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ON THE NILPOTENT RANK OF PARTIAL TRANSFORMATION SEMIGROUPS

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Synopsis. In [7] Sullivan proved that the semigroup SP_n of all strictly partial transformations on the set $X_n = \{1, ..., n\}$ is nilpotent-generated if n is even, and that if n is odd the nilpotents in SP_n generate $SP_n \backslash W_{n-1}$ where W_{n-1} consists of all elements in $[n-1, n-1]$ whose completions are odd permutations. We now show that whether n is even or odd both the rank and the nilpotent rank of the subsemigroup of SP_n generated by the nilpotents are equal to $n+2$.

1 – Introduction

Let P_n be the semigroup of all partial transformations on the set $X_n = \{1, ..., n\}$. An element α in P_n is said to have *projection characteristic* (k, r) or to belong to $[k, r]$ if $|\operatorname{dom}\alpha| = k$ and $|\operatorname{im}\alpha| = r$. Every element $\alpha \in [n-1, n-1]$ has domain $X_n \backslash \{i\}$ and image $X_n \backslash \{j\}$ for some i, j in X_n. Hence there is a unique element α^* in $[n, n]$ associated with α, defined by

$$i\,\alpha^* = j\,, \qquad x\,\alpha^* = x\,\alpha \ \text{ otherwise}\,,$$

and called the *completion* of α. In [3] Gomes and Howie proved that if n is even the subsemigroup SI_n of P_n consisting of all strictly partial one-to-one transformations is nilpotent generated. For n odd they showed that the nilpotents in SI_n generate $SI_n \backslash W_{n-1}$ where W_{n-1} consists of all $\alpha \in [n-1, n-1]$ whose completions are odd permutations.

Simultaneously and independently, Sullivan [7] investigated the corresponding question for SP_n, the subsemigroup of P_n consisting of all elements that are strictly partial, where the answer turns out to be similar: If N is the set of

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nilpotents in SP_n then

$$\langle N\rangle = \begin{cases} SP_n & \text{if } n \text{ is even,} \\ SP_n\backslash W_{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

In [4] Gomes and Howie raised the question of the rank of the semigroup SI_n. They showed that SI_n has rank $n+1$, and if n is even its nilpotent rank is also $n+1$. For n odd they showed that both the rank and the nilpotent rank of $SI_n\backslash W_{n-1}$ are equal to $n+1$. In this paper we show that SP_n has rank $n+2$, and if n is even its nilpotent rank is also $n+2$. The rank and the nilpotent rank of $SP_n\backslash W_{n-1}$ are also shown to be both equal to $n+2$, when n is odd.

2 – Preliminaries

The subsemigroup SP_n has n $\mathcal{J}$-classes, namely $J_{n-1}, J_{n-2}, ..., J_0$ (where J_0 consists of the empty map). For each r in $\{1, 2, ..., n-1\}$,

$$J_r = \bigcup_{k=1}^{n-1} [k, r] \ .$$

Lemma 2.1. *For each $\mathcal{J}$-class J_r in SP_n, where $r \le n$–3, we have $J_r \subseteq (J_{r+1})^2$.*

Proof: Suppose first that $\alpha = \begin{pmatrix} A_1 & A_2 & ... & A_r \\ b_1 & b_2 & ... & b_r \end{pmatrix} \in [k, r]$ where $r \le k \le n-2$. Then

$$\alpha = \begin{pmatrix} A_1 & A_2 & ... & A_r & x \\ a_1 & a_2 & ... & a_r & x \end{pmatrix} \begin{pmatrix} a_1 & a_2 & ... & a_r & y \\ b_1 & b_2 & ... & b_r & z \end{pmatrix},$$

a product of two elements in J_{r+1}, where $x, y \in X_n\backslash \operatorname{dom}\alpha$, $z \in X_n\backslash \operatorname{im}\alpha$ with $x \ne y$ and $\alpha_i \in A_i$ for all i.

Suppose now that $\alpha \in [n-1, r]$. We may suppose that A_1 contains more than one element. If $a_1, a_1' \in A_1$, then

$$\alpha = \begin{pmatrix} A_1\backslash\{a_1'\} & a_1' & A_2 & ... & A_r \\ a_1 & a_1' & a_2 & ... & a_r \end{pmatrix} \begin{pmatrix} \{a_1, a_1'\} & a_2 & ... & a_r & x \\ b_1 & b_2 & ... & b_r & y \end{pmatrix},$$

a product of two elements in J_{r+1}, where $x \in X_n\backslash \operatorname{dom}\alpha$, $y \in X_n\backslash \operatorname{im}\alpha$ and $a_i \in A_i$ for all i. ∎

Lemma 2.2. *For all $r \le n-2$, $[r, r] \subseteq ([r+1, r+1])^2$.*

Proof: Suppose that $\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix} \in [r, r]$. Then

$$\alpha = \begin{pmatrix} a_1 & \dots & a_r & x \\ a_1 & \dots & a_r & x \end{pmatrix} \begin{pmatrix} a_1 & \dots & a_r & y \\ b_1 & \dots & b_r & z \end{pmatrix},$$

where $x, y \in X_n \backslash \operatorname{dom} \alpha$, $z \in X_n \backslash \operatorname{im} \alpha$ with $x \neq y$. ∎

The following result follows from [6]

Lemma 2.3. *Every element $\alpha \in SP_n$ of height r is expressible as a product of nilpotents of the same height (where the height of α is defined to be $|\operatorname{im} \alpha|$).* ∎

Before considering the next result, we would like to clarify the notion of rank in an inverse semigroup and in a semigroup that is not necessarily inverse. By the rank of an inverse semigroup S we shall mean the cardinality of any subset A of minimal order in S such that $\langle A \cup V(A) \rangle = S$, where $V(A)$ is the set of inverses of elements in A. On the other hand, the rank of the semigroup S is simply the cardinality of any subset B of minimal order in S such that $\langle B \rangle = S$. If the subset A (or B) consists of nilpotents, the rank is called the nilpotent rank. We shall sometimes want to distinguish between the rank of an inverse semigroup S as an inverse semigroup and its rank as a semigroup.

Proposition 2.4. *Let $B = B(G, \{1, ..., n\})$ be a Brandt semigroup, where G is a finite group of rank r $(r \geq 1)$. Then the rank of B (as a semigroup) is $r + n - 1$.*

Proof: By Theorem 3.3 in [4] the rank of B as an inverse semigroup is $r + n - 1$. But the rank of B as a semigroup is potentially greater than its rank as an inverse semigroup. For if A is a generating set for B as a semigroup and $|A| = s$, then certainly A together with its inverses generates B, and so $s \geq r + n - 1$.

It now remains to show that we can select a generating set for B consisting of $r + n - 1$ elements. Let

$$A = \Big\{(1, g_1, 1), ..., (1, g_{r-1}, 1), (1, g_r, 2), (2, e, 3), ..., (n-1, e, n), (n, e, 1)\Big\},$$

where e is the identity of G and $\{g_1, ..., g_r\}$ is a generating set for G. We first show that $(1, g_r, 1)$ and $(1, e, 2)$ belong to $\langle A \rangle$.

$$(1, g_r, 1) = (1, g_r, 2)\,(2, e, 3) \cdots (n-1, e, n)\,(n, e, 1)\ .$$

Observe that

$$(1, g_r^2, 2) = (1, g_r, 2)\,(2, e, 3) \cdots (n-1, e, n)\,(n, e, 1)\,(1, g_r, 2)$$

and

$$(1, g_r^3, 2) = (1, g_r^2, 2)\,(2, e, 3) \cdots (n-1, e, n)\,(n, e, 1)\,(1, g_r, 2)\ .$$

Continuing in this way, we see that $(1, g_r^s, 2) \in \langle A \rangle$ for $s = 1, 2, \dots$. If t is the least integer for which $g_r^t = e$ then

$$(1, e, 2) = (1, g_r^t, 2) \in \langle A \rangle\ .$$

Let (i, g, j) be an arbitrary element in B. Then

$$(i, g, j) = (i, e, i+1) \cdots (n-1, e, n)\,(n, e, 1)\,(1, g, 1)\,(1, e, 2) \cdots (j-1, e, j)$$

and it is clear that $(1, g, 1)$ can be expressed as a product of the elements $(1, g_1, 1), \dots, (1, g_r, 1)$. Hence

$$\langle A \rangle = B\ .$$

Since $|A| = r + n - 1$ the proof is complete. ∎

As remarked in [4], the principal factor $PF_{n-1} = SP_n/(J_{n-2} \cup \dots \cup J_0)$ is a Brandt semigroup, where PF_{n-1} may be thought of in the usual way as $J_{n-1} \cup \{0\}$, and the product in PF_{n-1} of two elements of J_{n-1} is the product in SP_n if this lies in J_{n-1} and is 0 otherwise. The Brandt semigroup PF_{n-1} has the structure $B(G, I)$, where $G = S_{n-1}$, the symmetric group on $n-1$ symbols, and $I = \{1, \dots, n\}$. (See [6], section II.3.)

Let A be an irredundant set of generators of SI_n. Since SI_n is generated by the elements in J_{n-1}, we may choose to regard A as a subset of PF_{n-1}. The conclusion (as in [4]) is that *A generates SI_n if and only if it generates PF_{n-1}*.

The following Proposition now follows:

Proposition 2.5. *Let SI_n be the inverse semigroup of all strictly partial one-one maps on X_n, where $n \geq 3$. Then the rank of SI_n (as a semigroup) is $n+1$.* ∎

Proposition 2.6. *Let $n \geq 4$ be even. Then the nilpotent rank of SI_n (as a semigroup) is $n+1$.*

Proof: Define $H_{i,j}$ to consists of all elements α for which $\operatorname{dom}\alpha = X_n \backslash \{i\}$ and $\operatorname{im}\alpha = X_n \backslash \{j\}$. For $i = 4, \dots, n-1$ define a mapping $\xi_i \in H_{i,n}$ by

$$\xi_i = \begin{cases} \begin{pmatrix} 1 & 2 & \dots & i-1 & i+1 & \dots & n-i+1 & n-i+2 & n-i+3 & n-i+4 & \dots & n \\ i & i+1 & \dots & 2i-2 & 2i-1 & \dots & n-1 & 2 & 1 & 3 & \dots & i-1 \end{pmatrix} & \text{if } i \leq n/2, \\ \begin{pmatrix} 1 & 2 & \dots & (n/2)-1 & (n/2) & (n/2)+2 & (n/2)+3 & \dots & n \\ (n/2)+1 & (n/2)+2 & \dots & n-1 & 2 & 1 & 3 & \dots & (n/2) \end{pmatrix} & \text{if } i = (n/2)+1, \\ \begin{pmatrix} 1 & 2 & \dots & n-i & n-i+1 & n-i+2 & n-i+3 & \dots & i-1 & i+1 & \dots & n \\ i & i+1 & \dots & n-1 & 2 & 1 & 3 & \dots & 2i-n-1 & 2i-n & \dots & i-1 \end{pmatrix} & \text{if } i \geq (n/2)+2, \end{cases}$$

and

$$\xi_1 = \begin{pmatrix} 2 & 3 & \dots & n \\ 1 & 2 & \dots & n-1 \end{pmatrix},$$

$$\xi_2 = \begin{pmatrix} 1 & 3 & 4 & \dots & n-1 & n \\ 1 & 3 & 4 & \dots & n-1 & 2 \end{pmatrix},$$

$$\xi_3 = \begin{pmatrix} 1 & 2 & 4 & \dots & n-2 & n-1 & n \\ 3 & 4 & 5 & \dots & n-1 & 2 & 1 \end{pmatrix},$$

$$\xi_n = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 \\ 2 & 1 & 3 & \dots & n-1 \end{pmatrix}.$$

Then it is easy to verify that the mapping

$$\phi\colon\ B(S_{n-1}, \{1, ..., n\}) \to Q_{n-1}\ ,$$

defined by

$$(i, \eta, j)\, \phi = \xi_i\, \eta\, \xi_j^{-1}\ ,$$

is an isomorphism, where S_{n-1} is the symmetric group on X_{n-1}, and Q_{n-1} is the principal factor $SP_n/(J_{n-2} \cup ... \cup J_0)$.

From Proposition 2.4, the set

$$A = \Big\{(1, g_1, 1), (1, g_2, 2), (2, e, 3), ..., (n-1, e, n), (n, e, 1)\Big\}\ ,$$

where $g_1 = (1\ 2\ 3\ ...\ n-1)$, $g_2 = (1\ 2)$ and e is the identity permutation in S_{n-1}, generates $B(S_{n-1}, \{1, ..., n\})$. Thus $A\phi$ generates Q_{n-1} and hence SI_n. From [3] we borrow the notation $\|a_1\, a_2 \cdots a_n\|$ for the nilpotent α with domain $X_n \backslash \{a_n\}$ and image $X_n \backslash \{a_1\}$ for which $a_i\, \alpha = a_{i+1}$ $(i = 1, ..., n-1)$. Then it is easy to verify that

$$A\phi = \Big\{\beta, \alpha_1, \alpha_2, ..., \alpha_n\Big\}\ ,$$

where

$$\beta = \begin{pmatrix} 2 & 3 & \dots & n-1 & n \\ 3 & 4 & \dots & n & 2 \end{pmatrix}, \quad \alpha_n = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 \\ 3 & 2 & 4 & \dots & n \end{pmatrix},$$

$$\alpha_1 = \|2\ n\ n-1\ ...\ 3\ 1\|\,, \qquad \alpha_{n-1} = \|n\ n-2\ ...\ 1\ n-1\|$$

and

$$\alpha_i = \|i+1\ i-1\ i-2\ ...\ 1\ n\ n-1\ ...\ i+2\ i\| \quad \text{for}\ \ i = 2, ..., n-2\ ,$$

with $\beta = \xi_1 g_1 \xi_1^{-1}$, $\alpha_1 = \xi_1 g_2 \xi_2^{-1}$, $\alpha_i = \xi_i \xi_{i+1}^{-1}$ for $i = 2, ..., n-1$ and $\alpha_n = \xi_n \xi_1^{-1}$.

Now, bearing in mind that n is even, let

$$\delta_1 = \|1\ 4\ 6\ ...\ n-2\ n\ 3\ 5\ ...\ n-1\ 2\|\,, \qquad \delta_2 = \|2\ n-1\ n-2\ n-3\ ...\ 3\ 1\ n\|\ ;$$

then

$$\alpha_1\,\delta_1 = \beta \quad \text{and} \quad \delta_2\,\delta_1 = \alpha_n\ .$$

Hence the $n+1$ nilpotents

$$\alpha_1,\ ...,\ \alpha_{n-1},\ \delta_1,\ \delta_2$$

generate SI_n. The result follows from Proposition 2.5. ∎

Proposition 2.7. *Let N' be the set of all nilpotents in SI_n, where $n \geq 5$ is odd. Then the rank of $\langle N' \rangle$ (as a semigroup) is equal to its nilpotent rank and is $n+1$.*

Proof: As in Proposition 2.4, we first notice that the rank of $\langle N' \rangle$ must be greater or equal to $n+1$. For $i = 4, ..., n-1$ define a mapping $\lambda_i \in H_{i,n}$ (where $H_{i,n}$ consists of all elements α for which $\operatorname{dom}\alpha = X_n \backslash \{i\}$ and $\operatorname{im}\alpha = X_n \backslash \{n\}$) by

$$\lambda_i = \begin{cases} \begin{pmatrix} 1 & 2 & ... & i-1 & i+1 & ... & n-i+1 & n-i+2 & n-i+3 & n-i+4 & n-i+5 & ... & n \\ i & i+1 & ... & 2i-2 & 2i-1 & ... & n-1 & 2 & 3 & 1 & 4 & ... & i-1 \end{pmatrix} & \text{if } 4 \leq i < k, \\ \begin{pmatrix} 1 & 2 & ... & k-1 & k+1 & k+2 & k+3 & k+4 & ... & n \\ k & k+1 & ... & n-1 & 2 & 3 & 1 & 4 & ... & k-1 \end{pmatrix} & \text{if } i = k, \\ \begin{pmatrix} 1 & 2 & ... & k-2 & k-1 & k & k+2 & k+3 & ... & n \\ k+1 & k+2 & ... & n-1 & 2 & 3 & 1 & 4 & ... & k \end{pmatrix} & \text{if } i = k+1, \\ \begin{pmatrix} 1 & 2 & ... & n-i & n-i+1 & n-i+2 & n-i+3 & n-i+4 & ... & i-1 & i+1 & ... & n \\ i & i+1 & ... & n-1 & 2 & 3 & 1 & 4 & ... & 2i-n-1 & 2i-n & ... & i-1 \end{pmatrix} & \text{if } i \geq k+2, \end{cases}$$

where $k = (n+1)/2$, and

$$\lambda_1 = \begin{pmatrix} 2 & 3 & ... & n \\ 1 & 2 & ... & n-1 \end{pmatrix},$$

$$\lambda_2 = \begin{pmatrix} 1 & 3 & 4 & ... & n-1 & n \\ 3 & 1 & 4 & ... & n-1 & 2 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 2 & 4 & ... & n-2 & n-1 & n \\ 1 & 4 & 5 & ... & n-1 & 2 & 3 \end{pmatrix},$$

$$\lambda_n = \begin{pmatrix} 1 & 2 & 3 & 4 & ... & n-1 \\ 2 & 3 & 1 & 4 & ... & n-1 \end{pmatrix}.$$

Then the mapping

$$\psi\colon\ B(A_{n-1}, \{1, ..., n\}) \to Q_{n-1} \backslash W_{n-1}\ ,$$

defined by

$$(i, \mu, j)\,\psi = \lambda_i\,\mu\,\lambda_j^{-1}\ ,$$

is an isomorphism, where A_{n-1} is the alternating group on X_{n-1}. For if we let

$$\lambda_i^* = \lambda_i \cup (i, n) ,$$

then the total number of inversions in λ_i^* is

$$i(n-i)+2 \quad \text{for} \quad i \geq 4 ,$$

and is $n-1$, $2n-4$, $3n-11$ for $i = 1,2,3$ respectively (see for example [1], pp. 60–61). These numbers are clearly all even. Thus λ_i^* is an even permutation for all i. Hence $\lambda_i \in Q_{n-1} \backslash W_{n-1}$ and so the mapping ψ is well defined. It is easy to verify that ψ is a bijective homomorphism.

From Coxeter and Moser ([2], section 6.3) we find that A_{n-1} is of rank 2 (provided $n \geq 4$), being generated by

$$(1\ 2)\ (3\ \cdots\ n{-}1) \quad \text{and} \quad (1\ 2\ 3) .$$

From Proposition 2.4 the set

$$A = \Big\{(1,g_1,1), (1,g_2,2), (2,e,3), ..., (n-1,e,n), (n,e,1)\Big\} ,$$

where $g_1 = (1\ 2)\ (3 \cdots\ n{-}1)$, $g_2 = (1\ 2\ 3)$ and e is the identity permutation in A_{n-1}, generates $B(A_{n-1}, \{1,...,n\})$. Thus $A\psi$ generates $Q_{n-1} \backslash W_{n-1}$ and hence $\langle N' \rangle$.

It is easy to verify that

$$A\psi = \Big\{\beta, \alpha_1, ..., \alpha_n\Big\} ,$$

where

$$\beta = \begin{pmatrix} 2 & 3 & 4 & ... & n{-}1 & n \\ 3 & 2 & 5 & ... & n & 4 \end{pmatrix},$$

$$\alpha_1 = \|2\ n\ n{-}1\ ...\ 3\ 1\| , \quad \alpha_n = \|1\ 3\ 2\ 4\ 5\ ...\ n\| ,$$

and

$$\alpha_i = \|i{+}1\ i{-}1\ ...\ 1\ n\ n{-}1\ ...\ i{+}2\ i\| \quad \text{for} \quad i = 2, ..., n{-}1 ,$$

with $\beta = \lambda_1 g_1 \lambda_1^{-1}$, $\alpha_1 = \lambda_1 g_2 \lambda_2^{-1}$, $\alpha_i = \lambda_i \lambda_{i+1}^{-1}$ for $i = 2, ..., n{-}1$ and $\alpha_n = \lambda_n \lambda_1^{-1}$.

Now, let $\delta_1 = \|n\ 1\ 3\ 4\ 5\ ...\ n{-}1\ 2\|$; then

$$\beta = \alpha_1\, \delta_1\, \alpha_n .$$

Hence the $n+1$ nilpotents

$$\alpha_1,\ \alpha_2,\ ...,\ \alpha_n,\ \delta_1$$

generates $\langle N' \rangle$. ∎

We now close this section with a result that is of independent interest. Here $\{e\}$ denotes the trivial group.

Proposition 2.8. *The rank of* $B(\{e\},\{1,...,n\})$ *is* n.

Proof: The set of n elements

$$A=\Big\{(1,e,2),(2,e,3),...,(n-1,e,n),(n,e,1)\Big\}$$

generates $B(\{e\},\{1,...,n\})$. If (i,e,j) is an arbitrary element in $B(\{e\},\{1,...,n\})$, then

$$(i,e,j)=(i,e,i+1)\cdots(n-1,e,n)\,(n,e,1)\,(1,e,2)\cdots(j-1,e,j)\ .$$

Since any generating set must cover the $\mathcal{R}$-classes (as well as the $\mathcal{L}$-classes) by Lemma 3 in [5], and since the number of $\mathcal{R}$-classes in $B(\{e\},\{1,...,n\})$ is n, no set of fewer than n elements can generate $B(\{e\},\{1,...,n\})$. Hence the result. ∎

3 – Strictly partial maps

Theorem 3.1. *The rank of* SP_n *is* $n+2$.

Proof: We begin by showing that every generating set G of SP_n must contain at least $n+2$ elements. The top $\mathcal{J}$-class is $[n-1,n-1]$, and since this consists entirely of one-one maps it does not generate SP_n. From Lemmas 2.1 and 2.2 we have

$$SP_n=\Big\langle[n-1,n-1]\cup[n-1,n-2]\Big\rangle\ .$$

It is clear that in generating the elements of $[n-1,n-1]$ only elements of $[n-1,n-1]$ may be used, and by Propositon 2.5 and the remarks made just before at least $n+1$ elements are needed to generate $[n-1,n-1]$. Thus

$$\Big|G\cap[n-1,n-1]\Big|\geq n+1\ .$$

In generating the elements of $[n-1,n-2]$ at least one of the elements must be from $[n-1,n-2]$. That is

$$|G|\geq(n+1)+1=n+2\ .$$

To generate $[n-1,n-2]$ we now show that only one element from $[n-1,n-2]$ is needed. Let $\alpha\in[n-1,n-2]$ be given by

$$\alpha=\begin{pmatrix}\{a_1,a_2\} & a_3 & ... & a_{n-1}\\ b_1 & b_3 & ... & b_{n-1}\end{pmatrix}.$$

Then

$$\alpha = \gamma_1\,\gamma_2\,\gamma_3\ ,$$

where

$$\gamma_1 = \begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} \\ 1 & 2 & \dots & n-1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} \{1,2\} & 3 & \dots & n-1 \\ 3 & 4 & \dots & n \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 3 & 4 & \dots & n & 1 \\ b_1 & b_3 & \dots & b_{n-1} & b_2 \end{pmatrix}$$

and $b_2 \in X_n \backslash \operatorname{im}\alpha$. It is clear that $\gamma_1, \gamma_3 \in [n-1, n-1]$ and γ_2 is a fixed element in $[n-1, n-2]$. This completes the proof of the Theorem. ∎

Theorem 3.2. *Let $n \geq 4$ be even. Then the nilpotent rank of SP_n is $n+2$.*

Proof: From Proposition 2.6 and the proof of Theorem 3.1, the $n+2$ nilpotents

$$\alpha_1,\ \dots,\ \alpha_{n-1},\ \delta_1,\ \delta_2,\ \gamma_2$$

generate SP_n, where $\alpha_1, \dots, \alpha_{n-1}, \delta_1, \delta_2$ are as defined in Proposition 2.6 and γ_2 as in Theorem 3.1. ∎

Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ c_1 & c_2 & c_3 & \dots & c_n \end{pmatrix}$$

be a permutation on X_n, and define another permutation β on X_n by

$$1\beta = 2\alpha\,, \qquad 2\beta = 1\alpha \qquad \text{and} \qquad x\,\beta = x\,\alpha \ \ \text{otherwise.}$$

Thus

$$\beta = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ c_2 & c_1 & c_3 & \dots & c_n \end{pmatrix} = (1\ 2)\ \alpha\ .$$

So α is even if and only if β is odd and vise versa.

Theorem 3.3. *Let $n \geq 5$ be odd. Then the rank and the nilpotent rank of $SP_n \backslash W_{n-1}$ are both equal to $n+2$.*

Proof: $[n-1, n-1] \backslash W_{n-1}$ consists of one-one maps, so it does not generate $SP_n \backslash W_{n-1}$, as remarked in the proof of Theorem 3.1. From Lemma 2.1 above and Lemma 3.15 in [3]

$$SP_n \backslash W_{n-1} = \Big\langle \Big([n-1, n-1] \backslash W_{n-1}\Big) \cup [n-1, n-2] \Big\rangle\ .$$

From Proposition 2.7, at least $n+1$ elements are needed to generate $[n-1, n-1] \backslash W_{n-1}$. Moreover the $n+1$ elements may as well be all nilpotents.

As remarked in the proof of Theorem 3.1, to generate $[n-1, n-2]$, at least one of the elements must be from $[n-1, n-2]$. Thus if G is a set of generators of $SP_n \backslash W_{n-1}$. Then

$$|G| \geq n+2 \ .$$

It now remains to prove that every element $\alpha \in [n-1, n-2]$ is expressible as a product of nilpotents in $[n-1, n-1]$ and a fixed nilpotent from $[n-1, n-2]$. So let $\alpha \in [n-1, n-2]$ be

$$\begin{pmatrix} \{a_1, a_2\} & a_3 & \ldots & a_{n-1} \\ b_1 & b_3 & \ldots & b_{n-1} \end{pmatrix} .$$

Then α can be expressed as $\gamma_1 \, \beta \, \delta$, or alternatively as $\gamma_2 \, \beta \, \delta$, where

$$\gamma_1 = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_{n-1} \\ 1 & 2 & 3 & \ldots & n-1 \end{pmatrix} , \quad \gamma_2 = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_{n-1} \\ 2 & 1 & 3 & \ldots & n-1 \end{pmatrix} ,$$

$$\beta = \begin{pmatrix} \{1, 2\} & 3 & 4 & \ldots & n-1 \\ 3 & 4 & 5 & \ldots & n \end{pmatrix} , \quad \delta = \begin{pmatrix} 3 & 4 & 5 & \ldots & n \\ b_1 & b_3 & b_4 & \ldots & b_{n-1} \end{pmatrix} .$$

Here β is a fixed nilpotent in $[n-1, n-2]$, δ is an element in $[n-2, n-2]$ and by Lemma 3.15 in [3] is expressible as a product of two nilpotents in $[n-1, n-1]$. By the argument preceding the statement of the Theorem either the completion of γ_1 or that of γ_2 is even, and hence by Lemma 3.10 in [3] either γ_1 or γ_2 is expressible in terms of nilpotents in $[n-1, n-1]$. ∎

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REFERENCES

[1] Baumslag, B. and Chandler, B. – *Group theory*, Schaum's outline, McGraw-Hill, New York, 1968.

[2] Coxeter, H.S.M. and Moser, W.O.J. – *Generators and relations for discrete groups*, 3rd ed., Springer, 1972.

[3] Gomes, G.M.S. and Howie, J.M. – Nilpotents in finite symmetric inverse semigroups, *Proc. Edinburgh Math. Soc.*, 30 (1987), 383–395.

[4] Gomes, G.M.S. and Howie, J.M. – On the ranks of certain finite semigroups of transformations, *Math. Proc. Cambridge Phil. Soc.*, 101 (1987), 395–403.

[5] Howie, J.M. and McFadden, R.B. – Idempotent rank in finite full transformation semigroups, *Proc. Royal Soc. Edinburgh*, 114A (1990), 161–167.

[6] Petrich, M. – *Inverse semigroups*, Wiley, 1984.

[7] Sullivan, R.P. – Semigroups generated by nilpotent transformations, *J. Algebra*, 110 (1987), 324–345.

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