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SILVA DISTRIBUTIONS FOR CERTAIN LOCALLY COMPACT GROUPS

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We define axiomatically a space of distributions or generalized functions for a class of locally compact groups, the parametrizable groups (cf. Th. 1), using a method inspired in the papers by J.S. Silva concerning distributions on $\mathbb{R}^n$ (cf. [6]).

We establish an isomorphism between the space we define and the space of distributions by Bruhat (cf. [1]).

1 – Notations and preliminary results

Let G be a locally compact abelian and connected group. There exists an indexed family $F = \{K_\alpha : \alpha \in A\}$ of compact subgroups of G, such that $\bigcap_{\alpha\in A} K_\alpha = \{0\}$, $K_\alpha \cap K_\beta \in F$, $\forall\,\alpha,\beta \in A$ and each G/K_α is a Lie group L_α. We set $\alpha < \beta \Leftrightarrow K_\alpha \supset K_\beta$.

Let ϕ_α and $\phi_{\alpha\beta}$ be the canonical homomorphisms, $\phi_\alpha : G \to G/K_\alpha$, $\phi_{\alpha\beta}$: $G/K_\beta \to G/K_\alpha$. The projective limit of the projective system $(L_\alpha, \phi_{\alpha\beta})_{\alpha,\beta\in A}$ is identical with G and the canonical map from this limit onto each L_α is identical with ϕ_α.

Note that if we take $\alpha_0 \in A$ and consider the subset $A_0 = \{\alpha \in A\colon \alpha > \alpha_0\}$ we have $G = \lim\limits_{\substack{\longleftarrow \\ \alpha\in A}} L_\alpha = \lim\limits_{\substack{\longleftarrow \\ \alpha\in A_0}} L_\alpha$. So, without less generality, we can suppose that the set A has a first element α_0.

The continuous homomorphisms ϕ_α and $\phi_{\alpha\beta}$ are open and proper.

If we suppose that G is metrizable, we have $G = \lim\limits_{\substack{\longleftarrow \\ i\in\mathbb{N}_0}} L_i$.

As G is locally compact and connected, $G = \bigcup_{j\in\mathbb{N}} W_j$, where each W_j is an open subset of G which is relatively compact and such that $\overline{W}_j \subset W_{j+1}$ $(j \in \mathbb{N})$.

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If B is any compact subset of G, there exists $j \in \mathbb{N}$ such that $B \subset \overline{W}_j$.

For each $j \in \mathbb{N}$ we set $\Omega_j = K_{\alpha_0} + W_j$ and $\Delta_j = K_{\alpha_0} + \overline{W}_j$. We obtain a sequence of compact subsets of G such that, if B is any compact subset of G, $B \subset \Delta_j$ for a certain $j \in \mathbb{N}$. We also have $\phi_\alpha^{-1}(\phi_\alpha(\Delta_j)) = \Delta_j$ for each $\alpha \in A$ and each $j \in \mathbb{N}$.

We use the symbol $\mathcal{G}$ to denote the Lie algebra of G (cf. [2]) that is $\mathcal{G} = \lim\limits_{\substack{\leftarrow \\ \alpha\in A}} \mathcal{L}_\alpha$, each $\mathcal{L}_\alpha$ denoting the Lie algebra of L_α. The canonical homomorphisms $d\phi_\alpha\colon \mathcal{G} \to \mathcal{L}_\alpha$ and $d\phi_{\alpha\beta}\colon \mathcal{L}_\beta \to \mathcal{L}_\alpha$ are onto and induced by ϕ_α and $\phi_{\alpha\beta}$. The exponential map $\exp\colon \mathcal{G} \to G$ is defined by $\exp(X_\alpha)_{\alpha\in A} = (\exp X_\alpha)_{\alpha\in A}$.

We denote by n_α the dimensions of each L_α. If $\alpha < \beta$ we have $n_\alpha \le n_\beta$.

The exponential exp is an analytic diffeomorphic map from an open neighborhood B_α of zero in $\mathcal{L}_\alpha$ onto an open neighborhood V_α of zero in L_α. Let $U_\alpha \subset V_\alpha$ be such that $U_\alpha + U_\alpha \subset V_\alpha$; we consider the canonical identification of $\mathcal{L}_\alpha$ with $\mathbb{R}^{n_\alpha}$ and we call the couple (ξ_α, U_α) with $\xi_\alpha = (\exp|_{V_\alpha})^{-1}|_{U_\alpha}$, the canonical chart of L_α.

For each $a_\alpha \in L_\alpha$ we set $U_{a_\alpha} = a_\alpha + U_\alpha$ and $\xi_{a_\alpha}(u_\alpha) = \xi_\alpha(u_\alpha - a_\alpha)$. We call canonical atlas of L_α to $\{(\xi_{a_\alpha}, U_\alpha)\colon a_\alpha \in L_\alpha\}$.

Let $\mathbb{Q}_\alpha = \{(t_i) \in \mathbb{R}^{n_\alpha}\colon |t_i| < \delta_i\}$. We call $V_{a_\alpha} = a_\alpha + \xi_\alpha^{-1}(\mathbb{Q}_\alpha)$ an elementary neighborhood of $a_\alpha \in L_\alpha$.

For each $p = (p_i) \in \mathbb{N}_0^{n_\alpha}$, let $N^p(\mathbb{Q}_\alpha)$ be the subspace of the space $C(\mathbb{Q}_\alpha)$ of continuous complex-valued functions defined in $\mathbb{Q}_\alpha$ of the type

$$\theta_\alpha(t_1, ..., t_{n_\alpha}) = \sum_{i=1}^{n_\alpha} \sum_{k=0}^{p_i-1} t_i^k\, h_{ik}(t_1, ..., t_{n_\alpha}) \ ,$$

with $h_{ik} \in C(\mathbb{Q}_\alpha)$ and independent of t_i. The elements of $N^p(\mathbb{Q}_\alpha)$ are called pseudo-polynomials in $\mathbb{Q}_\alpha$ with degree less then p.

A function $f_\alpha \in C(V_{a_\alpha})$ such that $f \circ \exp \in N^p(\xi_\alpha(V_{a_\alpha} - a_\alpha))$ is called a pseudo-polynomial on V_{a_α} with degree less then p.

We call fundamental family of one parameter subgroups of L_α, the family of parameter subgroups determinated by the elements of one basis B_α of $\mathcal{L}_\alpha$, that is, $x_\alpha(t) = \exp tX_\alpha$, $t \in \mathbb{R}$, $X_\alpha \in B_\alpha$. Each basis B_α of $\mathcal{L}_\alpha$ determines a fundamental family S_α of one parameter subgroups of L_α.

Note that we can write

$$V_{a_\alpha} = \Big\{ a_\alpha + \sum_{i=1}^{n_\alpha} x_{i,\alpha}(t_i)\colon\ x_{i,\alpha} \in S_\alpha,\ (t_i) \in \mathbb{Q}_\alpha \Big\} \ .$$

Theorem 1. *Let G be an abelian locally compact group, connected and metrizable, $G = \lim\limits_{\substack{\leftarrow \\ i\in\mathbb{N}_0}} L_i$. There exists a family S of one parameter subgroups of*

G and fundamental families S_i of one parameter subgroups of L_i ($i \in \mathbb{N}_0$) such that, $\forall\, x \in S$, $\phi_i \circ x = x_i \in S_i$ or $\phi_i \circ x = 0$, each $x_i \in S_i$ being obtained as the image of just one element $x \in S$.

Proof: As $\ker d\phi_i \supset \ker d\phi_{i+1}$ ($i \in \mathbb{N}_0$), we consider H_i such that $H_i \oplus \ker d\phi_i = \ker d\phi_0$. We have $H_i \subset H_{i+1}$ ($i \in \mathbb{N}_0$).

We choose basis B_i^* of each H_i such that $B_i^* \subset B_{i+1}^*$: we have $d\phi_i(\ker d\phi_0) = \ker d\phi_{0i}$, so $B_{0i} = d\phi_i(B_i^*)$ is a basis for $\ker d\phi_{0i}$ ($i \in \mathbb{N}$).

If we take H_0 such that $H_0 \oplus \ker d\phi_0 = \mathcal{G}$ and a basis B_0^* of H_0 then $B_0 = d\phi_0(B_0^*)$ is a basis of $\mathcal{L}_0$. We obtain a sequence of basis (B_i) of $(\mathcal{L}_i)$ setting $B_0 = d\phi_0(B_0^*)$ and $B_i = d\phi_i(B_0^*) \cup B_{0i}$, $\forall\, i \in \mathbb{N}$: we have $d\phi_{ij}(B_j) = B_i \cup \{0\}$ ($i, j \in \mathbb{N}_0$, $i < j$).

Let's take $\mathcal{B}_i = B_i \cup \{0\}$ ($i \in \mathbb{N}_0$) and $\mathcal{B} = \bigcup_i B_i^*$. We have that $d\phi_i(\mathcal{B}) = \mathcal{B}_i$ ($i \in \mathbb{N}_0$), each $X_i \in \mathcal{B}_i$ is obtained as the image of just one element $X \in \mathcal{B}$ and $\mathcal{B} = \varprojlim_{i \in \mathbb{N}_0} \mathcal{B}_i$.

We complete our proof by taking $S = \{x(t) = \exp tX : X \in \mathcal{B}\}$ and $S_i = \{x_i(t) = \exp tX_i\colon X_i \in B_i\}$. ∎

Note: The construction of the family S, related with well choosen families S_i in the way expressed in the precedent theorem, can be done under more general hypothesis. In fact let $G = T^J \times \mathbb{R}^p$ with $p \in \mathbb{N}$ and J any set of indexes: if we take the family (N_α) of the finite parts of J and if we set $\alpha < \beta \Leftrightarrow N_\alpha \subset N_\beta$, we have $G = \varprojlim_\alpha (T^{N_\alpha} \times \mathbb{R}^p)$. In $\mathbb{R}^p$ we take, as usually, p one parameter subgroups $y_1(t) = (t, 0, ..., 0)$, ..., $y_p(t) = (0, ..., 0, t)$ and for each $j \in J$ and each N_α we set $z_{j,\alpha}(t) = (e^{i\delta_{jk}t})_{k \in N_\alpha}$ and $z_j(t) = (e^{i\delta_{jk}t})_{k \in J}$. The families

$$S_\alpha = \{(z_{j,\alpha}, 0)\}_{j \in N_\alpha} \cup \{(0, y_m)\}_{1 \le m \le p}$$

and

$$S = \{(z_j, 0)\}_{j \in J} \cup \{(0, y_m)\}_{1 \le m \le p} \ ,$$

with $(z_{j,\alpha}, 0)(t) = (z_{j,\alpha}(t), 0)$, $(z_j, 0)(t) = (z_j(t), 0)$, and $(0, y_m)(t) = (1, y_m(t))$ verify the statement of our theorem.

We call *parametrizable groups* the abelian locally compact and connected groups $G = \varprojlim_{\alpha \in A} L_\alpha$ for which we can find families S and S_α satisfying the conditions expressed on Theorem 1. We call the family S a fundamental family of one parameter subgroups of G.

Now on we are only going to deal with parametrizable groups (in particular with metrizable groups).

Let $G = \lim\limits_{\overleftarrow{\alpha\in A}} L_\alpha$ be a parametrizable group, $S = \{x_i :\ i \in I\}$ a fundamental family of one parameter subgroups of G, Ω an open set in G and $f \in C(\Omega)$: f is partially differentiable with respect to x_i in Ω if for every $a \in \Omega$, the real-valued function $t \to f(a + x_i(t))$ is differentiable in a neighborhood of $0 \in \mathbb{R}$.

We set

$$D_{x_i} f(a) = \frac{d}{dt} f(a + x_i(t))\Big|_{t=0}$$

an more generally

$$D_{x_i} f(a + x_i(t)) = \frac{d}{dt} f(a + x_i(t)) \ .$$

If $D_{x_i} D_{x_j} f$ and $D_{x_j} D_{x_i} f$ exists in $C(\Omega)$ we have $D_{x_i} D_{x_j} f = D_{x_j} D_{x_i} f$. If $p = (p_i) \in \mathbb{N}_0^I$, we set $D^p = \prod_{i\in I} D_{x_i}^{p_i}$.

If G is a lie group L, the partial-differentiation operators D_{x_i} coincides with the usual operators respecting to a fundamental family of one parameter subgroups of L.

Let Δ be the closure of an open relatively compact subset Ω of G, $f \in C(\Delta)$ such that $D_{x_i}(f|_\Omega)$ can be continuously extended to Δ. We also use the symbol $D_{x_i} f$ to denote the continuous function in Δ whose restriction to Ω coincides with $D_{x_i}(f|_\Omega)$.

We denote by $E(G)$ the subspace of $C(G)$ formed by the functions having the following property: for each $a \in G$ there exists $\alpha \in A$ and a function f_α continuous on a neighborhood V_{a_α} of $a_\alpha = \phi_\alpha(a)$ such that $f = f_\alpha \circ \phi_\alpha$ in $V_a = \phi_\alpha^{-1}(V_{a_\alpha})$.

If Δ is the closure of an open relatively compact subset of G, we denote by $E(\Delta)$ the subspace of $C(\Delta)$ formed by the functions that admits in Δ, for a certain $\alpha \in A$, a decomposition of the type $f = f_\alpha \circ \phi_\alpha$ with $f_\alpha \in C(\phi_\alpha(\Delta))$.

We take on $E(G)$ and $E(\Delta)$ the topologies induced by the usual topologies of $C(G)$ and $C(\Delta)$ respectively.

If G is not a Lie group, we have $E(G) \subsetneq C(G)$ (cf. [4]).

For each $\alpha \in A$ let $\mathcal{D}(L_\alpha)$ denote the space of infinitely differentiable complex-valued functions defined on L_α having compact support, with the usual topology and $\mathcal{D}_\alpha(G) = \{f = f_\alpha \circ \phi_\alpha :\ f_\alpha \in \mathcal{D}(L_\alpha)\}$. We define $\psi_\alpha : \mathcal{D}(L_\alpha) \to \mathcal{D}_\alpha(G)$ by setting $\psi_\alpha(f_\alpha) = f_\alpha \circ \phi_\alpha$ and we take on $\mathcal{D}_\alpha(G)$ the topology transported by ψ_α.

Following Bruhat, we define the space $\mathcal{D}(G)$ of infinitely differentiable complex-valued functions defined on G having compact support, by setting $\mathcal{D}(G) = \bigcup_{\alpha\in A} \mathcal{D}_\alpha(G)$ and we equip $\mathcal{D}(G)$ with the inductive limit topology of the spaces $\mathcal{D}_\alpha(G)$. We can write $\mathcal{D}(G) = \lim\limits_{\overrightarrow{\alpha\in A}} \mathcal{D}(L_\alpha)$.

Let us consider the sequence of compacts $(\Delta_j)_{j\in\mathbb{N}}$ (cf. p. 2) and let $\mathcal{D}(\Delta_j)$ be the subspace of $\mathcal{D}(G)$ formed by the functions having their support contained in

Δ_j. Let $\Delta_{j,\alpha} = \phi_\alpha(\Delta_j)$, $\forall\, \alpha \in A$: we have $\mathcal{D}(\Delta_j) = \bigcup_{\alpha\in A} \psi_\alpha(\mathcal{D}(A_{j,\alpha}))$ so we can write $\mathcal{D}(\Delta_j) = \lim\limits_{\substack{\rightarrow\\ \alpha\in A}} \mathcal{D}(\Delta_{j,\alpha})$ and $\mathcal{D}(G) = \lim\limits_{\substack{\rightarrow\\ j\in\mathbb{N}}} \mathcal{D}(\Delta_j)$.

The partial-differentiation operators corresponding to a fundamental family S of one parameter subgroups of G are related with the partial-differentiation operators corresponding to the fundamental families S_α of one parameter subgroups of each L_α (cf. Th. 1) in the following way: $D_{x_i}(f_\alpha \circ \phi_\alpha) = (D_{x_{i,\alpha}} f_\alpha) \circ \phi_\alpha$ if $\phi_\alpha \circ x_i = x_{i,\alpha}$.

We can easily show that those operators are continuous in $\mathcal{D}(G)$.

Let $G = L$, L a n-dimensional Lie group, and V_a an elementary neighborhood of $a \in L$. We can define in $C(V_a)$ an operator P_{x_i}, right inverse of D_{x_i} in $C(V_a)$, by setting

$$P_{x_i} f(u) = \int_{b_i}^{t_i} f\Big[u + \Big(x_1(t_1) + ... + x_i(s_i) + ... + x_n(t_n)\Big)\Big]\, ds_i\, \Big|_{t_1=...=t_i=0}$$
$$\forall\, f \in C(V_a), \quad \forall\, u \in V_a\ ,$$

with $(b_i) \in \mathbb{Q} = \xi_a(V_a)$. We easily verify that $P_{x_i} P_{x_j} = P_{x_j} P_{x_i}$, $1 \le i, j \le n$. If $p = (p_i) \in \mathbb{N}_0^n$, we set $P^p = \prod_{i\in I} P_{x_i}^{p_i}$.

2 – Axiomatic definition of a space $\widetilde{C}(G)$ of distributions on G

Let $S = \{x_i\colon\ i \in I\}$ be a fundamental family of one parameter subgroups of $G = \lim\limits_{\substack{\leftarrow\\ \alpha\in A}} L_\alpha$ and Δ the closure of an open relatively compact subset of G. We set

$$M = \Big\{p = (p_i) \in \mathbb{N}_0^I\colon\ p_i = 0 \text{ except for a finite number of indexes } i \in I\Big\}\ .$$

The following axiomatic is formulated in terms of "continuous functions", "addition" and "differentiation", its logical universe being the space $\widetilde{E}(\Delta)$ that we characterize as follows:

Axiom 1. $\widetilde{E}(\Delta) \supset E(\Delta)$.

Axiom 2. There exists an operation named "addition" that to every couple $T_1, T_2 \in \widetilde{E}(\Delta)$ associates an element $T \in \widetilde{E}(\Delta)$, named the addition of T_1 and T_2 and denoted by $T_1 + T_2$, such that if $T_1 = f_1 \in E(\Delta)$ and $T_2 = f_2 \in E(\Delta)$ then $T_1 + T_2$ is the addition in the usual sense.

Axiom 3. For each $i \in I$ there exists an operator $\widetilde{D}_{x_i}\colon\ \widetilde{E}(\Delta) \to \widetilde{E}(\Delta)$ such that:

1) If $T = f \in E(\Delta)$ and $D_{x_i} f \in E(\Delta)$ then $\widetilde{D}_{x_i} T = D_{x_i} f$.

2) $\widetilde{D}_{x_i}(T_1 + T_2) = \widetilde{D}_{x_i} T_1 + \widetilde{D}_{x_i} T_2$.

3) $\widetilde{D}_{x_i} \widetilde{D}_{x_j} T = \widetilde{D}_{x_j} \widetilde{D}_{x_i} T$, $\forall\, i, j \in I$.

We name the operator $\widetilde{D}_{x_i}$ $(i \in I)$ "generalized partial differentiation operator with respect to x_i". If $p = (p_i) \in \mathbb{N}_0^I$ we set $\widetilde{D}^p = \prod_{i \in I} \widetilde{D}_{x_i}^{p_i}$.

Axiom 4. For each $T \in E(\Delta)$ there exists a finite number of continuous functions $f_k \in E(\Delta)$ and a finite number of multi-indexes $p_k \in M$ such that

$$T = \sum_{k=1}^{m} \widetilde{D}^{p_k} f_k \ .$$

Axiom 5.

$$\sum_{k=1}^{m} \widetilde{D}^{p_k} f_k = \sum_{j=1}^{s} \widetilde{D}^{p_j} g_j$$

with $f_k = f_{\alpha_k} \circ \phi_{\alpha_k}$, $1 \le k \le m$, and $g_j = g_{\alpha_j} \circ \phi_{\alpha_j}$, $1 \le j \le s$, if and only if there exists $r \in M$, $r \ge p_1, ..., p_m, q_1, ..., q_s$ such that we have, for $\gamma > \alpha_1, ..., \alpha_m, \beta_1, ..., \beta_s$, for each $a \in \Delta$ and for each elementary neighborhood V_{a_γ} of $a_\gamma = \phi_\gamma(a)$,

$$\sum_{k=1}^{m} p^{r-p_k}\Big((f_{\alpha_k} \circ \phi_{\alpha_k \gamma})\Big|_{V_{a_\gamma}} \Big) - \sum_{j=1}^{s} p^{r-q_j}\Big((g_{\beta_j} \circ \phi_{\beta_j \gamma})\Big|_{V_{a_\gamma}} \Big) = h_\gamma \ ,$$

h_γ being a pseudo-polynomial in V_{a_γ} with degree less then r (cf. nº 1).

To construct a model for our axiomatic, we consider the set $\mathcal{P}(M \times E(\Delta))$ of the finite parts of $M \times E(\Delta)$ and the equivalence relation $*$ defined as follows:

$$\Big\{ (p_1, f_{\alpha_1} \circ \phi_{\alpha_1}), ..., (p_m, f_{\alpha_m} \circ \phi_{\alpha_m}) \Big\} * \Big\{ (q_1, g_{\beta_1} \circ \phi_{\beta_1}), ..., (q_s, q_{\beta_s} \circ \phi_{\beta_s}) \Big\}$$

if and only if there exists $r \in M$, $r \ge p_1, ..., p_m, q_1, ..., q_s$ such that we have, for $\gamma > \alpha_1, ..., \alpha_m, \beta_1, ..., \beta_s$, for each $a \in \Delta$ and each elementary neighborhood V_{a_γ} of $a_\gamma = \phi_\gamma(a)$,

$$\sum_{k=1}^{m} p^{r-p_k}\Big((f_{\alpha_k} \circ \phi_{\alpha_k \gamma})\Big|_{V_{a_\gamma}} \Big) - \sum_{j=1}^{s} p^{r-q_j}\Big((g_{\beta_j} \circ \phi_{\beta_j \gamma})\Big|_{V_{a_\gamma}} \Big) = h_\gamma \ ,$$

h_γ being a pseudo-polynomial in V_{a_γ} with degree less then r.

Denoting by $[\{(p_1, f_1), ..., (p_m, f_m)\}]$ the equivalence class of $\{(p_1, f_{\alpha_1} \circ \phi_{\alpha_1}),$ $..., (p_m, f_{\alpha_m} \circ \phi_{\alpha_m})\}$ we set

1) $[\{(p_1, f_1), ..., (p_m, f_m)\}] + [\{(q_1, g_1), ..., (q_s, g_s)\}] =$
$= [\{(p_1, f_1), ..., (p_m, f_m)\} \cup \{(q_1, g_1), ..., (q_s, g_s)\}]$;

2) $\lambda[\{(p_1, f_1), ..., (p_m, f_m)\}] = [\{(p_1, \lambda f_1), ..., (p_m, \lambda f_m)\}]$, $\forall \lambda \in \mathbb{C}$;

3) $\widetilde{D}_{x_j}[\{(p_1, f_1), ..., (p_m, f_m)\}] = [\{(q_1, f_1), ..., (q_m, f_m)\}]$,
with $p_k = (p_{k,i})_{i\in I}$, $1 \le k \le m$, and $q_k = (q_{k,i})_{i\in I}$, $1 \le k \le m$, such that

$$q_{k,i} = \begin{cases} p_{k,i} & \text{if } i \neq j, \\ p_{k,i} + 1 & \text{if } i = j. \end{cases}$$

Taking $\widetilde{E}(\Delta) = \mathcal{P}(M \times E(\Delta))/*$ with the operations defined above, we easily verify that $\widetilde{E}(\Delta)$ satisfies the precedent axioms.

For each multi-index $p \in M$ we set

$$\widetilde{E}_p(\Delta) = \left\{T \in \widetilde{E}(\Delta)\colon\ T = \widetilde{D}^p f \ \text{ with } f \in E(\Delta)\right\}$$

and

$$N_p(\Delta) = \left\{f \in E(\Delta)\colon\ \widetilde{D}^p f = 0\right\}$$

and define an isomorphism $\widetilde{D}^p f \to f + N_p(\Delta)$ between each $\widetilde{E}_p(\Delta)$ and each quotient vector space $E(\Delta)/N_p(\Delta)$. We take on each $\widetilde{E}_p(\Delta)$ the quotient topology. The sets

$$B_{p,n} = \left\{T = \widetilde{D}^p f\colon\ f \in E(\Delta) \ \text{ and } \ \|f\| < 1/n\right\}, \quad n \in \mathbb{N}\ ,$$

form a basis of balanced neighborhoods of zero in $\widetilde{E}_p(\Delta)$.

We take on $\widetilde{E}(\Delta)$ the finest locally compact topology for which the natural injections of $\widetilde{E}_p(\Delta)$ into $\widetilde{E}(\Delta)$, $p \in M$, are continuous.

The operators $\widetilde{D}$ are continuous on $\widetilde{E}(\Delta)$.

Let us consider the sequence of compacts $(\Delta_j)_{j\in\mathbb{N}}$ (cf. p. 2). We can define homomorphisms $\rho_{ij}\colon \widetilde{E}(\Delta_j) \to \widetilde{E}(\Delta_i)$, using the restriction mapping, that is

$$\rho_{ij}\Big(\sum_{k=1}^{m} \widetilde{D}^{p_k} f_k\Big) = \sum_{k=1}^{m} \widetilde{D}^{p_k}(f_k|_{\Delta_i})\,, \quad i \le j\ .$$

We obtain a projective system of topological spaces, $(\widetilde{E}(\Delta_j), \rho_{ij})_{i,j\in\mathbb{N}}$, and we set $\widetilde{E}(G) = \varprojlim_{j\in\mathbb{N}} \widetilde{E}(\Delta_j)$. As $E(G) \subsetneq C(G)$, we call distribution on G to every element of the completion $\widetilde{C}(G)$ of $\widetilde{E}(G)$. Observe that if we denote by $\widetilde{C}(\Delta)$ the completion of each $\widetilde{E}(\Delta)$, we have $\widetilde{C}(G) = \varprojlim_{j\in\mathbb{N}} \widetilde{C}(\Delta_j)$. When $G = \mathbb{R}^n$ the space $\widetilde{C}(G)$ coincides with the space of distributions constructed in [6].

3 – Relation between the spaces $\widetilde{C}(G)$ and $\mathcal{D}'(G)$

The space $\mathcal{D}'(G)$ of the Bruhat distributions, being the strong dual of $\mathcal{D}(G)$, is the topological projective limit of the spaces $\mathcal{D}'(L_\alpha)$, $\mathcal{D}'(G) = \varprojlim_{\alpha\in A} \mathcal{D}'(L_\alpha)$ and the canonical maps are the transposed ${}^{\rm t}\psi_\alpha$ and ${}^{\rm t}\psi_{\alpha\beta}$ of ψ_α and $\psi_{\alpha\beta}$ respectively. We also have $\mathcal{D}'(\Delta_j) = \varprojlim_{\alpha\in A} \mathcal{D}'(\Delta_{j,\alpha})$ and $\mathcal{D}'(G) = \varprojlim_{j\in\mathbb{N}} \mathcal{D}'(\Delta_j)$.

Let Δ be the closure of an open relatively compact subset of G, μ the Haar measure of G, $\theta \in C(G)$ and $T \in \widetilde{E}(\Delta)$. From Axiom 4 we have $T = \sum_{k=1}^m \widetilde{D}^{p_k} f_k$ with $f_k \in E(\Delta)$. As $\mathcal{D}(G)$ is dense in $C_0(G)$ (cf. [1]), there exist sequences $(h_{k,r})_{r\in\mathbb{N}}$ in $\mathcal{D}(G)$, $1 \le k \le m$, such that $h_{k,r} \to f_k$, uniformly on Δ. Then, the sequence $(h_r)_{r\in\mathbb{N}}$ in $\mathcal{D}(G)$, $h_r = \sum_{k=1}^m \widetilde{D}^{p_k} h_{k,r}$ converges to T in $\widetilde{E}(\Delta)$.

We set, by definition:

$$\int_\Delta T(u)\,\theta(u)\,d\mu(u) = \lim_r \int_\Delta h_r(u)\,\theta(u)\,d\mu(u) \ ,$$

when this limit exists and is independent of the sequence $(h_r)_{r\in\mathbb{N}}$.

Lemma. *Each space $\widetilde{E}(\Delta_j)$ is isomorphic to a topological subspace of $\mathcal{D}'(\Delta_j)$.*

Proof: If $G = \mathbb{R}^n$ or G is a n-dimensional Lie group, the expression

$$\mathcal{T}(\theta) = \int_\Delta T(u)\,\theta(u)\,d\mu(u)$$

defines a topological isomorphism $\mathcal{T} \leftrightarrow T$ between $\widetilde{E}(\Delta_j) = \widetilde{C}(\Delta_j)$ and $\mathcal{D}'(\Delta_j)$. The first case was studied in [6]. For the second case, we can cover the compact Δ_j by a finite number of elementary neighborhoods and take a partition of unity in $\mathcal{D}(G)$ subordinated to this covering. It is then sufficient to use again [6] and the canonical atlas of G.

In the general case $G = \varprojlim_{\alpha\in A} L_\alpha$, we take for each $\alpha \in A$ the Haar measure $\mu_\alpha = \phi_\alpha(\mu)$ of L_α, and we define a topological isomorphism $H_{j,\alpha}$ from $\widetilde{C}(\Delta_{j,\alpha})$ onto $\mathcal{D}'(\Delta_{j,\alpha})$, $T_\alpha \leftrightarrow \mathcal{T}_\alpha$, by setting

$$\mathcal{T}_\alpha(\theta_\alpha) = \int_{\Delta_{j,\alpha}} T_\alpha(u_\alpha)\,\theta_\alpha(u_\alpha)\,d\mu_\alpha(u_\alpha)\,, \qquad \forall\,\theta_\alpha \in \mathcal{D}(\Delta_{j,\alpha}) \ .$$

Detailing, if $T_\alpha = \sum_{k=1}^m \widetilde{D}^{p_k} f_{\alpha,k}$, we have for each $\theta_\alpha \in \mathcal{D}(\Delta_{j,\alpha})$,

$$\begin{aligned}(H_{j,\alpha}(T_\alpha))(\theta_\alpha) = \mathcal{T}_\alpha(\theta_\alpha) &= \sum_{k=1}^m \int_{\Delta_{j,\alpha}} (\widetilde{D}^{p_k} f_{\alpha,k})(u_\alpha)\,\theta_\alpha(u_\alpha)\,d\mu_\alpha(u_\alpha)\\ &= \sum_{k=1}^m (-1)^{|p_k|} \int_{\Delta_{j,\alpha}} f_{k,\alpha}(u_\alpha)\,(\widetilde{D}^{p_k}\theta_\alpha)(u_\alpha)\,d\mu_\alpha(u_\alpha)\ .\end{aligned}$$

We consider continuous homomorphisms $\widetilde{\psi}_{\alpha\beta}\colon \widetilde{C}(\Delta_{j,\alpha}) \to \widetilde{C}(\Delta_{j,\beta})$ defined by $\widetilde{\psi}_{\alpha\beta}(\widetilde{D}^{p_\alpha} f_\alpha) = \widetilde{D}^{p_\beta}(f_\alpha \circ \phi_{\alpha\beta})$ with $p_\alpha = (p_{\alpha,i}) \in \mathbb{N}_0^{n_\alpha}$ and $p_\beta = (p_{\beta,i}) \in \mathbb{N}_0^{n_\beta}$ such that

$$p_{\beta,i} = \begin{cases} p_{\alpha,i} & \text{if } \phi_{\alpha\beta}\circ x_{i,\beta} = x_{i,\alpha} \in S_\alpha,\\ 0 & \text{if } \phi_{\alpha\beta}\circ x_{i,\beta} = 0\ ,\end{cases}$$

We also consider the continuous homomorphisms $\widetilde{\psi}_\alpha : \widetilde{C}(\Delta_{j,\alpha}) \to \widetilde{E}(\Delta_j)$ defined by $\widetilde{\psi}_\alpha(\widetilde{D}^{p_\alpha} f_\alpha) = \widetilde{D}^{p}(f_\alpha \circ \phi_\alpha)$ with $p = (p_i) \in M$ such that

$$p_i = \begin{cases} p_{\alpha,i} & \text{if } \phi_\alpha\circ x_i = x_{i,\alpha} \in S_\alpha,\\ 0 & \text{if } \phi_\alpha\circ x_i = 0\ ,\end{cases}$$

and we obtain

$$\widetilde{E}(\Delta_j) = \bigcup_{\alpha\in A} \widetilde{\psi}_\alpha(\widetilde{C}(\Delta_{j,\alpha}))\ .$$

If $\alpha < \beta$

$$\begin{aligned}&(H_{j,\beta}(\widetilde{D}^{p_\beta}(f_\alpha\circ\phi_{\alpha\beta})))(\theta_\alpha\circ\phi_{\alpha\beta}) =\\ &\qquad = \int_{\Delta_{j,\beta}} (\widetilde{D}^{p_\beta}(f_\alpha\circ\phi_{\alpha\beta}))(u_\beta)\,(\theta_\alpha\circ\phi_{\alpha\beta})(u_\beta)\,d\mu_\beta(u_\beta)\\ &\qquad = \int_{\Delta_{j,\alpha}} (\widetilde{D}^{p_\alpha} f_\alpha)(u_\alpha)\,\theta_\alpha(u_\alpha)\,d\mu_\alpha(u_\alpha)\\ &\qquad = (H_{j,\alpha}(\widetilde{D}^{p_\alpha} f_\alpha))(\theta_\alpha)\ ,\end{aligned}$$

and so

$$(H_{j,\beta}(\widetilde{\psi}_{\alpha\beta}(T_\alpha)))(\theta_\alpha\circ\phi_{\alpha\beta}) = (H_{j,\alpha}(T_\alpha))(\theta_\alpha)\,, \quad \forall\, T_\alpha \in \widetilde{C}(\Delta_{j,\alpha}),\ \forall\,\theta_\alpha \in \mathcal{D}(\Delta_{j,\alpha})\ .$$

Then we can consider a homomorphism $H_j : \widetilde{E}(\Delta_j) \to \mathcal{D}'(\Delta_j)$ defined by $H_j(\widetilde{\psi}_\alpha(T_\alpha)) = (H_{j,\beta}(\widetilde{\psi}_{\alpha\beta}(T_\alpha)))_{\beta>\alpha}$. As each $H_{j,\alpha}$ is one-to-one, H_j is also one-to-one. We easily verify that

$$\begin{aligned}&(H_j(T))(\theta) = \Big(H_j\Big(\sum_{k=1}^m \widetilde{D}^{p_k} f_k\Big)\Big)(\theta) = \int_{\Delta_j} T(u)\,\theta(u)\,d\mu(u) =\\ &\quad = \sum_{k=1}^m (-1)^{|p_k|} \int_{\Delta_j} f_k(u)\,(\widetilde{D}^{p_k}\theta)(u)\,d\mu(u), \quad \forall\, T \in \widetilde{E}(\Delta_{j,\alpha}),\ \forall\,\theta \in \mathcal{D}(\Delta_{j,\alpha})\ .\end{aligned}$$

To prove that H_j is continuous it is sufficient to verify that if $(T_r)_{r\in\mathbb{N}}$ is a sequence in $\widetilde{E}_p(\Delta_j)$ such that $T_r \to 0$ there, that is $T_r = \widetilde{D}^p f_r$ and $f_r \to 0$ uniformly on Δ_j, then $H_j(T_r) \to 0$ on every bounded subset B of $\mathcal{D}(\Delta_j)$. From [1] there exists $\alpha \in A$ such that $B = \psi_\alpha(B_\alpha)$ with B_α bounded subset of $\mathcal{D}(\Delta_{j,\alpha})$. Then there exists $b \in \mathbb{R}^+$ such that

$$\sup_{u\in\Delta_j} |(D^p\theta)(u)| = \sup_{u_\alpha\in\Delta_{j,\alpha}} |(\widetilde{D}^{p_\alpha}\theta_\alpha)(u_\alpha)| < b\,, \qquad \forall\,\theta \in B$$

and

$$\begin{aligned}(H_j(T_r))(\theta) &= \Big|\int_{\Delta_j} (\widetilde{D}^p f_r)(u)\,\theta(u)\,d\mu(u)\Big| \\ &= \Big|(-1)^{|p|}\int_{\Delta_j} f_r(u)(D^p\theta)(u)\,d\mu(u)\Big| \\ &\le \sup_{u\in\Delta_j} |f_r(u)|\,b\,\mu(\Delta_j)\,, \qquad \forall\,\theta\in B\ ,\end{aligned}$$

so $H_j(T_r) \to 0$ in $\mathcal{D}'(\Delta_j)$.

Let us prove that H_j is bicontinuous.

We consider a basis of neighborhoods of zero in $\widetilde{E}_p(\Delta_j)$, $p \in M$,

$$B_{p,n} = \Big\{T = \widetilde{D}^p f\colon\ f \in \widetilde{E}(\Delta_j) \ \text{ and } \ \|f\| < 1/n\Big\}\,, \qquad n \in \mathbb{N}\ .$$

The balanced convex hulls of the sets $\bigcup_{p\in M} B_{p,n}$, $n \in \mathbb{N}$, form a basis of neighborhoods of zero in $\widetilde{E}(\Delta_j)$, $\{B_n\colon\ n\in\mathbb{N}\}$. We consider in each $\widetilde{C}(\Delta_{j,\alpha})$ the corresponding basis of neighborhoods $\{B_{n,\alpha}\colon\ n\in\mathbb{N}\}$: we have $B_n = \bigcup_{\alpha\in A}\widetilde{\psi}_\alpha(B_n)$.

As each $H_{j,\alpha}$ is bicontinuous, $W_{n,\alpha} = H_{j,\alpha}(B_{n,\alpha})$ is a neighborhood of zero in $\mathcal{D}'(\Delta_j)$. Let us take $W_n = {}^{\mathrm{t}}\psi_\alpha^{-1}(W_{n,\alpha}) \cap H_j(\widetilde{E}(\Delta_j))$: W_n is a neighborhood of zero in $H_j(\widetilde{E}(\Delta_j))$ for the topology induced by $\mathcal{D}'(\Delta_j)$. We must prove that $H_j^{-1}(W_n) \subset B_n$.

Noting that

$$\begin{aligned}({}^{\mathrm{t}}\psi_\alpha((H_j\circ\widetilde{\psi}_\alpha)(T_\alpha)))(\theta_\alpha) &= ((H_j\circ\widetilde{\psi}_\alpha)(T_\alpha))(\theta_\alpha\circ\phi_\alpha) \\ &= \int_{\Delta_j} (\widetilde{\psi}_\alpha(T_\alpha))(u)\,(\theta_\alpha\circ\phi_\alpha)(u)\,d\mu(u) \\ &= \int_{\Delta_{j,\alpha}} T_\alpha(u_\alpha)\,\theta_\alpha(u_\alpha)\,d\mu_\alpha(u_\alpha) = (H_{j,\alpha}(T_\alpha))(\theta_\alpha)\ , \\ &\qquad \forall\,T_\alpha \in \widetilde{C}(\Delta_{j,\alpha}),\ \ \forall\,\theta_\alpha \in \mathcal{D}(\Delta_{j,\alpha})\ ,\end{aligned}$$

we have ${}^{\mathrm{t}}\psi_\alpha \circ H_j \circ \widetilde{\psi}_\alpha = H_{j,\alpha}$, $\forall\, \alpha \in A$, so

$$B_{n,\alpha} = H_{j,\alpha}^{-1}(W_{n,\alpha}) = \widetilde{\psi}_\alpha^{-1}\Big[H_j^{-1}\Big({}^{\mathrm{t}}\psi_\alpha^{-1}(W_{n,\alpha}) \cap H_j(\widetilde{E}(\Delta_j))\Big) \cap \widetilde{\psi}_\alpha(\widetilde{C}(\Delta_{j,\alpha}))\Big]$$

and

$$\widetilde{\psi}_\alpha(B_{n,\alpha}) = H_j^{-1}\Big[{}^{\mathrm{t}}\psi_\alpha^{-1}(W_{n,\alpha}) \cap H_j(\widetilde{E}(\Delta_j))\Big] \cap \widetilde{\psi}_\alpha(\widetilde{C}(\Delta_{j,\alpha}))\ .$$

Finally,

$$\begin{aligned} H_j^{-1}(W_n) &= \bigcup_{\alpha\in A} \Big[H_j^{-1}(W_n) \cap \widetilde{\psi}_\alpha(\widetilde{C}(\Delta_{j,\alpha}))\Big] \\ &= \bigcup_{\alpha\in A} \Big[H_j^{-1}\Big({}^{\mathrm{t}}\psi_\alpha^{-1}(W_{n,\alpha}) \cap H_j(\widetilde{E}(\Delta_j))\Big) \cap \widetilde{\psi}_\alpha(\widetilde{C}(\Delta_{j,\alpha}))\Big] \\ &= \bigcup_{\alpha\in A} \widetilde{\psi}_\alpha(B_{n,\alpha}) = B_n\ . \blacksquare \end{aligned}$$

Corollary. *The space $\widetilde{E}(G)$ is isomorphical to a topological subspace of $\mathcal{D}'(G)$.* ∎

Theorem 2. *The space $\widetilde{C}(G)$ is isomorphical to the space $\mathcal{D}'(G)$.*

Proof: As $\mathcal{D}(G)$ is dense in $\mathcal{D}'(G)$ (cf. [1]) and $\mathcal{D}(G) \subset \widetilde{E}(G)$, $\widetilde{E}(G)$ is also dense in $\mathcal{D}'(G)$. As $\mathcal{D}'(G)$ induces on $\widetilde{E}(G)$ its own topology, the completion $\widetilde{C}(G)$ of $\widetilde{E}(G)$ is isomorphical to $\mathcal{D}'(G)$. ∎

Final Remark. As J.S. Silva refers, namely in the introduction of his paper "Integrals and orders of growth of distributions" (Proceedings of an International Summer Institute held in Lisbon, September 1964), we believe that the characterization of the distributions T on G in terms of "continuous functions", "addition" and "differentiation" can afford the introduction of the Fourier transformation for distributions by means of the integral

$$\int_G T(u)\, \chi(-u)\, d\mu(u)\,, \qquad \chi \in \widehat{G}\ ,$$

without assuming any previous theory of the same transformation for functions.

We hope to publish soon a paper about the Fourier transformation theory for distributions on parametrizable groups.

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