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MAPPING i_2 ON THE FREE PARATOPOLOGICAL GROUPS

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Abstract. Let $FP(X)$ be the free paratopological group over a topological space X. For each nonnegative integer $n \in \mathbb{N}$, denote by $FP_n(X)$ the subset of $FP(X)$ consisting of all words of reduced length at most n, and i_n by the natural mapping from $(X \oplus X^{-1} \oplus \{e\})^n$ to $FP_n(X)$. We prove that the natural mapping $i_2 : (X \oplus X_d^{-1} \oplus \{e\})^2 \to FP_2(X)$ is a closed mapping if and only if every neighborhood U of the diagonal Δ_1 in $X_d \times X$ is a member of the finest quasi-uniformity on X, where X is a T_1-space and X_d denotes X when equipped with the discrete topology in place of its given topology.

1. Introduction

In 1941, free topological groups were introduced by Markov in [**9**] with the clear idea of extending the well-known construction of a free group from group theory to topological groups. Now, free topological groups have become a powerful tool of study in the theory of topological groups and serve as a source of various examples and as an instrument for proving new theorems, see [**1**].

As in free topological groups, Romaguera, Sanchis and Tkachenko in [**12**] defined free paratopological groups and proved the existence of the free paratopological group $FP(X)$ for every topological space X. Recently, Elfard, Lin, Nickolas, and Pyrch have investigated some properties of free paratopological groups, see [**2**, **3**, **7**, **8**, **10**, **11**].

For each nonnegative integer $n \in \mathbb{N}$, denote by $FP_n(X)$ the subset of $FP(X)$ consisting of all words of reduced length at most n, and i_n by the natural mapping from $(X \oplus X^{-1} \oplus \{e\})^n$ to $FP_n(X)$. Here we mainly improve some results of Elfard and Nickolas. The main result is that the natural mapping $i_2\colon (X \oplus X_d^{-1} \oplus \{e\})^2 \to FP_2(X)$ is a closed mapping if and only if every neighborhood U of the diagonal Δ_1 in $X_d \times X$ is a member of the finest quasi-uniformity on X, where X is a T_1-space and X_d denotes X when equipped with the discrete topology in place of its given topology.

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2. Preliminaries

All mappings are continuous. We denote by $\mathbb{N}$ and $\mathbb{Z}$ the sets of all natural numbers and the integers, respectively. The letter e denotes the neutral element of a group. Readers may consult [**1**, **4**–**6**] for notations and terminology not explicitly given here.

Recall that a *topological group* G is a group G with a (Hausdorff) topology such that the product mapping of $G \times G$ into G is jointly continuous and the inverse mapping of G onto itself associating x^{-1} with an arbitrary $x \in G$ is continuous. A *paratopological group* G is a group G with a topology such that the product mapping of $G \times G$ into G is jointly continuous.

Definition 2.1. [**12**] Let X be a subspace of a paratopological group G. Assume that

(1) The set X generates G algebraically, that is $< \langle X \rangle = G$;
(2) Each continuous mapping $f\colon X \to H$ to a paratopological group H extends to a continuous homomorphism $\hat{f}\colon G \to H$.

Then G is called the *Markov free paratopological group on* X and is denoted by $FP(X)$.

Again, if all the groups in the above definitions are Abelian, then we get the definition of the *Markov free Abelian paratopological group on* X which will be denoted by $AP(X)$.

By [**12**], FPX and $AP(X)$ exist for every space X and the underlying abstract groups of FPX and $AP(X)$ are the free groups on the underlying set of the topological space X respectively. We denote these abstract groups by $FP_a(X)$ and $AP_a(X)$ respectively.

Since X generates the free group $FP_a(X)$, each element $g \in FP_a(X)$ has the form $g = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$, where $x_1, \dots, x_n \in X$ and $\varepsilon_1, \dots, \varepsilon_n = \pm 1$. This word for g is called *reduced* if it contains no pair of consecutive symbols of the form xx^{-1} or $x^{-1}x$. It follows that if the word g is reduced and nonempty, then it is different from the neutral element of $FP_a(X)$. For every nonnegative integer n, denote by $FP_n(X)$ and $AP_n(X)$ the subspace of paratopological groups $FP(X)$ and $AP(X)$ that consists of all words of reduced length $\leqslant n$ with respect to the free basis X, respectively.

Let X be a T_1-space. For each $n \in \mathbb{N}$, denote by i_n the multiplication mapping from $\left(X \oplus X_d^{-1} \oplus \{e\}\right)^n$ to $B_n(X)$, $i_n(y_1, \dots, y_n) = y_1 \cdots y_n$ for every point $(y_1, \dots, y_n) \in \left(X \oplus X_d^{-1} \oplus \{e\}\right)^n$, where X_d^{-1} denotes the set X^{-1} equipped with the discrete topology and $B_n(X)$ denotes $FP_n(X)$ or $AP_n(X)$.

By a *quasi-uniform space* $(X, \mathcal{U})$ we mean the natural analog of a *uniform space* obtained by dropping the symmetry axiom. For each quasi-uniformity $\mathcal{U}$ the filter $\mathcal{U}^{-1}$ consisting of the inverse relations $U^{-1} = \{(y, x) : (x, y) \in U\}$ where $U \in \mathcal{U}$ is called the *conjugate quasi-uniformity* of $\mathcal{U}$.

Let X be a topological space. Then X_d denotes X when equipped with the discrete topology in place of its given topology. We denote the diagonals of $X_d \times X$ and $X \times X_d$ by Δ_1 and Δ_2, respectively. In [**10**], the authors proved that X^{-1} is discrete in free paratopological group $FP(X)$ and $AP(X)$ over X if X is a T_1-space. We denote the sets $\{(x^{-1}, y) : (x, y) \in X \times X\}$ and $\{(x, y^{-1}) : (x, y) \in X \times X\}$ by Δ_1^* and Δ_2^*, respectively.

3. Main results

First, we recall some results in the free paratopological groups.

THEOREM 3.1. [**3**] *If X is a T_1-space, then the mapping*

$$i_2|_{i_2^{-1}(FP_2(X)\setminus FP_1(X))} : i_2^{-1}(FP_2(X) \setminus FP_1(X)) \to FP_2(X) \setminus FP_1(X)$$

is a homeomorphism.

THEOREM 3.2. [**2**] *Let X be a T_1-space and let $w = x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_n^{\epsilon_n}$ be a reduced word in $FP_n(X)$, where $x_i \in X$ and $\epsilon_i = \pm 1$, for all $i = 1, 2, \dots, n$, and if $x_i = x_{i+1}$ for some $i = 1, 2, \dots, n-1$, then $\epsilon_i = \epsilon_{i+1}$. Then the collection $\mathcal{B}$ of all sets of the form $U_1^{\epsilon_1} U_2^{\epsilon_2} \dots U_n^{\epsilon_n}$, where, for all $i = 1, 2, \dots, n$, the set U_i is a neighborhood of x_i in X when $\epsilon_i = 1$ and $U_i = \{x_i\}$ when $\epsilon_i = -1$ is a base for the neighborhood system at w in $FP_n(X)$.*

THEOREM 3.3. [**2**] *Let X be a T_1-space and let $w = \epsilon_1 x_1 + \epsilon_2 x_2 + \dots + \epsilon_n x_n$ be a reduced word in $AP_n(X)$, where $x_i \in X$ and $\epsilon_i = \pm 1$, for all $i = 1, 2, \dots, n$, and if $x_i = x_j$ for some $i, j = 1, 2, \dots, n$, then $\epsilon_i = \epsilon_j$. Then the collection $\mathcal{B}$ of all sets of the form $\epsilon_1 U_1 + \epsilon_2 U_2 + \dots + \epsilon_n U_n$, where, for all $i = 1, 2, \dots, n$, the set U_i is a neighborhood of x_i in X when $\epsilon_i = 1$ and $U_i = \{x_i\}$ when $\epsilon_i = -1$ is a base for the neighborhood system at w in $AP_n(X)$.*

THEOREM 3.4. *If X is a T_1-space, then the mapping*

$$f = i_2 |_{i_2^{-1}(AP_2(X)\setminus AP_1(X))} : i_2^{-1}(AP_2(X) \setminus AP_1(X)) \to AP_2(X) \setminus AP_1(X)$$

is a 2 to 1, open and perfect mapping.

PROOF. Obviously, f is a 2 to 1 mapping. Next, we shall prove that f is open and closed. Let $C_2(X) = AP_2(X) \setminus AP_1(X)$ and $C_2^*(X) = i_2^{-1}(AP_2(X) \setminus AP_1(X))$. Obviously, we have

$$C_2^*(X) = (X \times X) \oplus (X_d^{-1} \times X_d^{-1}) \oplus (X_d^{-1} \times X) \setminus \Delta_1^* \oplus (X \times X_d^{-1}) \setminus \Delta_2^*.$$

(1) *The mapping f is open.* Let $(x_1^{\epsilon_1}, x_2^{\epsilon_2}) \in C_2^*(X)$, where $x_1, x_2 \in X$ and $x_1 \neq x_2$ if $\epsilon_1 \neq \epsilon_2$. Let U be a neighborhood of $(x_1^{\epsilon_1}, x_2^{\epsilon_2})$ in $C_2^*(X)$. By Theorem 3.3, $f(U)$ is a neighborhood of $x_1^{\epsilon_1} x_2^{\epsilon_2}$ in $C_2(X)$. (Indeed, the argument is similar to the proof of [**3**, Theorem 3.4].) Therefore, f is open.

(2) *The mapping f is closed.* Let E be a closed subset of $C_2^*(X)$. To show that $i_2(E)$ is closed in $C_2(X)$ take $w \in \overline{i_2(E)}$. Next, we shall show that $w \in i_2(E)$. Indeed, it is obvious that w has a reduced form $w = \epsilon_1 x_1 + \epsilon_2 x_2$, where $\epsilon_i = 1$ or -1 $(i = 1, 2)$, $x_1, x_2 \in X$ and $x_1 \neq x_2$ if $\epsilon_1 \neq \epsilon_2$.

Suppose that $w = x + y \notin i_2(E)$, where $x = \epsilon_1 x_1$ and $y = \epsilon_2 x_2$. Then $\{(x, y), (y, x)\} \cap E = \emptyset$. Since E is closed, we can pick open neighborhoods $V(x)$ of x in $X \cup X_d^{-1}$, $V(y)$ of y in $X \cup X_d^{-1}$ such that $(V(x) \times V(y)) \cap E = \emptyset$ and

$(V(y) \times V(x)) \cap E = \emptyset$. Let $U = (V(x) \times V(y)) \cup (V(y) \times V(x))$. Then U is open. Since f is an open map, we have $f(U)$ is a neighborhood of w and $f(U) \cap \overline{i_2(E)} = \emptyset$. This contradicts with $w \in \overline{i_2(E)}$. □

For an arbitrary space X, the mapping $f\colon X \to \mathbb{Z}$ defined by setting $f(x) = 1$ for all $x \in X$ is continuous, and thus extends to a continuous homomorphism $\widehat{f}\colon AP(X) \to \mathbb{Z}$. Therefore, the collection of sets $Z_n(X) = \widehat{f}^{-1}(\{n\})$ for $n \in \mathbb{Z}$ forms a partition of $AP(X)$ into clopen subspaces.

For a T_1-space, define

$$g\colon (X_d \times X) \oplus (X \times X_d) \oplus (\{e\} \times \{e\}) \to AP_2(X) \cap Z_0(X)$$

by

$$g(x,y) = \begin{cases} -x+y, & \text{if } (x,y) \in X_d \times X; \\ x-y, & \text{if } (x,y) \in X \times X_d; \\ e, & \text{if } x = y. \end{cases}$$

Let $g_j = i_2 \mid_{i_2^{-1}(AP_2(X) \cap Z_j(X))}$ for $j = -2, \ldots, 2$, where

$$i_2\colon (X \oplus X_d^{-1} \oplus \{e\})^2 \to AP_2(X).$$

Obviously, $i_2 = \bigoplus_{j=-2}^{j=2} \{g_j\}$, and i_2 is a closed (resp. quotient) mapping if and only if each g_j is a closed (resp. quotient) mapping, where $j = -2, \ldots, 2$. By Theorem 3.4, it is easy to see that g_{-2} and g_2 are open and closed. Moreover, since $-X$ occurs with the discrete topology and X occurs with its original topology in $AP(X)$, the mappings g_{-1} and g_1 are open and closed. Obviously, g is a closed (resp. quotient) mapping if and only if g_0 is a closed (resp. quotient) mapping. Therefore, we have the following result:

Lemma 3.1. *Let X be a T_1-space. Then i_2 is a closed (resp. quotient) mapping if and only if g is a closed (resp. quotient) mapping.*

Lemma 3.2. [**3**] *Let X be a space and let Δ_1 be the diagonal in the space $X_d \times X$. Then Δ_1 is closed if and only if X is T_1. Similarly for the diagonal Δ_2 in the space $X \times X_d$.*

Suppose that $\mathcal{U}^*$ is the finest quasi-uniformity of a space X. We say that $P = \{U_i\}_{i \in \mathbb{N}}$ is a sequence of $\mathcal{U}^*$ if each $U_i \in \mathcal{U}^*$. Put

$${}^{\omega}\mathcal{U}^* = \{P : P \text{ is a sequence of } \mathcal{U}^*\}.$$

For each $n \in \mathbb{N}$ and $P = \{U_i\}_{i \in \mathbb{N}} \in {}^{\omega}\mathcal{U}^*$, let $\mathcal{Q}_n(\mathbb{N}) = \{A \subset \mathbb{N} : |A| = n\}$,

$$W_n(P) = \{-x_1 + y_1 - \cdots - x_n + y_n : (x_j, y_j) \in U_{i_j}$$

for $j = 1, 2, \ldots, n, \{i_1, i_2, \ldots, i_n\} \in \mathcal{Q}_n(\mathbb{N})\}$, and $\mathcal{W}_n = \{W_n(P) : P \in {}^{\omega}\mathcal{U}^*\}$.

Remark 3.1. In the above definition, for $P = \{U_i\}_{i \in \mathbb{N}} \in {}^{\omega}\mathcal{U}^*$, there may exist $i \neq j$ such that $U_i = U_j$. In particular, for every $U \in \mathcal{U}^*$, we have $\{U_i = U\}_{i \in \mathbb{N}}$ is also in ${}^{\omega}\mathcal{U}^*$. Moreover, the reader should note that the representation of elements of $W_n(P)$ need not be a reduced representation.

THEOREM 3.5. [**7**] *For each $n \in \mathbb{N}$, the family $\mathcal{W}_n$ is a neighborhood base of e in $AP_{2n}(X)$.*

The proof of the following Theorem is a modification of [**3**, Theorem 3.10].

THEOREM 3.6. *Let X be a T_1-space. Then the mapping*

$$i_2\colon (X \oplus X_d^{-1} \oplus \{e\})^2 \to AP_2(X)$$

is a quotient mapping if and only if every neighborhood U of the diagonal Δ_1 in $X_d \times X$ is a member of the finest quasi-uniformity $\mathcal{U}^*$ on X.

PROOF. Put $Z = (X_d \times X) \oplus (X \times X_d) \oplus (\{e\} \times \{e\})$.

NECESSITY. Suppose that i_2 is a quotient mapping. It follows from Lemma 3.1 that $g\colon Z \to AP_2(X) \cap Z_0(X)$ is a quotient mapping. Let U be a neighborhood of Δ_1 in $X_d \times X$. Obviously, $U \cup (-U)$ is a neighborhood of $\Delta_1 \cup \Delta_2$ in Z. Let $P = \{U_n\}_{n\in\mathbb{N}}$, where $U_n = U$ for each $n \in \mathbb{N}$. Let $W_1(P) = \{-x + y : (x, y) \in U\}$. Then $g^{-1}(W_1(P)) = U \cup (-U) \cup \{(e, e)\}$ that is a neighborhood of $\Delta_1 \cup \Delta_2 \cup \{(e, e)\}$ in Z, then $W_1(P)$ is a neighborhood of e in $AP_2(X) \cap Z_0(X)$, and hence in $AP_2(X)$. By Theorem 3.5, there exists $Q \in {}^{\omega}\mathcal{U}^*$ such that $W_1(Q) \subset W_1(P)$, where $Q = \{V_n\}_{n\in\mathbb{N}}$. Then $V_1 \subset U$, hence $U \in \mathcal{U}^*$.

SUFFICIENCY. Suppose that every neighborhood U of the diagonal Δ_1 in $X_d \times X$ is a member of the finest quasi-uniformity $\mathcal{U}^*$ on X. To show that i_2 is a quotient mapping, it follows from Lemma 3.1 that it suffices to show that the mapping $g\colon Z \to AP_2(X) \cap Z_0(X)$ is a quotient mapping. Take a subset $A \subset AP_2(X) \cap Z_0(X)$ such that $g^{-1}(A)$ is open in Z. Put $U = g^{-1}(A) \cap (X_d \times X)$ and $V = g^{-1}(A) \cap (X \times X_d)$. Firstly, we show the following claim:

CLAIM: If $e \notin A$, then A is open in $AP_2(X) \cap Z_0(X)$. Indeed, since $e \notin A$, $U \cap \Delta_1 = \emptyset$ and $V \cap \Delta_2 = \emptyset$. By Lemma 3.2, Δ_1 and Δ_2 are closed in $X_d \times X$ and $X \times X_d$, respectively, and $X_d \times X \setminus \Delta_1$ and $X \times X_d \setminus \Delta_2$ are open in $X_d \times X$ and $X \times X_d$, respectively. Hence $U \cup V$ is open in the space $i_2^{-1}(AP_2(X) \setminus AP_1(X))$, and by Theorem 3.4, $g(U \cup V) = A$ is open in $AP_2(X) \cap Z_0(X)$.

Next we shall show that A is open in $AP_2(X) \cap Z_0(X)$. Take arbitrary $a \in A$. Then it suffices to show that A is an open neighborhood of a.

CASE 1: $a = e$. Obviously, U and V are open neighborhoods of Δ_1 and Δ_2 in $X_d \times X$ and $X \times X_d$, respectively. Therefore, $S = U \cap (V^{-1})$ is an open neighborhood of Δ_1 in $X_d \times X$, and thus $S \in \mathcal{U}^*$. Let $W_1(R) = \{-x+y : (x, y) \in S\}$, where $R = \{S_n\}_{n\in\mathbb{N}}$ and $S_n = S$ for each $n \in \mathbb{N}$. By Theorem 3.5, $W_1(R)$ is a neighborhood of e in $AP_2(X)$. Since $S = U \cap (V^{-1})$ and the definition of g, it is easy to see that $W_1(R) \subset A$. Therefore, A is a neighborhood of e in $AP_2(X)$, hence in $AP_2(X) \cap Z_0(X)$.

CASE 2: $a \neq e$. Let W be an open neighborhood of a in $AP_2(X) \cap Z_0(X)$ such that $e \notin W$. Then the set $g^{-1}(A \cap W)$ is open in Z, and it follows from the claim that $A \cap W$ is an open neighborhood of a in $AP_2(X) \cap Z_0(X)$. Hence A is open in $AP_2(X) \cap Z_0(X)$. □

The following theorem is the main result in [**3**], and some related concepts can be seen in [**5**]. Next, we shall improve this result in Theorem 3.9.

THEOREM 3.7. [**3**] *Let X be a T_1-space. Then the following statements are equivalent:*

(1) *The mapping $i_2\colon (X \oplus X_d^{-1} \oplus \{e\})^2 \to FP_2(X)$ is a quotient mapping;*
(2) *Every neighborhood U of the diagonal Δ_1 in $X_d \times X$ is a member of the finest quasi-uniformity $\mathcal{U}^*$ on X;*
(3) *Every neighbornet of X is normal;*
(4) *The finest quasi-uniformity $\mathcal{U}^*$ on X consists of all neighborhoods of the diagonal Δ_1 in $X_d \times X$;*
(5) *If N_x is a neighborhood of x for all $x \in X$, then there exists a neighborhood M_x of x such that $\bigcup_{y\in M_x} M_y \subset N_x$ for all $x \in X$;*
(6) *If N_x is a neighborhood of x for all $x \in X$, then there exists a quasi-pseudometric d on X such that d_x is upper semi-continuous and $B_d(x,1) \subset N_x$ for all $x \in X$.*

Let X be a set. Define $j_2, k_2\colon X \times X \to F_a(X)$ by $j_2(x,y) = x^{-1}y$ and $k_2(x,y) = yx^{-1}$.

THEOREM 3.8. [**3**] *Let X be a topological space. Then the collection $\mathcal{B}$ of sets $j_2(U) \cup k_2(U)$ for $U \in \mathcal{U}^*$ is a base of neighborhoods at the identity e in $FP_2(X)$.*

Now we can prove the main theorem in this paper.

THEOREM 3.9. *Let X be a T_1-space. Then the following statements are equivalent:*

(1) *The mapping $i_2\colon (X \oplus X_d^{-1} \oplus \{e\})^2 \to FP_2(X)$ is a quotient mapping;*
(2) *The mapping $i_2\colon (X \oplus X_d^{-1} \oplus \{e\})^2 \to AP_2(X)$ is a quotient mapping;*
(3) *The mapping $i_2\colon (X \oplus X_d^{-1} \oplus \{e\})^2 \to FP_2(X)$ is a closed mapping;*
(4) *The mapping $i_2\colon (X \oplus X_d^{-1} \oplus \{e\})^2 \to AP_2(X)$ is a closed mapping.*

PROOF. Obviously, we have (3) ⇒ (1) and (4) ⇒ (2). Moreover, it follows from Theorems 3.6 and 3.7 that we have (2) ⇒ (1). It suffices to show that (1) ⇒ (3) and (2) ⇒ (4).

(1) ⇒ (3). Clearly, both $FP_2(X) \setminus FP_1(X)$ and $FP_1(X) \setminus \{e\}$ are open in $FP_2(X)$. Let E be a closed subset in $(X \oplus X_d^{-1} \oplus \{e\})^2$. To show that $i_2(E)$ is closed in $FP_2(X)$ take $w \in \overline{i_2(E)}$.

CASE a1: $w \in FP_1(X) \setminus \{e\}$. Suppose $w \notin i_2(E)$, then $(w,e) \notin E$ and $(e,w) \notin E$. Since E is closed, there is an open neighborhood U (open in $X \cup X_d^{-1}$) of w such that $(U \times \{e\}) \cap E = \emptyset$ and $(\{e\} \times U) \cap E = \emptyset$. Obviously, we have $(U \times \{e\}) \cup (\{e\} \times U) = i_2^{-1}(U)$. Then U is open in $FP_2(X)$ since $(U \times \{e\}) \cup (\{e\} \times U)$ is open in $(X \oplus X_d^{-1} \oplus \{e\})^2$ and i_2 is a quotient map. Hence $U \cap i_2(E) = \emptyset$, which contradicts $w \in \overline{i_2(E)}$.

CASE a2: $w \in FP_2(X) \setminus FP_1(X)$. Let $w = w_1^{\epsilon_1} w_2^{\epsilon_2}$, where $w_i \in X$ and $\epsilon_i = 1$ or -1 $(i = 1, 2)$. Suppose that $w \notin i_2(E)$. Then $(w_1^{\epsilon_1}, w_2^{\epsilon_2}) \notin E$.

SUBCASE a21: $\epsilon_1 = \epsilon_2 = 1$. Since $(w_1, w_2) \notin E$ and E is closed in $(X \oplus X_d^{-1} \oplus \{e\})^2$, there exist neighborhoods U and V of w_1 and w_2 in X, respectively, such that $(U \times V) \cap E = \emptyset$. Therefore, it is easy to see that $UV \cap i_2(E) = \emptyset$. From

Theorem 3.2 it follows that UV is a neighborhood of w, hence $w \notin \overline{i_2(E)}$, which is a contradiction.

SUBCASE a22: $\epsilon_1 = \epsilon_2 = -1$. From Theorem 3.2 it follows that $\{w_1^{-1}w_2^{-1}\}$ is a neighborhood of w, then $w \notin \overline{i_2(E)}$, which is a contradiction.

SUBCASE a23: $\epsilon_1 \neq \epsilon_2$. Without loss of generality, we may assume that $\epsilon_1 = 1$ and $\epsilon_2 = -1$. Then since $(w_1, w_2^{-1}) \notin E$ and E is closed in $(X \oplus X_d^{-1} \oplus \{e\})^2$, there exists a neighborhood U of w_1 in X such that $(U \times \{w_2^{-1}\}) \cap E = \emptyset$ and $w_2 \notin U$. (This is possible since X is T_1.) Obviously, $Uw_2^{-1} \subset FP_2(X) \smallsetminus FP_1(X)$. Therefore, it is easy to see that $Uw_2^{-1} \cap i_2(E) = \emptyset$. From Theorem 3.2 it follows that Uw_2^{-1} is a neighborhood of w, hence $w \notin \overline{i_2(E)}$, which is a contradiction.

Therefore, we have $w \in i_2(E)$.

CASE a3: $w = e$. Suppose that $e \notin i_2(E)$. Then $E \cap (\Delta_1 \cup \Delta_2 \cup \{(e,e)\}) = \emptyset$. For any $x \in X$, since E does not contain points (x^{-1}, x) and (x, x^{-1}), there exists an open neighborhood $U(x)$ of x in X such that $(\{x^{-1}\} \times U(x)) \cap E = \emptyset$ and $(U(x) \times \{x^{-1}\}) \cap E = \emptyset$. Let $U = \bigcup_{x \in X}(\{x^{-1}\} \times U(x))$ and $V = \bigcup_{x \in X}(U(x) \times \{x^{-1}\})$. Then $U \cap E = \emptyset$ and $V \cap E = \emptyset$. Let $W = U \cup V \cup \{e\} \times \{e\}$. Then W is open in $(X \oplus X_d^{-1} \oplus \{e\})^2$ by (2) of Theorem 3.7. Obviously, we have $W \cap E = \emptyset$. It is easy to see that $i_2^{-1}(i_2(W)) = W$, then $i_2(W)$ is open since i_2 is a quotient map. Hence $i_2(W) \cap i_2(E) = \emptyset$, this is a contradiction.

(2) $\Rightarrow$ (4). (Note: The proof is almost similar to (1) $\Rightarrow$ (3). However, we give out the proof for the convenience of readers.) Clearly, both $AP_2(X) \smallsetminus AP_1(X)$ and $AP_1(X) \smallsetminus \{e\}$ are open in $AP_2(X)$. Let E be a closed subset in $(X \oplus -X_d \oplus \{e\})^2$. To show that $i_2(E)$ is closed in $AP_2(X)$ take $w \in \overline{i_2(E)}$.

CASE b1: $w \in AP_1(X) \smallsetminus \{e\}$. Suppose $w \notin i_2(E)$, then $(w, e) \notin E$ and $(e, w) \notin E$. Since E is closed, there is an open neighborhood U (open in $X \cup -X_d$) of w such that $(U \times \{e\}) \cap E = \emptyset$ and $(\{e\} \times U) \cap E = \emptyset$. Obviously, we have $(U \times \{e\}) \cup (\{e\} \times U) = i_2^{-1}(U)$. Then U is open in $AP_2(X)$ since $(U \times \{e\}) \cup (\{e\} \times U)$ is open in $(X \oplus -X_d \oplus \{e\})^2$ and i_2 is a quotient map by Theorems 3.6 and 3.7. Then $U \cap i_2(E) = \emptyset$, that contradicts $w \in \overline{i_2(E)}$.

CASE b2: $w \in AP_2(X) \smallsetminus AP_1(X)$. Let $w = \epsilon_1 w_1 + \epsilon_2 w_2$, where $w_i \in X$ and $\epsilon_i = 1$ or -1 ($i = 1, 2$). Suppose that $w \notin i_2(E)$. Then $(\epsilon_1 w_1, \epsilon_2 w_2) \notin E$ and $(\epsilon_2 w_2, \epsilon_1 w_1) \notin E$.

SUBCASE b21: $\epsilon_1 = \epsilon_2 = 1$. Since $\{(w_1, w_2), (w_2, w_1)\} \notin E$ and E is closed in $(X \oplus -X_d \oplus \{e\})^2$, there exist neighborhoods U and V of w_1 and w_2 in X, respectively, such that $(U \times V \cup V \times U) \cap E = \emptyset$. Therefore, it is easy to see that $(U + V) \cap i_2(E) = \emptyset$. From Theorem 3.3 it follows that $U + V$ is a neighborhood of w, hence $w \notin \overline{i_2(E)}$, which is a contradiction.

SUBCASE b22: $\epsilon_1 = \epsilon_2 = -1$. From Theorem 3.2 it follows that $\{-w_1 - w_2\}$ is a neighborhood of w, then $w \notin \overline{i_2(E)}$, which is a contradiction.

SUBCASE b23: $\epsilon_1 \neq \epsilon_2$. Without loss of generality, we may assume that $\epsilon_1 = 1$ and $\epsilon_2 = -1$. Then since $\{(w_1, -w_2), (-w_2, w_1)\} \notin E$ and E is closed in $(X \oplus -X_d \oplus \{e\})^2$, there exists a neighborhood U of w_1 in X such that $(U \times \{w_2^{-1}\} \cup$

$\{w_2^{-1}\} \times U) \cap E = \emptyset$ and $w_2 \notin U$. (This is possible since X is T_1.) Obviously, $U - w_2 \subset AP_2(X) \setminus AP_1(X)$. Therefore, it is easy to see that $(U - w_2) \cap \overline{i_2(E)} = \emptyset$. From Theorem 3.3 it follows that $U - w_2$ is a neighborhood of w, hence $w \notin \overline{i_2(E)}$, which is a contradiction.

Therefore, we have $w \in i_2(E)$.

Case b3: $w = e$. Suppose that $e \notin i_2(E)$. Then $E \cap (\Delta_1 \cup \Delta_2 \cup \{(e, e)\}) = \emptyset$. For any $x \in X$, since E does not contain points $(-x, x)$ and $(x, -x)$, there exists an open neighborhood $U(x)$ of x in X such that $(\{-x\} \times U(x)) \cap E = \emptyset$ and $(U(x) \times \{-x\}) \cap E = \emptyset$. Let $U = \bigcup_{x \in X}(\{-x\} \times U(x))$ and $V = \bigcup_{x \in X}(U(x) \times \{-x\})$. Then $U \cap E = \emptyset$ and $V \cap E = \emptyset$. Let $W = U \cup V \cup \{e\} \times \{e\}$. Then W is open in $(X \oplus -X_d \oplus \{e\})^2$ by Theorem 3.7. Obviously, we have $W \cap E = \emptyset$. It is easy to see that $i_2^{-1}(i_2(W)) = W$, then $i_2(W)$ is open in $AP_2(X)$ since i_2 is a quotient map by Theorems 3.6 and 3.7. Hence $i_2(W) \cap i_2(E) = \emptyset$, which is a contradiction. □

Proposition 3.1. *Let X be a T_1-space. Then, for some $n \geqslant 3$,*

$$i_n \colon (X \oplus X_d^{-1} \oplus \{e\})^n \to FP_n(X)$$

is a closed map if and only if X is discrete.

Proof. If X is discrete, then $FP(X)$ is discrete, so it is easy to see that each i_n is a closed map.

Let i_n be a closed map for some $n \geqslant 3$. Since X is T_1, then X^{-1} is discrete. Suppose that X is not discrete, then there exists $x \in X$ such that $x \in \overline{X \setminus \{x\}}$. Let

$$A = \{(x_\alpha, x_\alpha, x_\alpha^{-1}, e, \ldots, e) \in (X \oplus X_d^{-1} \oplus \{e\})^n : x_\alpha \in X \setminus \{x\}\}.$$

Then A is a closed discrete subset of $(X \oplus X_d^{-1} \oplus \{e\})^n$, and therefore, $i_n(A) = X \setminus \{x\}$ is a closed discrete subset, which is a contradiction. Hence X is discrete. □

Note. Therefore, we can improve all results in [**3**, Sections 4 and 5] from quotient mappings to closed mappings. For example, we have the following proposition.

Proposition 3.2. *The mapping i_2 is a closed mapping for any countable T_1-space. In particular, the mapping i_2 is a closed mapping for any countable subspace of the real line $\mathbb{R}$.*

Corollary 3.1. *$FP_2(\mathbb{Q})$ and $AP_2(\mathbb{Q})$ are Fréchet, where $\mathbb{Q}$ is the rational number of real line $\mathbb{R}$.*

Proof. By Proposition 3.2, i_2 is a closed mapping. Then $FP_2(\mathbb{Q})$ and $AP_2(\mathbb{Q})$ are Fréchet since $(X \oplus X_d^{-1} \oplus \{e\})^2$ is Fréchet and closed mappings preserve the property of Fréchet. □

By [**5**, Proposition 6.26], we also have the following proposition.

Proposition 3.3. *For an arbitrary compact first-countable Hausdorff space X, the mapping i_2 is closed if and only if X is countable.*

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