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NEIGHBOURHOOD CONDITIONS FOR FRACTIONAL ID-[A, B]-FACTOR-CRITICAL GRAPHS

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Abstract. A graph G is fractional ID-$[a, b]$-factor-critical if $G - I$ has a fractional $[a, b]$-factor for every independent set I of G. We extend a result of Zhou and Sun concerning fractional ID-k-factor-critical graphs.

1. Introduction

The graphs considered in this paper will be finite and undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex x of G, we use $d_G(x)$ and $N_G(x)$ to denote the degree of x in G and the neighbourhood of x in G, respectively.

Let a and b be two integers with $2 \leqslant a \leqslant b$. A spanning subgraph F of G is called an $[a, b]$-factor if $a \leqslant d_F(x) \leqslant b$ for each $x \in V(G)$. If $a = b = k$, then an $[a, b]$-factor is called a k-factor.

Let $h\colon E(G) \to [0, 1]$ be a function. If $a \leqslant \sum_{x \in e} h(e) \leqslant b$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional $[a, b]$-factor of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. If $a = b = k$, then a fractional $[a, b]$-factor is called a fractional k-factor. A graph G is fractional ID-$[a, b]$-factor-critical if $G - I$ has a fractional $[a, b]$-factor for every independent set I of G.

Many authors have investigated $[a, b]$-factors [**3**, **4**, **6**–**8**] and fractional factors [**9**, **11**]. The following results on fractional ID-k-factor-critical and fractional ID-$[a, b]$-factor-critical graphs are known.

Theorem 1.1. *Let G be a graph, and let k be an integer with $k \geqslant 1$. If*

$$\alpha(G) \leqslant \frac{4k(\delta(G) - k + 1)}{k^2 + 6k + 1},$$

then G is fractional ID-k-factor-critical.

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THEOREM 1.2. *Let k be an integer with $k \geqslant 2$, and let G be a graph of order n with $n \geqslant 9k-14$. Suppose for any subset $X \subset V(G)$, we have*

$$N_G(X) = V(G) \quad \text{if} \quad |X| \geqslant \left\lfloor \frac{kn}{3k-1} \right\rfloor; \quad \text{or}$$
$$|N_G(X)| \geqslant \frac{3k-1}{k}|X| \quad \text{if} \quad |X| < \left\lfloor \frac{kn}{3k-1} \right\rfloor.$$

Then G is fractional ID-k-factor-critical.

THEOREM 1.3. *Let G be a graph of order n, and let a and b be two integers with $1 \leqslant a \leqslant b$. If $n \geqslant \frac{(a+2b)(a+b-2)+1}{b}$ and $\delta(G) \geqslant \frac{(a+b)n}{a+2b}$, then G is fractional ID-$[a,b]$-factor-critical.*

In this paper, we study fractional ID-$[a,b]$-factor-critical graph, and then get a neighbourhood condition for a graph to be fractional ID-$[a,b]$-factor-critical, which is an extension of Theorem 1.2.

2. Main result

We first show the main result in this paper.

THEOREM 2.1. *Let a and b be two integers with $2 \leqslant a \leqslant b$, and let G be a graph of order n with $n > 1 + \frac{(a+2b-1)(2a+b-4)}{b}$. Suppose that for any subset $X \subset V(G)$, we have*

$$N_G(X) = V(G) \quad \text{if} \quad |X| \geqslant \left\lfloor \frac{bn}{a+2b-1} \right\rfloor; \quad \text{or}$$
$$|N_G(X)| \geqslant \frac{a+2b-1}{b}|X| \quad \text{if} \quad |X| < \left\lfloor \frac{bn}{a+2b-1} \right\rfloor.$$

Then G is fractional ID-$[a,b]$-factor-critical.

We give two lemmas which play an important role in the proof of Theorem 2.1.

LEMMA 2.1. *Let G be a graph. Then G has a fractional $[a,b]$-factor if and only if for every subset S of $V(G)$, we have $\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \geqslant 0$, where $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \leqslant a-1\}$ and $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$.*

LEMMA 2.2. *Let G be a graph which satisfies the assumption of Theorem 2.1. Then $\delta(G) \geqslant \frac{(a+b-1)n+b}{a+2b-1}$.*

PROOF. Suppose that x is a vertex of G with degree $\delta(G)$. Set $X = V(G) \setminus N_G(x)$. Notice that $x \notin N_G(X)$ and $N_G(X) \neq V(G)$. Hence, we obtain

$$n-1 \geqslant |N_G(X)| \geqslant \frac{a+2b-1}{b}|X|,$$

which implies

$$|X| \leqslant \frac{b(n-1)}{a+2b-1}. \tag{2.1}$$

Using (2.1) and $|X| = n - \delta(G)$, we have $n - \delta(G) \leqslant \frac{b(n-1)}{a+2b-1}$, wherefrom

$$\delta(G) \geqslant n - \frac{b(n-1)}{a+2b-1} = \frac{n(a+b-1)+b}{a+2b-1}.$$

Thus, Lemma 2.2 holds. □

PROOF OF THEOREM 2.1. Let I be an independent set of G and $H = G - I$. We prove the theorem by contradiction. Suppose that H has no fractional $[a, b]$-factor. By Lemma 2.1, there exists some subset $S \subset V(H)$ such that

$$\delta_H(S,T) = b|S| + d_{H-S}(T) - a|T| \leqslant -1. \tag{2.2}$$

We choose S and T such that $|T|$ is as small as possible. According to (2.2) and $H = G - I$, we obtain

$$\delta_H(S,T) = b|S| + d_{G-I-S}(T) - a|T| \leqslant -1, \tag{2.3}$$

where $T = \{x : x \in V(G) \setminus (I \cup S),\ d_{G-I-S}(x) \leqslant a-1\}$.

If $T = \phi$, then from (2.3), we have $-1 \geqslant \delta_H(S,T) = b|S| \geqslant 0$, a contradiction. Hence, $T \neq \phi$. Define $h = \min\{d_{G-I-S}(x) : x \in T\}$ and choose $x_1 \in T$ subject to $d_{G-I-S}(x_1) = h$. According to the definition of T, we have $0 \leqslant h \leqslant a-1$. Obviously, it holds that

$$\delta(G) \leqslant d_G(x_1) \leqslant d_{G-I-S}(x_1) + |I| + |S| = h + |I| + |S|.$$

So, $|S| \geqslant \delta(G) - h - |I|$. Combining this with Lemma 2.2, we get

$$|S| \geqslant \delta(G) - h - |I| \geqslant \frac{n(a+b-1)+b}{a+2b-1} - h - |I|. \tag{2.4}$$

Claim 1. $|I| \leqslant \frac{b(n-1)}{a+2b-1}$. Indeed, For arbitrary $x \in I$, we have $d_G(x) \geqslant \delta(G)$. Since I is independent, we obtain $n \geqslant d_G(x) + |I| \geqslant \delta(G) + |I|$, which implies $|I| \leqslant n - \delta(G)$. Combining this with Lemma 2.2, we get

$$|I| \leqslant n - \delta(G) \leqslant n - \frac{n(a+b-1)+b}{a+2b-1} = \frac{b(n-1)}{a+2b-1}.$$

Claim 2. $1 \leqslant h \leqslant a-1$. Assume $h = 0$ and set $\lambda = |\{x : x \in T,\ d_{G-I-S}(x) = 0\}|$. Then, we have $\lambda \geqslant 1$. Let $Y = V(G) \setminus (I \cup S)$. In view of $h = 0$ and $T \subset Y$, we get

$$N_G(Y) \neq V(G). \tag{2.5}$$

If $|Y| \geqslant \lfloor \frac{bn}{a+2b-1} \rfloor$, then by the assumption of Theorem 2.1 we obtain $N_G(Y) = V(G)$, which contradicts (2.5). Therefore, $|Y| < \lfloor \frac{bn}{a+2b-1} \rfloor$. Combining this with the condition of Theorem 2.1, we get

$$n - \lambda \geqslant |N_G(Y)| \geqslant \frac{a+2b-1}{b}|Y| = \frac{a+2b-1}{b}(n - |I| - |S|),$$

and then,

$$|S| \geqslant n - |I| - \frac{b(n-\lambda)}{a+2b-1} = \frac{n(a+b-1)+b\lambda}{a+2b-1} - |I|. \tag{2.6}$$

Using (2.6), $2 \leqslant a \leqslant b$, Claim 1, $\lambda \geqslant 1$ and the fact that $|I| + |S| + |T| \leqslant n$, we obtain

$$\begin{aligned}
-1 &\geqslant \delta_H(S,T) = b|S| + d_{G-I-S}(T) - a|T| \\
&\geqslant b|S| + |T| - \lambda - a|T| = b|S| - (a-1)|T| - \lambda \\
&\geqslant b|S| - (a-1)(n - |I| - |S|) - \lambda \\
&= (a+b-1)|S| + (a-1)|I| - (a-1)n - \lambda \\
&\geqslant (a+b-1)\Big[\frac{n(a+b-1)+b\lambda}{a+2b-1} - |I|\Big] + (a-1)|I| - (a-1)n - \lambda \\
&= \frac{b^2 n}{a+2b-1} - b|I| + \frac{\lambda(ab+b^2-a-3b+1)}{a+2b-1} \\
&\geqslant \frac{b^2 n}{a+2b-1} - \frac{b^2(n-1)}{a+2b-1} + \frac{ab+b^2-a-3b+1}{a+2b-1} \\
&= \frac{2b^2+ab-3b-a+1}{a+2b-1} = \frac{(b-1)(2b-1)+a(b-1)}{a+2b-1} = b-1 \geqslant 1 > 0,
\end{aligned}$$

a contradiction. So $1 \leqslant h \leqslant a-1$.

Next we shall consider two cases and derive contradictions in each case.

Case 1. $|N_G(T)| < \frac{a+2b-1}{b}|T|$. First, we prove the following claim.

Claim 3. $N_G(T) = V(G)$. Suppose, to the contrary, that $N_G(T) \neq V(G)$. By the assumption of Theorem 2.1, we get $|T| < \lfloor \frac{bn}{a+2b-1} \rfloor$. (Or, $|T| \geqslant \lfloor \frac{bn}{a+2b-1} \rfloor$ and so $N_G(T) = V(G)$, which contradicts $N_G(T) \neq V(G)$.) Since $|T| < \lfloor \frac{bn}{a+2b-1} \rfloor$, according to the assumption of Theorem 2.1 we have $|N_G(T)| \geqslant \frac{a+2b-1}{b}|T|$, which contradicts $|N_G(T)| < \frac{a+2b-1}{b}|T|$. Therefore, we obtain $N_G(T) = V(G)$.

From Claim 2, we know that $1 \leqslant h \leqslant a-1$. In the following we consider two subcases with $h = 1$ and $2 \leqslant h \leqslant a-1$.

Subcase 1.1. $h = 1$. According to $|N_G(T)| < \frac{a+2b-1}{b}|T|$ and Claim 3, we get

$$|T| > \frac{b|N_G(T)|}{a+2b-1} = \frac{bn}{a+2b-1}.$$

Note that $\lfloor \frac{bn}{a+2b-1} \rfloor \leqslant \frac{bn}{a+2b-1}$. Then we have $|T| > \frac{bn}{a+2b-1} \geqslant \left\lfloor \frac{bn}{a+2b-1} \right\rfloor$. By the integrity of $|T|$, we obtain

$$|T| \geqslant \left\lfloor \frac{bn}{a+2b-1} \right\rfloor + 1. \tag{2.7}$$

From (2.7), $T \subset V(G) - I - S$ and $d_{G-I-S}(x_1) = h = 1$, we get

$$|T \setminus N_G(x_1)| = |T \setminus N_{G-I-S}(x_1)| \geqslant |T| - 1 \geqslant \left\lfloor \frac{bn}{a+2b-1} \right\rfloor. \tag{2.8}$$

In view of (2.8) and the assumption of Theorem 2.1, we have

$$N_G(T \setminus N_G(x_1)) = V(G). \tag{2.9}$$

However, it is obvious that $x_1 \notin N_G(T \setminus N_G(x_1))$, which contradicts (2.9).

Subcase 1.2. $2 \leqslant h \leqslant a-1$. By $N_G(T) < \frac{a+2b-1}{b}|T|$, Claim 3 and $n > 1 + \frac{(a+2b-1)(2a+b-4)}{b}$, we have

$$|T| > \frac{bn}{a+2b-1} > \frac{b}{a+2b-1}\Big[1 + \frac{(a+2b-1)(2a+b-4)}{b}\Big] > a-1 \geqslant h,$$

and so $T \setminus N_G(x_1) = T \setminus N_{G-I-S}(x_1) \neq \phi$. Obviously, $x_1 \notin N_G(T \setminus N_G(x_1))$ and so $N_G(T \setminus N_G(x_1)) \neq V(G)$. Hence, we have $|T \setminus N_G(x_1)| < \lfloor \frac{bn}{a+2b-1} \rfloor$(Otherwise, $|T \setminus N_G(x_1)| \geqslant \lfloor \frac{bn}{a+2b-1} \rfloor$, then from Theorem 2.1 we have $N_G(T \setminus N_G(x_1)) = V(G)$, a contradiction).

According to the assumption of Theorem 2.1 and $T \subset V(G) - I - S$, we get

$$\begin{aligned} n-1 &\geqslant |N_G(T \setminus N_G(x_1))| \geqslant \frac{a+2b-1}{b}|T \setminus N_G(x_1)| \\ &= \frac{a+2b-1}{b}|T \setminus N_{G-I-S}(x_1)| \geqslant \frac{a+2b-1}{b}(|T|-h), \end{aligned}$$

which implies

$$|T| \leqslant h + \frac{b(n-1)}{a+2b-1}. \tag{2.10}$$

According to Claim 1, (2.3), (2.4) and (2.10), we have

$$\begin{aligned} -1 &\geqslant \delta_H(S,T) = b|S| + d_{G-I-S}(T) - a|T| \\ &\geqslant b|S| + h|T| - a|T| = b|S| - (a-h)|T| \\ &\geqslant b\Big[\frac{n(a+b-1)+b}{a+2b-1} - h - |I|\Big] - (a-h)\Big[h + \frac{b(n-1)}{a+2b-1}\Big] \\ &\geqslant b\Big[\frac{n(a+b-1)+b}{a+2b-1} - h - \frac{b(n-1)}{a+2b-1}\Big] - (a-h)\Big[h + \frac{b(n-1)}{a+2b-1}\Big] \\ &= h^2 - h\Big[a+b-\frac{b(n-1)}{a+2b-1}\Big] + \frac{b(a+2b-n)}{a+2b-1}, \end{aligned}$$

that is,

$$-1 \geqslant \delta_H(S,T) \geqslant h^2 - h\Big[a+b-\frac{b(n-1)}{a+2b-1}\Big] + \frac{b(a+2b-n)}{a+2b-1}. \tag{2.11}$$

Set

$$f(h) = h^2 - h\big[a+b-\frac{b(n-1)}{a+2b-1}\big] + \frac{b(a+2b-n)}{a+2b-1}.$$

Then by $2 \leqslant h \leqslant a-1$, $n > 1 + \frac{(a+2b-1)(2a+b-4)}{b}$ and $a \geqslant 2$, we obtain

$$f'(h) = 2h - \Big[a+b-\frac{b(n-1)}{a+2b-1}\Big] \geqslant 4-a-b+\frac{b(n-1)}{a+2b-1} \geqslant 4-a-b+2a+b-4 = a \geqslant 2.$$

It is obvious that the function $f(h)$ attains its minimum value at $h=2$, since $2 \leqslant h \leqslant a-1$. Thus, by (2.11), we have

$$\begin{aligned} -1 &\geqslant \delta_H(S,T) \geqslant f(h) \geqslant f(2) \\ &= 4 - 2(a+b) + \frac{2b(n-1)}{a+2b-1} + \frac{b(a+2b-n)}{a+2b-1} \end{aligned}$$

$$= 4 - 2(a+b) + \frac{2b(n-1)}{a+2b-1} + \frac{b(a+2b-1) - b(n-1)}{a+2b-1}$$
$$= 4 - 2(a+b) + b + \frac{b(n-1)}{a+2b-1}$$
$$\geqslant 4 - 2(a+b) + b + 2a + b - 4 = 0,$$

a contradiction.

Case 2. $|N_G(T)| \geqslant \frac{a+2b-1}{b}|T|$. In this case, we have

$$|T| \leqslant \frac{b}{a+2b-1}|N_G(T)| \leqslant \frac{bn}{a+2b-1}. \tag{2.12}$$

Subcase 2.1. $h = 1$.

Subsubcase 2.1.1. $|T| > \frac{b(n-1)}{a+2b-1}$. From (2.4) and $h = 1$, we get

$$|S| + |I| + |T| > \frac{n(a+b-1)+b}{a+2b-1} - 1 + \frac{b(n-1)}{a+2b-1} = n - 1. \tag{2.13}$$

On the other hand, $|S| + |I| + |T| \leqslant n$. Combining this with (2.13), we get

$$|S| + |I| + |T| = n. \tag{2.14}$$

Using Claim 1, (2.12) and (2.14), we have

$$\begin{aligned}\delta_H(S,T) &= b|S| + d_{G-I-S}(T) - a(T)\\ &= b(n - |I| - |T|) + d_{G-I-S}(T) - a|T|\\ &\geqslant b(n - |I| - |T|) + |T| - a|T|\\ &= b(n - |I|) - (a+b-1)|T|\\ &\geqslant b\Big[n - \frac{b(n-1)}{a+2b-1}\Big] - (a+b-1)\cdot\frac{bn}{a+2b-1}\\ &= \frac{b^2}{a+2b-1} > 0,\end{aligned}$$

which contradicts (2.3).

Subsubcase 2.1.2. $|T| \leqslant \frac{b(n-1)}{a+2b-1}$. We write $m = |\{x : x \in T, d_{G-I-S}(x) = 1\}|$. Clearly, $|T| \geqslant m$. In view of (2.3), (2.4), Claim 1, $h = 1$, $|T| \leqslant \frac{b(n-1)}{a+2b-1}$ and $|T| \geqslant m$, we get

$$\begin{aligned}-1 &\geqslant \delta_H(S,T) = b|S| + d_{G-I-S}(T) - a|T|\\ &\geqslant b|S| + 2|T| - m - a|T| = b|S| - (a-2)|T| - m\\ &\geqslant b\Big[\frac{n(a+b-1)+b}{a+2b-1} - 1 - |I|\Big] - (a-2)\cdot\frac{b(n-1)}{a+2b-1} - m\\ &\geqslant b\Big[\frac{n(a+b-1)+b}{a+2b-1} - 1 - \frac{b(n-1)}{a+2b-1}\Big] - (a-2)\cdot\frac{b(n-1)}{a+2b-1} - m\\ &= \frac{b(n-1)}{a+2b-1} - m \geqslant |T| - m \geqslant 0,\end{aligned}$$

a contradiction.

Subcase 2.2. $2 \leqslant h \leqslant a-1$. According to (2.4), (2.12) and Claim 1, $n > 1 + \frac{(a+2b-1)(2a+b-4)}{b}$ and $h \geqslant 2$, we have

$$\begin{aligned}
\delta_H(S,T) &= b|S| + d_{G-I-S}(T) - a|T| \\
&\geqslant b\Big[n - \frac{b(n-1)}{a+2b-1} - |I| - h\Big] + h|T| - a|T| \\
&\geqslant b\Big[n - \frac{2b(n-1)}{a+2b-1} - h\Big] - \frac{bn(a-h)}{a+2b-1} \\
&= bn - \frac{2b^2(n-1)}{a+2b-1} - \frac{bn(a-h)}{a+2b-1} - bh \\
&= bn - \frac{2b^2(n-1)}{a+2b-1} - \frac{b(a-h)(n-1)}{a+2b-1} - \frac{b(a-h)}{a+2b-1} - bh \\
&= bn - \frac{2b^2(n-1)}{a+2b-1} - \frac{ab(n-1)}{a+2b-1} + \frac{bh(n-1)}{a+2b-1} - \frac{b(a-h)}{a+2b-1} - bh \\
&= bn - (n-1)\cdot\frac{b(a+2b)}{a+2b-1} + h\Big[\frac{b(n-1)}{a+2b-1} - b\Big] - \frac{b(a-h)}{a+2b-1} \\
&\geqslant bn - (n-1)\cdot\frac{b(a+2b)}{a+2b-1} + 2\Big[\frac{b(n-1)}{a+2b-1} - b\Big] - \frac{b(a-2)}{a+2b-1} \\
&= b(n-1) - b + \frac{b(n-1)(2-a-2b)}{a+2b-1} - \frac{b(a-2)}{a+2b-1} \\
&= b(n-1)\Big(1 - \frac{a+2b-2}{a+2b-1}\Big) - b - \frac{b(a-2)}{a+2b-1} \\
&\geqslant \frac{(a+2b-1)(2a+b-4) - b(a+2b-1) - b(a-2)}{a+2b-1} \\
&= \frac{(a-2)(2a+3b-2)}{a+2b-1} \geqslant 0,
\end{aligned}$$

which contradicts (2.3).

Now, we deduce the contradictions. Therefore, H has a fractional $[a,b]$-factor. That is, G is fractional ID-$[a,b]$-factor-critical. □

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