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ANNIHILATOR TOPOLOGICAL ALGEBRAS (*)

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Abstract. In this paper we study annihilator *topological algebras*, not necessarily Banach or even locally convex ones, along with their structure theory. We also refer to (D)-algebras a convenient variant for Q-algebras. In fact, for semisimple annihilator algebras, (D) and Q'-algebras coincide. Topological algebras with proper closed ideals having non zero annihilators are also considered.

0 – Introduction

Annihilator algebras have been introduced by F.F. Bonsall and A.W. Goldie into the framework of Banach algebra theory. Later M.A. Naĭmark extended the previous context by considering semisimple annihilator topological Q-algebras with continuous quasi-inversion. In both cases a structure theory has been established. Furthermore, topological annihilator algebras in the sense of this paper have been also studied in the past, as e.g. in [5], [17], [19] (although continuous multiplication is rather understood therein). In this paper we study annihilator *topological algebras*, not necessarily Banach or even locally convex ones, along with their structure theory.

In a semisimple annihilator Q'-algebra we prove the existence of minimal ideals, hence, equivalently, of non trivial primitive idempotents (minimal elements; cf. Theorems 3.8 and 3.9). These ideals contribute to an identification of the structure of the algebra (cf. Theorem 4.3). In particular, one characterizes annihilator Q'-algebras as semisimple, through the density of the socle. Furthermore, one has another structural information through the minimal closed 2-sided ide-

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als. In the latter case these ideals are semisimple topologically simple algebras being also annihilator ones, while they are dual algebras if this is the case for the given algebra (Theorem 4.12). We also single out algebras having the property that every ideal of the algebra contains a minimal one ((D)-algebras, cf. Definition 3.1). For semisimple annihilator algebras the notions (D) and Q' coincide (Theorem 3.6). Finally, we consider topological algebras with proper closed ideals having non zero annihilators (cf. Definition 2.4 and Section 4; see also [9]).

1 – Notation and preliminaries

Let E be a $\mathbb{C}$-algebra. If $(\emptyset \neq) S \subseteq E$, $\mathcal{A}_\ell(S)$ (resp. $\mathcal{A}_r(S)$) denotes the left (right) annihilator of S; viz. $\mathcal{A}_\ell(S) = \{x \in E \colon xS = \{0\}\}$, resp. $\mathcal{A}_r(S) = \{x \in E \colon Sx = \{0\}\}$. $\mathcal{A}_\ell(S)$ (resp. $\mathcal{A}_r(S)$) is a left (resp. right) ideal of E, which is closed, if E is a topological algebra. $\mathcal{L}_\ell$ ($\mathcal{L}_r$, $\mathcal{L}(E) \equiv \mathcal{L}$) denotes the set of all closed left (right, 2-sided) ideals in a topological algebra E. Besides, $\mathcal{M}_\ell(E)$ (resp. $\mathcal{M}_r(E)$) stands for the set of all maximal closed regular left (right) ideals of E, while $\mathbf{m}_\ell(E)$ denotes that of all minimal closed left ideals. We write $m_\ell(E)$ (resp. $m_r(E)$) for the set of all minimal left (right) ideals of an algebra E. $R(E)$ denotes the Jacobson radical of an algebra E. If $R(E) = (0)$, then E is said to be semisimple. Besides, $\mathcal{I}\mathrm{d}(E)$ denotes the set of all non zero idempotent elements of an algebra E, i.e., the set of all $x \in E$ with $0 \neq x = x^2$. A *minimal* element of an algebra E, is a non zero idempotent, such that xEx is a division algebra. A non zero element of an algebra E is called *primitive* if it can not be expressed as the sum of two orthogonal idempotents viz. of some $y, z \in \mathcal{I}\mathrm{d}(E)$ with $yz = zy = 0$. We denote by $\mathcal{P}(E)$ the set of primitive elements of E, while $\mathcal{IP}(E)$ that of primitive idempotents.

Definition 1.1. A topological algebra E [14] is called a Q'_ℓ (resp. Q'_r)-*algebra*, if every maximal regular left (resp. right) ideal is closed. E is said to be a Q'-*algebra*, if it is both a Q'_ℓ and a Q'_r-algebra.

We note that a Q'-ring is not in general, a Q-ring (see e.g. [18]). On the other hand, every Q_ℓ-algebra (its group of left quasi-regular elements is open) is a Q'_ℓ-algebra (see also [14: p. 67, Theorem 6.1]).

The following observation for a Q-algebra is due to A. Mallios.

Lemma 1.2. *Let E be a Q'_ℓ-algebra and let I be a proper regular left ideal. Then $\overline{I}$ is still a proper (regular left) ideal.*

Proof: I is contained in a maximal regular left ideal, say M (Krull). By hypothesis, M is closed, hence $\overline{I} \subseteq \overline{M} = M$. ∎

The above lemma holds for right ideals, as well. The results in the remainder of the paper stated for left (right) ideals remain true, by interchanging the notions left and right.

Now, we give a characterization of Q'_ℓ-algebras.

Proposition 1.3. *A topological algebra E is a Q'_ℓ-algebra, if and only, if it has no proper dense regular left ideals.*

Proof: By Lemma 1.2 we only have to prove that the condition is sufficient. So if M is a maximal regular left ideal of E, then $M \subseteq \overline{M} \neq E$ so that $M = \overline{M}$. ∎

2 – Annihilator and torsion algebras

An algebra E is called *left* (resp. *right*) *preannihilator*, if $\mathcal{A}_\ell(E) = (0)$ (resp. $\mathcal{A}_r(E) = (0)$). If $\mathcal{A}_\ell(E) = \mathcal{A}_r(E) = (0)$, E is called *preannihilator*.

In particular, a topological algebra E is said to be an *annihilator* algebra, if it is preannihilator with $\mathcal{A}_r(I) \neq (0)$ for every $I \in \mathcal{L}_\ell$, $I \neq E$, and $\mathcal{A}_\ell(J) \neq (0)$ for every $J \in \mathcal{L}_r$, $J \neq E$.

An algebra without nilpotent elements and a fortiori without divisors of zero is preannihilator.

A topological algebra is called *topologically semiprime*, when the following holds:

$$\text{If } I \in \mathcal{L} \text{ satisfies } I^2 = (0), \quad \text{then } I = (0)\,. \tag{2.1}$$

Concerning the previous notion we actually have the following result.

Theorem 2.1. *In every topological algebra E the condition (2.1) holds equivalently in $\mathcal{L}_\ell$ (or in $\mathcal{L}_r$).*

The proof of the theorem is derived from the following lemmas.

Lemma 2.2. *Let E be a topologically semiprime algebra and S a non empty subset of E. Moreover, let I be the closed left ideal generated by $S+\mathcal{A}_\ell(S)$. Then I is a left preannihilator algebra.*

Proof: By definition of I, $\mathcal{A}_\ell(S) \subseteq I$. Hence $\mathcal{A}_\ell(I) \subseteq \mathcal{A}_\ell(\mathcal{A}_\ell(S))$, i.e., $\mathcal{A}_\ell(I)\,\mathcal{A}_\ell(S) = (0)$ and in particular, $\mathcal{A}_\ell(I)^2 = (0)$. Thus, since $\mathcal{A}_\ell(I) \in \mathcal{L}$, $\mathcal{A}_\ell(I) = (0)$. ∎

The preceding lemma and the fact that an algebra E is left preannihilator if and only if there exists $(\emptyset \neq) S \subseteq E$, such that $\mathcal{A}_\ell(S) = (0)$, yield now the following.

Lemma 2.3. *Every topologically semiprime algebra is preannihilator.* ∎

A sort of inverse of Lemma 2.3 is given in [10] which extends in our case a result of W. Ambrose [2: p. 372]. In fact, *in a preannihilator Hausdorff locally convex H^*-algebra which is also orthogonally complemented there are no nilpotent ideals* (cf. [10: Theorem 1.2, Lemma 3.13 and comments following it]). Such an algebra is, of course, topologically semiprime.

Proof of Theorem 2.1: It is enough to show that, if (2.1) holds in $\mathcal{L}$, then also holds in $\mathcal{L}_\ell$. In fact, let $I \in \mathcal{L}_\ell$ with $I^2 = (0)$. Consider the left ideal IE, being a right ideal, as well. Thus $\overline{IE} \in \mathcal{L}$. On the other hand, $\overline{IE}\,\overline{IE} \subseteq \overline{I^2E} = (0)$ (see also [14: p. 6, Lemma 1.5]). Hence $\overline{IE} = (0)$ from which one gets $I \subseteq \mathcal{A}_\ell(E)$, that is (Lemma 2.3) $I = (0)$. ∎

We next consider some classes of topological algebras which exhibit a certain kind of "torsion", in the sense that they contain proper closed ideals (left or right) with non zero left or right annihilators. Among them we single out those given by the following.

Definition 2.4. A topological algebra E is called an $(\mathcal{M}_\ell)$-*algebra* if

$$\mathcal{A}_\ell(I) = (0), \quad \text{with } I \in \mathcal{L}_\ell, \qquad \text{implies} \quad I = E\,. \tag{2.2}$$

A topological algebra E is called an $(\mathcal{M}'_\ell)$ (resp. $(\mathcal{M}'_r)$)-*algebra* if

$$\mathcal{A}_\ell(I) = (0), \quad \text{with } I \in \mathcal{L}_r, \qquad \text{implies} \quad I = E\,, \tag{2.3}$$

respectively

$$\mathcal{A}_r(I) = (0), \quad \text{with } I \in \mathcal{L}_\ell, \qquad \text{implies} \quad I = E\,. \tag{2.4}$$

The proof of the next lemma is obtained by applying a standard argument; cf. e.g. [16: p. 99, lemmas 2.8.10, 2.8.11].

Lemma 2.5. *Let E be a topologically semiprime algebra. Then $I \cap \mathcal{A}_\ell(I) = (0)$ and $J \cap \mathcal{A}_r(J) = (0)$ for every $I \in \mathcal{L}_\ell$, $J \in \mathcal{L}_r$ and $\mathcal{A}_\ell(K) = \mathcal{A}_r(K)$ for every $K \in \mathcal{L}$.*

Now, suppose moreover, that E satisfies either one of the following two equivalent conditions

$$\mathcal{A}_\ell(J) = (0), \quad \textit{with } J \in \mathcal{L}, \qquad \textit{then} \quad J = E\ , \tag{2.5}$$

$$\mathcal{A}_r(J) = (0), \quad \textit{with } J \in \mathcal{L}, \qquad \textit{then} \quad J = E\ . \tag{2.6}$$

Then for every $I \in \mathcal{L}$, one gets $E = \overline{I \oplus \mathcal{A}_\ell(I)} = \overline{I \oplus \mathcal{A}_r(I)}$, moreover, $\mathcal{L}_\ell(I) \subseteq \mathcal{L}_\ell(E)$ and $\mathcal{L}_r(I) \subseteq \mathcal{L}_r(E)$. ∎

The previous lemma holds also for topological algebras without nilpotent elements. On the other hand, (2.5) and (2.6) are fulfilled, in case E is an annihilator algebra.

Lemma 2.6. *Let E be a right preannihilator $(\mathcal{M}'_r)Q'_\ell$-algebra. Moreover, let $x \in \mathcal{I}\mathrm{d}(E)$, such that the closed right ideal xE is minimal closed. Then the regular left ideal $E(1-x)$ is maximal.*

Proof: Since $x \in \mathcal{I}\mathrm{d}(E)$, $E(1-x)$ is proper. Let $E(1-x) \subseteq M \subset E$, ("$\subset$" means *proper subset*), for some (regular) left ideal M of E. Then, since E is a right preannihilator algebra, $\mathcal{A}_r(M) \subseteq xE$. If $\mathcal{A}_r(M) = (0)$, then $\mathcal{A}_r(\overline{M}) = (0)$ and thus $\overline{M} = E$, which is a contradiction (cf. Proposition 1.3). Therefore, $\mathcal{A}_r(M) = xE$ and $M \subseteq \mathcal{A}_\ell(\mathcal{A}_r(M)) = E(1-x)$, which implies $M = E(1-x)$ and this completes the proof. ∎

Now, a topological algebra E is called *dual*, if

$$\mathcal{A}_\ell(\mathcal{A}_r(I)) = I \qquad \text{for every} \quad I \in \mathcal{L}_\ell\ , \tag{2.7}$$

and

$$\mathcal{A}_r(\mathcal{A}_\ell(J)) = J \qquad \text{for every} \quad J \in \mathcal{L}_r\ . \tag{2.8}$$

If (2.7) (resp. (2.8)) holds, then E is called a *left* (resp. *right*) *dual algebra*. Every dual algebra is an annihilator algebra (see also [15: p. 321]). The converse is not, in general, true (see e.g. [4], [8] and [12]). There are, however, some special cases for which the previous two classes coincide (see e.g. [1], [7], [9] and [15]).

3 – (D)-algebras and minimal ideals

Many of our later results are based on the following notion.

Definition 3.1. An algebra E is said to be a (D_ℓ) (resp. (D_r))-*algebra*, if the following holds:

(D_ℓ) (resp. (D_r)) Every left (right) ideal contains a minimal left (right) ideal.

A (D_ℓ) and (D_r)-algebra is called a (D)-algebra.

Concerning the following results see also [15: p. 322, II].

Lemma 3.2. *Let E be a right preannihilator $(\mathcal{M}'_r)Q'_\ell$-algebra. Moreover, let x_0 be a left quasi-singular element of E. Then, there exists an element $0 \neq z \in E$, such that $z = x_0 z$.*

Proof: By assumption for x_0, $I \equiv E(1-x_0)$ is proper, with $x_0 \notin I$. Besides (Lemma 1.2) $\overline{I}$ is a (proper closed) regular left ideal of E. Therefore $\mathcal{A}_r(\overline{I}) \neq (0)$, thus there is $0 \neq z \in \mathcal{A}_r(E(1-x_0))$, such that $(x - xx_0)z = 0$ for every $x \in E$. Hence, $z = x_0 z$, since E is right preannihilator. ∎

Lemma 3.3. *Let E be a left preannihilator (topological) algebra and M a maximal (closed) right ideal of E. Moreover, let $0 \neq z \in \mathcal{A}_\ell(M)$. Then*

$$M = \mathcal{A}_r((z)_\ell) \quad (\textit{resp. } M = \mathcal{A}_r(\overline{(z)_\ell})) \,, \tag{3.1}$$

where $(z)_\ell$ (resp. $\overline{(z)_\ell}$) is the left (resp. closed left) ideal of E generated by z.

Proof: Since $(z)_\ell \subseteq \mathcal{A}_\ell(M)$, $M \subseteq \mathcal{A}_r((z)_\ell)$. Besides, $\mathcal{A}_r((z)_\ell) \neq E$, otherwise $(z)_\ell E = (0)$. Thus, since E is left preannihilator, $z = 0$, which is a contradiction. Therefore, $M = \mathcal{A}_r((z)_\ell)$. In a similar way we get $M = \mathcal{A}_r(\overline{(z)_\ell})$. ∎

The following result is given in [7: p. 155, Theorem 1] for annihilator Banach algebras and in [15: p. 322, Theorem 1 and p. 323, Corollary 1] for annihilator Waelbroeck algebras (i.e. Q-algebras with a continuous quasi-inversion [14]). However, the gist of the proof in the latter case is the *Q'-property* (see Definition 1.1). So we have.

Theorem 3.4. *Let E be an annihilator Q'-algebra. Moreover, let M be a maximal closed right ideal of E, such that*

$$\mathcal{A}_\ell(M) \cap R(E) = (0) \,. \tag{3.2}$$

Then $M = (1-x)E$ with $x \in \mathcal{I}\mathrm{d}(E) \cap \mathcal{A}_\ell(M)$.

Moreover, M is a maximal right ideal of E and the (closed) left ideal $\mathcal{A}_\ell(M)$ is minimal and thus minimal closed. In particular, x in $(1-x)E$ is primitive (idempotent).

Proof: The first part of the assertion follows from the above comments.

Let now I be a proper right ideal of E with $M \subseteq I$. Then $M = \overline{M} \subseteq \overline{I} \neq E$ (see also Lemma 1.2). Thus $M = \overline{I} = I$, i.e., M is a maximal (regular) right ideal. Now $\mathcal{A}_\ell(M) = Ex$. Besides, xE is minimal (see for instance [3: p. 221, Proposition 8.9 and p. 220, Property 8.6] and/or [11: p. 117, Theorem 3.9]) and therefore minimal closed. Hence (Lemma 2.6) $E(1-x)$ is maximal and thus Ex is minimal. Now, if $x \notin \mathcal{P}(E)$, then $x = y + z$ with $y, z \in \mathcal{I}\mathrm{d}(E)$ and $yz = zy = 0$. Consider the left ideal Ez. If $w \in Ey$ then $w = wy$ and thus $wz = wyz = 0$. Therefore, $w = wx \in Ex$, i.e., $Ey \subseteq Ex$. Besides, $Ey \neq Ex$, otherwise $Exz = Eyz$ and hence $xz = 0$. Therefore $yz + z = 0$, that is a contradiction. Thus $(0) \neq Ey \subset Ex$ in contradiction with what we stated above. ∎

The conclusion of Theorem 3.4 is still in force, when (3.2) is replaced by $\mathcal{A}_\ell(M) \not\subseteq R(E)$ (cf. also [6: p. 161, Theorem 3]). In this connection we remark that the last relation follows from (3.2), since E is an annihilator algebra so that $\mathcal{A}_\ell(M) \neq (0)$.

Corollary 3.5. *In every annihilator Q'-algebra E a maximal right ideal that satisfies (3.2) (take, for instance, E semisimple) is regular if and only if it is closed.*

Proof: Supposing simply that the given algebra is Q', the condition is, obviously, necessary. It is also sufficient by Theorem 3.4. ∎

We come now to the main result of this section.

Theorem 3.6. *Let E be a semisimple topological algebra. Consider the assertions:*

1) *E is a (D)-algebra;*

2) *E is a Q'-algebra.*

Then 1)⇒2). The above two assertions are equivalent if E is an annihilator algebra.

Proof: **1)⇒2):** Let M be a maximal left ideal of E. Consider the right ideal $\mathcal{A}_r(M)$. By hypothesis, there exists a minimal right ideal I, such that $I \subseteq \mathcal{A}_r(M)$. But $I = xE$, with x minimal and $x \in \mathcal{IP}(E)$ (see for instance [15: p. 326, III], where just semisimplicity of the algebra suffices, as well as [16: p. 45, Lemma 2.1.5]). Now, $M \subseteq \mathcal{A}_\ell(\mathcal{A}_r(M)) \subseteq \mathcal{A}_\ell(xE)$ and, since $x \in \mathcal{I}\mathrm{d}(E)$, $\mathcal{A}_\ell(xE) = E(1-x) \neq E$. Thus $M = E(1-x)$ that is closed; i.e., E is a (Q'_ℓ)-algebra. Likewise, we prove that E is a (Q'_r)-algebra as well.

2)⇒1): Let $I \neq (0)$ be a left ideal of E. If I is minimal, there is nothing to prove. So assume that I is not minimal and does not contain minimal left ideals. By semisimplicity and Theorem 3.4 there is a maximal (closed) regular right ideal M, such that $M = (1-x)E$ with $x \in \mathcal{IP}(E)$. Since E is preannihilator and $x \in \mathcal{I}\mathrm{d}(E)$, $\mathcal{A}_\ell(M) = Ex$ which is minimal (ibid.). Thus, for $z \in I$, Exz is either minimal or (0) (see, for instance, [6: p. 155, Lemma 7]). The first case yields $Exz \subseteq I$, that is a contradiction. Thus $Exz = (0)$ for every $z \in I$. Hence $I \subseteq \mathcal{A}_r(Ex) = M$. That is I is contained in every maximal (closed) regular right ideal of E. Thus (see, for instance [15: p. 163, III′] and/or [16: p. 55, Theorem 2.3.2]) $I \subseteq R(E)$ which is a contradiction. (For another proof based on structure theory see comments following Theorem 4.3). ∎

By the previous proof we get that: *in a semisimple topological* (D)*-algebra every maximal (left) ideal is regular*.

Corollary 3.7. *Let E be a topologically semiprime algebra and I a minimal left ideal of E. Then $I = Ex$ with x minimal primitive idempotent. Therefore, if E is, moreover, a (D_ℓ)-algebra, then $\mathcal{IP}(E) \neq \emptyset$.*

Proof: Claim that $\overline{I^2} \neq (0)$. Otherwise, by hypothesis, and since $\overline{I}^2 = \overline{I}\,\overline{I} \subseteq \overline{I^2}$ (cf. [14: p. 6, Lemma 1.5]), $\overline{I} = (0)$ and a fortiori $I = (0)$, that is a contradiction, since I is minimal. Now $\overline{I^2} \neq (0)$ implies $I^2 \neq (0)$. Otherwise we get a contradiction. Thus, by $I^2 \neq (0)$ we have $I = Ex$, with x minimal and $x \in \mathcal{IP}(E)$ (cf. [15: p. 326, III] and [16: p. 45, Lemma 2.1.5]) and this completes the proof. ∎

We state the following immediate consequence of Theorem 3.6 and Corollary 3.7.

Theorem 3.8. *Every semisimple annihilator Q-(topological) algebra E (but Q' suffices, as well) "has enough minimal ideals" (viz. it is a (D)-algebra). Moreover, each one of the latter is of the form Ex, with x a minimal primitive idempotent element in E.* ∎

The following result extends [15: p. 326, V, and p. 327, VI].

Theorem 3.9. *Let E be a topological algebra and $x \in \mathcal{I}\mathrm{d}(E)$. Consider the assertions:*

1) *The (closed) left ideal Ex is minimal (thus minimal closed).*

2) $x \in \mathcal{P}(E)$.

Then 1)⇒2). The above assertions are equivalent in case E is a topologically semiprime (D_ℓ)-algebra.

Proof: 1)⇒2): Cf. the relevant argument in the proof of Theorem 3.4.

Now, if $(0) \neq J \subset Ex$ for some left ideal J of E, then by assumption, there is a minimal left ideal I, such that $(0) \neq I \subseteq J$. Thus $(0) \neq I \subset Ex$ with $I^2 \neq (0)$. Hence $I = Ey$ with $y \in \mathcal{I}\mathrm{d}(E)$ (see Corollary 3.7) and therefore $(0) \neq Ey \subset Ex$ with $y = y^2 = yx$. Consider the element $z = xy$. Then $z = xyx$ and $zx = z = xz$. On the other hand, $z^2 = zxy = z$ and $yz = yxy = y$. Thus $z \in \mathcal{I}\mathrm{d}(E)$. Moreover, $x - z \in \mathcal{I}\mathrm{d}(E)$. For, if $x = z$, $Ex = Ez \subset Ex$ which is a contradiction. Thus $x - z \neq 0$. Furthermore, $(x-z)^2 = x - z$ and $z(x-z) = (x-z)z$. The previous argument yields $x \notin \mathcal{P}(E)$ which is a contradiction. Thus 2)⇒1). ∎

If E is semiprime and x minimal, then Ex and xE are minimal left resp. right ideals of E. See for instance [16: p. 46, Corollary 2.1.9]. In this regard, in view of the proof of Corollary 3.7 *the last assertion is true for E a topologically semiprime algebra.*

Now we give conditions such that the notions minimal element and primitive idempotent coincide.

Corollary 3.10. *Let E be a topologically semiprime algebra and $x \in E$. Consider the statements:*

1) *x is minimal;*

2) *$x \in \mathcal{IP}(E)$.*

Then 1)⇒2). The above assertions are equivalent, if E is a (D_ℓ) (or a (D_r))-algebra.

Proof: 1)⇒2): Since x is minimal, Ex and xE are minimal ideals. Hence $x \in \mathcal{IP}(E)$ (see Theorem 3.9).

If $x \in \mathcal{IP}(E)$, the ideal Ex (or the ideal xE) is minimal (cf. Theorem 3.9). Thus the algebra xEx is a division algebra (see for instance [13: p. 103, proof of Lemma 1] and/or [16: p. 45, Lemma 2.1.5]). Hence x is minimal and this completes the proof. ∎

Theorem 3.11. *Let E be a locally convex algebra with a continuous quasi-inversion, such that the left ideal Ex or the right ideal xE is minimal and $x \in \mathcal{I}\mathrm{d}(E)$. Then*

$$xEx = \mathbb{C}\ , \tag{3.3}$$

within an isomorphism of topological algebras.

Proof: Since E is a locally convex algebra with a continuous quasi-inversion, xEx is a *locally convex algebra with a continuous inversion*. So (3.3) is true, by Gel'fand–Mazur, xEx being also a division algebra (proof of Corollary 3.10). Cf. also [14: p. 62, Corollary 5.1]. ∎

Lemma 3.12. *Let E be a left preannihilator $(\mathcal{M}'_\ell)$-algebra and $x \in \mathcal{I}\mathrm{d}(E)$. Consider the assertions:*

1) *$Ex \in \mathbf{m}_\ell(E)$;*

2) *$(1-x)E \in \mathcal{M}_r(E)$.*

Then 1)⇒2). The converse statement holds in each one of the following cases:

i) *E is a topologically semiprime (D_ℓ)-algebra;*

ii) *E is a left dual algebra.*

Proof: 1)⇒2): Suppose $(1-x)E \subseteq M \neq E$ for some closed (regular) right ideal M. Then, since E is left preannihilator and $x \in \mathcal{I}\mathrm{d}(E)$, $\mathcal{A}_\ell(M) \subseteq Ex$. Moreover, $\mathcal{A}_\ell(M) \neq (0)$. Thus, by the minimality of Ex, $\mathcal{A}_\ell(M) = Ex$, which implies $M \subseteq \mathcal{A}_r(\mathcal{A}_\ell(M)) = (1-x)E$. Hence $(1-x)E = M$.

2)⇒1): i) Let $(0) \neq I \subseteq Ex$ for some $I \in \mathcal{L}_\ell$. Then, by hypothesis, there exists a minimal left ideal, say J, such that $J \subseteq I \subseteq Ex$. Moreover, $J = Ey$ with $y \in \mathcal{IP}(E)$ (see Corollary 3.7). Hence $\mathcal{A}_r(Ex) \subseteq \mathcal{A}_r(Ey)$ and since $x, y \in \mathcal{I}\mathrm{d}(E)$, $(1-x)E \subseteq (1-y)E$. Since y is right quasi-singular, $(1-y)E$ is proper. Therefore, $(1-x)E = (1-y)E$. Hence, since a topologically semiprime algebra is left preannihilator (see Lemma 2.3), $Ex = Ey$, i.e., $Ex \in \mathbf{m}_\ell(E)$.

ii) As in i) $(1-x)E \subseteq \mathcal{A}_r(I)$. If $\mathcal{A}_r(I) = E$, then $IE = (0)$, which implies $I = (0)$, a contradiction. Therefore $(1-x)E = \mathcal{A}_r(I)$ and thus $I = \mathcal{A}_\ell(\mathcal{A}_r(I)) = Ex$. ∎

Since semisimple implies semiprime, case i) in the above lemma, is a fortiori satisfied for semisimple annihilator Q'-algebras (see Theorem 3.6).

4 – Structure theorems

If E is a $\mathbb{C}$-algebra, we denote by $\mathcal{S}_\ell$ (resp. $\mathcal{S}_r$) the left (resp. right) socle of E. In case $\mathcal{S}_\ell = \mathcal{S}_r \equiv \mathcal{S}$, the resulted 2-sided ideal $\mathcal{S}$ is called the *socle* of E (see [16: p. 46]). Let $(L_i)_{i\in\Lambda}$ (resp. $(R_j)_{j\in K}$) be the family of all minimal left (resp. right) ideals of E; these families have the same set of indices, i.e., the set of all minimal elements, if, for instance, E is a topologically semiprime (D_ℓ) or (D_r)-algebra (cf. Corollaries 3.7 and 3.10). Moreover, $\mathcal{S}_\ell = \sum_{i\in\Lambda} Ex_i$, $\mathcal{S}_r = \sum_{i\in\Lambda} x_i E$. Hence the socle is defined and is given by

$$\mathcal{S} = \sum_{i\in\Lambda} Ex_i = \sum_{i\in\Lambda} x_i E \tag{4.1}$$

(see also [16: p. 46, Lemma 2.1.11 and Lemma 2.1.12]).

Lemma 4.1. *Let E be a left preannihilator topological algebra with dense socle. Then E is semiprime.*

Proof: Let $I \subseteq E$ be a 2-sided ideal with $I^2 = (0)$. If $J \in m_\ell(E)$, then either $J \cap I = J$ or $J \cap I = (0)$. Thus $IJ = (0)$, that is $J \subseteq \mathcal{A}_r(I)$. Therefore $\overline{\mathcal{S}} \subseteq \mathcal{A}_r(I)$, hence, by hypothesis, $I = (0)$. ∎

For the following result see also [6: p. 162, Proposition 5].

Proposition 4.2. *Let E be a non-radical topologically semiprime annihilator Q'-algebra. Then*

- **i**) $\mathcal{A}_\ell(\mathcal{S}) = \mathcal{A}_r(\mathcal{S}) = R(E)$;
- **ii**) $\overline{\mathcal{S}} \cap R(E) = (0)$.

If, moreover, $\mathcal{S}$ is dense in E, then

- **iii**) *E is semisimple.*

Proof: Since E is non-radical, it contains a right quasi-singular element, say z (cf. [16: p. 42 and p. 55, Theorem 2.3.2]), so that the regular right ideal $J = \{y - zy\colon\ y \in E\} \equiv (1-z)E$ is proper (see also [14: p. 66, Lemma 6.4]). Now, if M is a maximal regular right ideal, with $J \subseteq M$ (Krull), we claim that $\mathcal{A}_\ell(M) \nsubseteq R(E)$. Otherwise, and since $R(E) \subseteq M$ (see, for instance [16: p. 55, Theorem 2.3.2]), it would be $\mathcal{A}_\ell(M)^2 = (0)$. Therefore, since E is topologically semiprime (see also Theorem 2.1) $\mathcal{A}_\ell(M) = (0)$ and hence $M = E$, that is a contradiction. On the other hand, $M = (1-x)E$, $x \in \mathcal{I}\mathrm{d}(E) \cap \mathcal{A}_\ell(M)$ (cf. also the comments following Theorem 3.4). Now, since M is maximal, Ex is minimal with x minimal (see proofs of Theorem 3.4 and Corollary 3.10). Thus

$$R(E) = \bigcap_{x,\min} (1-x)E = \bigcap_{x,\min} \mathcal{A}_r(Ex) = \mathcal{A}_r(\mathcal{S})\ .$$

Likewise, $R(E) = \mathcal{A}_\ell(\mathcal{S})$ and this finishes the proof of i). By i) one gets $\mathcal{S} \subseteq \mathcal{A}_r(\mathcal{A}_\ell(\mathcal{S})) = \mathcal{A}_r(R(E))$, so that $\overline{\mathcal{S}} \subseteq \mathcal{A}_r(R(E))$ and hence (see also Lemma 2.5) $\overline{\mathcal{S}} \cap R(E) = (0)$, which proves ii). Finally, since $E = \overline{\mathcal{S}}$, iii) is a direct consequence of ii). ∎

By Theorem 3.8, a semisimple annihilator Q'-algebra contains minimal left and minimal right ideals. So we now get the following result (see also [6: p. 162, Corollary 6]).

Theorem 4.3. *An annihilator Q'-algebra E is semisimple if and only if it has a dense socle.*

Proof: By Lemma 4.1 and Proposition 4.2 we only have to prove that the condition is necessary. Indeed, let $x \in E$ such that $\mathcal{S}_\ell x = 0$. Then, by (4.1), $Ex_i x = (0)$, hence $x_i x = 0$ for every $i \in \Lambda$. Therefore,

$$x = x - x_i x \in (1 - x_i)E \equiv M_i \,, \qquad i \in \Lambda \,. \tag{4.2}$$

Moreover, by Lemma 2.6, the ideal M_i is maximal and thus maximal closed. Let now M be a maximal (closed) regular right ideal of E. Then (Theorem 3.4) $M = (1 - y)E$ with $y \in \mathcal{IP}(E) \cap \mathcal{A}_\ell(M)$. Now, the left ideal $\mathcal{A}_\ell(M) = Ey$ is minimal (ibid.), i.e., Ey is one among L_i and thus M is one of M_i, $i \in \Lambda$ (cf. (4.2)). Therefore, M_i, $i \in \Lambda$, exhaust all maximal (closed) regular right ideals of E. Hence $x \in \bigcap_{i\in\Lambda} M_i = R(E)$, so $x = 0$ by semisimplicity. Therefore, $\mathcal{A}_r(\overline{\mathcal{S}}_\ell) = (0)$, hence by hypothesis, $\overline{\mathcal{S}}_\ell = E$. Similarly, $\overline{\mathcal{S}}_r = E$. ∎

As follows from the previous proof, one gets the following (set-theoretic) bijections

$$\mathcal{M}_r(E) \cong m_\ell(E) \quad \text{ and } \quad \mathcal{M}_\ell(E) \cong m_r(E) \,.$$

Based on Theorem 4.3, one can get another proof of prop. 2)⇒1) of Theorem 3.6: Let $I \neq (0)$ be a left ideal of E that does not contain minimal left ideals. If $(L_i)_{i\in\Lambda}$ is the family of all minimal left ideals of E, then $I \cap L_i = I \cap Ex_i = (0)$, $x_i \in \mathcal{IP}(E)$ for every $i \in \Lambda$. This follows from Corollary 3.7 and the fact that E is topologically semiprime, as semisimple (cf. also [6: p. 155, Proposition 5]). Now let $z \in E$ and $i \in \Lambda$. Then either $Ex_i z = (0)$ or $Ex_i z = Ex_\lambda$ for some $\lambda \in \Lambda$ (see for instance [ibid: p. 155, Lemma 7]). Thus $Ex_i z \cap I = (0)$ for every $z \in E$, $i \in \Lambda$ and $x_i E \cap I = (0)$ for every $i \in \Lambda$. Hence, by $x_i EI \subseteq x_i E \cap I$, $x_i EI = (0)$ for every $i \in \Lambda$. On the other hand (Theorem 4.3), $E = \overline{\sum_{i\in\Lambda} x_i E}$. Therefore, $EI \subseteq \overline{\sum_{i\in\Lambda} x_i EI}$ and hence $I = (0)$, a contradiction. A similar proof establishes the analogous result for right ideals. Therefore, E is a (D)-algebra.

Concerning the following result see also [6: p. 163, Theorem 9].

Theorem 4.4. *Let E be a topologically semiprime annihilator algebra and $I \in \mathcal{L}$ such that $\overline{EI} = \overline{IE} = I$. Then I is a topologically semiprime annihilator algebra. Moreover, I is semisimple, if $I \cap R(E) = (0)$.*

Proof: Let $N \in \mathcal{L}_\ell(I)$ with $N^2 = (0)$. By Lemma 2.5 and the comments following it, $N \in \mathcal{L}_\ell$. Thus $N = (0)$ and so I is topologically semiprime. Let now

$K \in \mathcal{L}_\ell(I)$ with $K \neq I$. Then $\mathcal{A}_r(K) \cap I \neq (0)$, otherwise $IE \subseteq K$. Indeed, by considering the left ideal $J \equiv K \oplus \mathcal{A}_\ell(I)$ of E, one gets in turn

$$IJ \subseteq IK + I\mathcal{A}_\ell(I) \subseteq K \ . \tag{4.3}$$

On the other hand, $\mathcal{A}_r(J)I \subseteq \mathcal{A}_r(J) \cap I = (0)$. Hence $\mathcal{A}_r(J) \subseteq \mathcal{A}_\ell(I) \subseteq J$ and thus $\mathcal{A}_r(J)^2 \subseteq J\mathcal{A}_r(J) = (0)$. Therefore $\mathcal{A}_r(J) = (0)$ (see also Theorem 2.1) and hence $J = E$. Now, by (4.3) and the hypothesis, one gets $I = \overline{IE} \subseteq K$, that is a contradiction. Thus $\mathcal{A}^I_r(K) \equiv \mathcal{A}_r(K) \cap I \neq (0)$. Similarly, $\mathcal{A}^I_\ell(L) \neq (0)$ for $L \in \mathcal{L}_r(I)$ with $L \neq I$. Moreover, $\mathcal{A}^I_\ell(I) \equiv \mathcal{A}_\ell(I) \cap I = (0)$ (see Lemma 2.5). Analogously, $\mathcal{A}^I_r(I) = (0)$. The foregoing prove now that I *is an annihilator algebra*. Finally if $I \cap R(E) = (0)$, then by [6: p. 126, Corollary 20] $R(I) = (0)$ and this completes the proof. ∎

Corollary 4.5. *Let E be an annihilator topologically semiprime algebra. Then $\overline{\mathcal{S}}$ is an algebra of the same type. In particular, $\overline{\mathcal{S}}$ is also semisimple, if E is a non-radical Q'-algebra, as well.*

Proof: Let I be a minimal left ideal of E. Then (Corollary 3.7), $I = Ex$, $x \in \mathcal{I}\mathrm{d}(E)$, so that $Ex = Ex^2 \subseteq E\mathcal{S}$; hence, $\overline{\mathcal{S}} \subseteq \overline{E\overline{\mathcal{S}}}$, thus $\overline{\mathcal{S}} = \overline{E\overline{\mathcal{S}}}$. Likewise, $\overline{\mathcal{S}} = \overline{\overline{\mathcal{S}}E}$. So, by Proposition 4.2 and Theorem 4.4, we get the assertion. ∎

Now, by Theorem 4.3 we get the next. Cf. also [6: p. 163, Corollary 10].

Corollary 4.6. *In every semisimple annihilator Q'-algebra (with socle $\mathcal{S}$), $\overline{\mathcal{S}}$ is an algebra of the same type.* ∎

Proposition 4.7. *Let E be an $(\mathcal{M}_\ell)$-algebra, such that $\mathcal{A}_\ell(I) = (0)$ for every $I \in \mathcal{L}_\ell - \{(0)\}$ and $\mathcal{S}_\ell \equiv \sum_{i\in\Lambda} L_i \neq (0)$. Then $E = \overline{\mathcal{S}}_\ell$.* ∎

Now we obtain a second structure theorem for a semisimple annihilator Q'-algebra. For this we apply Theorem 4.3. We will make use of the following terminology: *Let E be a semiprime topological algebra and $(K_\alpha)_{\alpha\in A}$ the family of all minimal closed 2-sided ideals of E. Then the sum of K_α, $\alpha \in A$, $K \equiv \sum_{\alpha\in A} K_\alpha$ is direct*, so we have $K = \bigoplus_{\alpha\in A} K_\alpha$. Moreover, if $K_\alpha \neq K_\beta$, then $K_\alpha K_\beta = \{0\}$; in this case $\bigoplus_{\alpha\in A} K_\alpha$ is called *direct sum of the ideals K_α*, while $\overline{\bigoplus_{\alpha\in A} K_\alpha}$ is said to be the *topological direct sum of the ideals K_α*. For the proof we use the argument of [15: p. 328, Theorem 5].

The next lemma specializes to a similar result in [7: p. 158, Theorem 5], whose proof can be adapted to our case.

Lemma 4.8. *Let E be a topologically semiprime algebra and I a minimal (closed) left or right ideal of E. If $\mathcal{RL}(I)$ is the closed 2-sided ideal generated by I, then $\mathcal{RL}(I)$ is minimal closed.* ∎

Lemma 4.9. *Let E be a topologically semiprime algebra which, moreover, is a $(D_r\mathcal{M}'_r)$-algebra. If I is a minimal closed 2-sided ideal of E, then I is a left preannihilator $(\mathcal{M}'_r)$-algebra.*

Proof: Since $\mathcal{A}^I_\ell(I) \equiv I \cap \mathcal{A}_\ell(I) = (0)$ (see also Lemma 2.5) the algebra I is left preannihilator. Let now $K \in \mathcal{L}_\ell(I)$ with $K \neq I$. Then (ibid.) $\mathcal{L}_\ell(I) \subseteq \mathcal{L}_\ell(E)$ and $\mathcal{A}_\ell(I) = \mathcal{A}_r(I)$. Consider the left ideal $J \equiv K + \mathcal{A}_\ell(I) = K + \mathcal{A}_r(I)$. Then either $E = \overline{J}$ or $E \neq \overline{J}$. If $E = \overline{J}$, then for $x \in I$, $xJ = xK + x\mathcal{A}_r(I)$ which implies $xJ \subseteq K$. Therefore, for every $x \in I$, $xE = x\overline{J} \subseteq \overline{xJ} \subseteq K$, that yields $I^2 \subseteq K$. Now $I^2 \neq (0)$, otherwise $I = (0)$, which is a contradiction. Besides, $(0) \neq \mathcal{RL}(I^2) \subseteq I$ and thus $\mathcal{RL}(I^2) = I$. On the other hand, $\mathcal{RL}(I^2) \subseteq K \subseteq I$. Therefore $I = K$, a contradiction. Hence $E \neq \overline{J}$. Moreover, $\overline{J} \in \mathcal{L}_\ell$, thus $\mathcal{A}_r(\overline{J}) \neq (0)$ and a fortiori $\mathcal{A}_r(J) \neq (0)$. Now, by hypothesis, $\mathcal{A}_r(J)$ contains a minimal right ideal and hence contains a primitive idempotent, say x_0. Consider (Theorem 3.9 and Lemma 4.8) the minimal closed 2-sided ideal $\mathcal{RL}(x_0E)$. Then, since $\mathcal{RL}(x_0E) \cap I \in \mathcal{L}$, either $\mathcal{RL}(x_0E) \cap I = (0)$ or $I = \mathcal{RL}(x_0E) \cap I = \mathcal{RL}(x_0E)$. In the first case, $x_0 I = (0)$ and hence $x_0 \in \mathcal{A}_\ell(I) \subseteq J$. Therefore, since $x_0 \in \mathcal{A}_r(J)$, $x_0^2 = 0$, a contradiction. Thus, since $x_0^2 = x_0 \in \mathcal{RL}(x_0E)$, $x_0 \in I$ and therefore $x_0 \in \mathcal{A}_r(J) \cap I$, namely $\mathcal{A}^I_r(J) \neq (0)$ and thus $\mathcal{A}^I_r(K) \neq (0)$. I.e., the algebra I is an $(\mathcal{M}'_r)$-algebra. ∎

By Theorem 3.6, Lemma 4.9 and [6: p. 126, Corollary 20] we get the following.

Corollary 4.10. *Every minimal closed 2-sided ideal of a semisimple annihilator Q'-algebra is an annihilator semisimple algebra.* ∎

Lemma 4.11. *Let E be a dual topologically semiprime algebra. Then every minimal closed 2-sided ideal I of E is a dual algebra, as well.*

Proof: We prove that the algebra I is left dual; i.e., $\mathcal{A}^I_\ell(\mathcal{A}^I_r(K)) = \mathcal{A}_\ell(\mathcal{A}_r(K) \cap I) \cap I = K$, $K \in \mathcal{L}_\ell(I)$. Similarly, we prove the right duality of I. It suffices to show that $\mathcal{A}_\ell(\mathcal{A}_r(K) \cap I) \cap I \subseteq K$ for every $K \in \mathcal{L}_\ell(I)$. By Lemma 2.5, $\mathcal{L}_\ell(I) \subseteq \mathcal{L}_\ell(E)$, thus $K \in \mathcal{L}_\ell(E)$. Therefore (cf., for instance, [15: p. 231, (5_α)]),

$$\mathcal{A}_\ell(\mathcal{A}_r(K) \cap I) = \overline{\mathcal{A}_\ell(\mathcal{A}_r(K)) + \mathcal{A}_\ell(I)} = \overline{K + \mathcal{A}_\ell(I)} = \overline{K + \mathcal{A}_r(I)}$$

(see also Lemma 2.5). Moreover, since $K \in \mathcal{L}_\ell(I)$ and $I\mathcal{A}_r(I) = (0)$, $I(K + \mathcal{A}_r(I)) \subseteq IK \subseteq K$, i.e., $I(\overline{K + \mathcal{A}_r(I)}) \subseteq K$. Thus, if $x \in \mathcal{A}_\ell(\mathcal{A}_r(K) \cap I) \cap I$, $x \in \overline{K + \mathcal{A}_r(I)}$ and therefore $Ix \subseteq K$. Hence, $Ix\mathcal{A}_r(K) \subseteq K\mathcal{A}_r(K) = (0)$. On the other hand, since $\mathcal{A}_r(I)I = \mathcal{A}_\ell(I)I = (0)$ and $x \in I$, $\mathcal{A}_r(I)x\mathcal{A}_r(K) = (0)$.

Therefore, $(I+\mathcal{A}_r(I))x\mathcal{A}_r(K)=(0)$. Besides, $Ex\mathcal{A}_r(K)\subseteq\overline{(I+\mathcal{A}_r(I))x\mathcal{A}_r(K)}=(0)$ (ibid.) which implies $x\in\mathcal{A}_\ell(\mathcal{A}_r(K))=K$. ∎

The next theorem is the analogue in our case of the classical *second structure theorem of Wedderburn*. It thus extends previous ones in [7: p. 158, Theorem 6] and [15: p. 328, Theorem 5].

Theorem 4.12 (Second structure theorem)**.** *Every semisimple annihilator Q'-algebra E is the topological direct sum of its minimal closed 2-sided ideals, i.e.,*

$$E=\overline{\bigoplus_{\alpha\in A}K_\alpha}\,. \tag{4.4}$$

Moreover, each K_α is a semisimple topologically simple annihilator algebra. In particular, if E is a dual algebra, every K_α is a dual algebra too.

Proof: By Theorem 3.6 there exists a minimal left ideal, say L_i^α, such that $L_i^\alpha\subseteq K_\alpha$. Hence $L_i^\alpha\subseteq\mathcal{RL}(L_i^\alpha)\subseteq K_\alpha$, that is $\mathcal{RL}(L_i^\alpha)=K_\alpha$. Now, if $(L_i)_{i\in\Lambda}$ is the family of all minimal left ideals of E, then (Lemma 4.8) $\mathcal{RL}(L_i)$ is one of K_α's; therefore, for every $i\in\Lambda$, $L_i\subseteq\mathcal{RL}(L_i)\subseteq\bigcup_{\alpha\in A}K_\alpha$. Thus $\sum_{i\in\Lambda}L_i\subseteq\sum_{\alpha\in A}K_\alpha$ and $E=\overline{\bigoplus_{\alpha\in A}K_\alpha}$ (cf. Theorem 4.3). Moreover (Corollary 4.10), every K_α is an annihilator semisimple algebra. Now, if $(0)\neq J\in\mathcal{L}(K_\alpha)$, then (Lemma 2.5) $J\in\mathcal{L}$; hence $J=K_\alpha$, that is K_α is topologically simple (there are no closed 2-sided ideals) and this along with Lemma 4.11 completes the proof. ∎

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