, q-1\}.$$

It follows from (2.1) that we have for every $n \in \mathbb{N}$, $z \in \mathbb{R}_{\delta_0}$

$$|g_{n+1}(z)| \leqslant \exp(Le^{C|\operatorname{Re}(z)|}) \max_{u \in z + K_1} |g_n(u)|,$$
$$|h_{n+1}(z)| \leqslant \exp(Le^{C|\operatorname{Re}(z)|}) \max_{u \in z - K_2} |h_n(u)|$$

where $L > 1$ is a constant. Then we have for all $n \in \mathbb{N}^*$, $z \in \mathbb{R}_{\delta_0}$

$$\begin{aligned} |g_n(z)| &\leqslant \exp\Big(\sum_{j=0}^{n-1} Le^{C\ (|\operatorname{Re}(z)| + j\beta_q)}\Big) \max_{u \in z + K_1^{(n)}} |g_0(u)| \\ &\leqslant \exp\big(nLe^{C(|\operatorname{Re}(z)| + n\beta_q)}\big) \max_{u \in z + K_1^{(n)}} |\chi_+(u)|, \\ |h_n(z)| &\leqslant \exp\Big(\sum_{j=0}^{n-1} Le^{C(|\operatorname{Re}(z)| + j\gamma_1|)}\Big) \max_{u \in z - K_2^{(n)}} |h_0(u)| \\ &\leqslant \exp\big(nLe^{C(|\operatorname{Re}(z)| + n\gamma_1)}\big) \max_{u \in z - K_2^{(n)}} |\chi_-(u)|. \end{aligned}$$

Let $a > 0$. There exists $N_a \in \mathbb{N}^*$ such that $(\beta_2 + \gamma_{q-1})N_a \geqslant a$ and

$$z + K_1^{(n)} \subset \mathbb{R}_{\delta_0}^{+}, \quad n \geqslant N_a, \quad z \in \mathbb{R}_{\delta_0,a},$$
$$z - K_2^{(n)} \subset \mathbb{R}_{\delta_0}^{-}, \quad n \geqslant N_a, \quad z \in \mathbb{R}_{\delta_0,a}.$$

It follows then from (3.1) that we have for all $n \geqslant N_a$, $z \in \mathbb{R}_{\delta_0,a}$

$$\begin{aligned}\max_{u\in z+K_1^{(n)}} |\chi_+(u)| &\leqslant D_1 \exp\Big(- \cos(C_1\delta_0) \exp\Big(C_1 \min_{u\in z+K_1^{(n)}} |\operatorname{Re}(u)|\Big)\Big)\\ &\leqslant D_1 \exp\big(- \cos(C_1\delta_0) e^{C_1(-a+n\beta_2)}\big),\\ \max_{u\in z-K_1^{(n)}} |\chi_-(u)| &\leqslant D_1 \exp\Big(- \cos(C_1\delta_0) \exp\Big(C_1 \min_{u\in z-K_1^{(n)}} |\operatorname{Re}(u)|\Big)\Big)\\ &\leqslant D_1 \exp\big(- \cos(C_1\delta_0) e^{C_1(-a+n\gamma_{q-1})}\big).\end{aligned}$$

Consequently we have for all $n \geqslant N_a$, $z \in \mathbb{R}_{\delta_0,a}$

$$|g_n(z)| \leqslant D_1 \exp\big(nLe^{C(a+n\beta_q)} - \cos(C_1\delta_0) e^{C_1(-a+n\beta_2)}\big),$$
$$|h_n(z)| \leqslant D_1 \exp\big(nLe^{C(a+n\gamma_1)} - \cos(C_1\delta_0) e^{C_1(-a+n\gamma_{q-1})|}\big).$$

On the other hand we have

$$nLe^{C(a+n\beta_q)} = \underset{n\to+\infty}{o}\big[\cos(C_1\delta_0) e^{C_1(-a+n\beta_2)}\big],$$
$$nLe^{C(a+n\gamma_1)} = \underset{n\to+\infty}{o}\big[\cos(C_1\delta_0) e^{C_1(-a+n\gamma_{q-1})|}\big].$$

So, there exist real constants $D_a > 0$ and $E_a > 0$ and an integer $P_a \geqslant N_a$ such that the following inequalities hold

$$|g_n(z)| \leqslant D_a \exp\big(- E_a e^{C_1(-a+n\beta_2)}\big), \quad z \in \mathbb{R}_{\delta_0,a}, \quad n \geqslant P_a,$$
$$|h_n(z)| \leqslant D_a \exp\big(- E_a e^{C_1(-a+n\gamma_{q-1})|}\big), \quad z \in \mathbb{R}_{\delta_0,a}, \quad n \geqslant P_a.$$

It follows that the function series $\sum g_{n|\mathbb{R}_{\delta_0}}$ and $\sum h_{n|\mathbb{R}_{\delta_0}}$ are uniformly convergent on every compact subset of $\mathbb{R}_{\delta_0}$ and that the functions $\sum_{n=P_a}^{+\infty} g_n$ and $\sum_{n=P_a}^{+\infty} h_n$ are holomorphic on $\mathbb{R}_{\delta_0,a}$ for every $a > 0$. Let G_+ and G_- be the sums of $\sum g_n|_{\mathbb{R}_{\delta_0}}$ and $\sum h_n|_{\mathbb{R}_{\delta_0}}$, respectively. Since all the functions $g_{n|\mathbb{R}}$ and $h_{n|\mathbb{R}}$ belong to $C_M\{\mathbb{R}\}$, it follows that the functions $g_+ := G_+|_{\mathbb{R}}$ and $g_- := G_-|_{\mathbb{R}}$ belong to $C_M\{\mathbb{R}\}$. Elementary computations show that

$$\sum_{j=1}^{q} a_j(x)\, g_+(x+\alpha_j) = \chi_+(x), \quad x \in \mathbb{R},$$
$$\sum_{j=1}^{q} a_j(x)\, g_-(x+\alpha_j) = \chi_-(x), \quad x \in \mathbb{R}.$$

Then it follows from (3.1) that the function $g := g_+ + g_-$ is a solution on the interval $[-\delta_0, \delta_0]$ of the difference equation (1.1). But the function

$$x \mapsto \sum_{j=1}^{q} a_j(x)\, g(x+\alpha_j) - \chi(x)$$

belongs to the quasianalytic Carleman class $C_M\{\mathbb{R}\}$. Consequently the function $g \in C_M\{\mathbb{R}\}$ is a solution on $\mathbb{R}$ of difference equation (1.1). The proof of the main result is complete.

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