PORTUGALIAE MATHEMATICA
Vol. 52 Fasc. 1 – 1995

ON THE INDICATOR OF GROWTH OF ENTIRE FUNCTIONS OF EXPONENTIAL TYPE IN INFINITE DIMENSIONAL SPACES AND THE LEVI PROBLEM IN INFINITE DIMENSIONAL PROJECTIVE SPACES

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Abstract: Let E be a separable complex Fréchet space with the bounded approximation property, or a complex DFN-space and $\mathbf{P}(E)$ be the complex projective space induced from E. Then we solve affirmatively the Levi problem in a Riemann domain over the projective space $\mathbf{P}(E)$. By using this result, we give the infinite dimensional version of the indicator theorem of entire functions of exponential type on $\mathbf{C}^n$.

1 – Introduction

Let E be a locally convex space, here always assumed to be complex and Hausdorff. Let f be an entire function of exponential type on E. Then the *indicator* I_f of the entire function f is the function on E with values in $[-\infty,\infty)$ defined by

$$I_f(z) = \limsup_{z'\to z} \limsup_{t\to\infty} \frac{1}{t} \log|f(tz')|$$

for every $z \in E$. The indicator has the following properties:

(1) I_f is plurisubharmonic.

(2) I_f is positively homogeneous of order 1, that is, $I_f(tz) = tI_f(z)$ for every positive number t and every $z \in E$.

Conversely when given a plurisubharmonic function p on E which is positively homogeneous of order 1, we consider the problem to ask whether or not there exists an entire function f of exponential type on E with $I_f = p$. Kiselman

Received: January 12, 1993.

[18], Lelong [19] and Martineau [22] solved affirmatively this problem in case the dimension of E is finite. Their results are called the *indicator theorem* of entire functions of exponential type on $\mathbf{C}^n$.

This paper is concerned with the Levi problem in infinite dimensional projective spaces and with the indicator theorem of entire functions of exponential type in infinite dimensional spaces. The main theorems of this paper are the following two theorems.

Theorem 1. *Let E be a separable Fréchet space with the bounded approximation property or a DFN-space and (ω, φ) be a Riemann domain over the complex projective space $\mathbf{P}(E)$ induced from E. Assume that ω is not homeomorphic to $\mathbf{P}(E)$ through φ. Then the following statements (1), (2), (3), (4) and (5) are equivalent. Moreover if ω is an open subset of $\mathbf{P}(E)$, the statements (1), (2), (3), (4), (5) and (6) are equivalent:*

(1) *ω is pseudoconvex;*

(2) *For any finite dimensional subspace F of E, $\varphi^{-1}(\mathbf{P}(F))$ is a Stein manifold;*

(3) *ω is a domain of holomorphy;*

(4) *ω is a domain of holomorphy and holomorphically separated;*

(5) *ω is a domain of existence;*

(6) *There exists a non-constant holomorphic function f on ω such that, for every connected open neighbourhood V of an arbitrary point on the boundary of ω, each component of $\omega \cap V$ contains zero of f of arbitrarily high order.*

Theorem 2. *Let E be a separable Fréchet space with the bounded approximation property or a DFN-space, and p be a plurisubharmonic function in E which is positively homogeneous of order 1. Then there exists an entire function f of exponential type on E such that*

$$p(z) = \limsup_{z' \to z} \limsup_{t \to \infty} \frac{1}{t} \log |f(tz')|$$

for every $z \in E$.

Corollary 3. *If E is a nuclear Fréchet space with the bounded approximation property or a DFN-space, there exists an analytic functional μ on the strong dual space E' of the space E such that*

$$p(z) = \limsup_{z' \to z} \limsup_{t \to \infty} \frac{1}{t} \log |\mu(\exp(tz'))|$$

for every $z \in E$.

The proof of Theorem 2 is based on the characterization of pseudoconvex domains of the projective space $\mathbf{P}(E)$ in Theorem 1. This method of the proof was first given by Kiselman [18] in case $E = \mathbf{C}^n$.

The Levi problem was first solved by Oka [35] in $\mathbf{C}^2$. Moreover Oka [36] extended his result to Riemann domains over $\mathbf{C}^n$. At the same time, Bremermann [3] and Norguet [32] solved this problem in $\mathbf{C}^n$. The Levi problem in infinite dimensional spaces is also the important object of study in infinite dimensional complex analysis, and has been solved affirmatively in various infinite dimensional spaces (cf. Aurich [1], Colomeau and Mujica [5], Dineen [6], [8, Appendix 1], Dineen, Noverraz and Schottenloher [9], Grumann [11], Grumann and Kiselman [12], Hervier [14], Hirchowitz [15], Matos [23], Mujica [25], [27], Noverraz [33], [34], Pomes [39], Popa [40], Schottenloher [41]). Josefson [16] gave an example of a non-separable Banach space in which the Levi problem is negative. Fujita [10], Kiselman [18] and Takeuchi [42] extended the result of the Levi problem in Riemann domains over $\mathbf{C}^n$ to those over the complex projective space $\mathbf{P}(\mathbf{C}^{n+1})$ of dimension n. Kajiwara [17] and Nishihara [31] investigated the Levi problem in Riemann domains over infinite dimensional projective spaces. In case E is a topological vector space with the finite open topology, Kajiwara [17] solved affirmatively the Levi problem in the projective space $\mathbf{P}(E)$. In case E is a Banach space with a Schauder basis, Nishihara [31] solved affirmatively this problem in Riemann domains over the projective spaces $\mathbf{P}(E)$. Therefore Theorem 1 is the extension of Nishihara [31].

2 – Notations and preliminaries

Let E be a locally convex spaces and $\mathrm{cs}(E)$ be the set of all nontrivial continuous seminorms on E.

A Hausdorff space M is called a *complex manifold* modeled on the space E if there exists a family $\mathcal{F} = \{(U_i, \varphi_i);\ i \in I\}$ of pairs (U_i, φ_i) of open sets U_i of M and homeomorphisms φ_i of open sets U_i onto open sets of E satisfying the following conditions.

(1) For any $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, the mappings $\varphi_i \circ \varphi_j^{-1} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ between open sets in E are holomorphic.

(2) $\bigcup_{i \in I} U_i = M$.

$\mathcal{F}$ is called the *atlas* of M, and an element of $\mathcal{F}$ is called a *chart* of M.

Let M and N be complex manifolds with atlases $\{(U_i, \varphi_i);\ i \in I\}$ and $\{(U'_\alpha, \varphi'_\alpha);\ \alpha \in A\}$ respectively. Then a mapping $f:\ M \to N$ is said to be *holomorphic* if, for any $i \in I$ and any $\alpha \in A$ with $f(U_i) \cap U'_\alpha \neq \emptyset$, the mapping $\varphi'_\alpha \circ f \circ \varphi_i^{-1}$ is holomorphic when it is defined. Particularly, if $N = \mathbf{C}$, f is called a *holomorphic function* on M. We denote by $H(M)$ the vector space of all holomorphic functions in M. A function $p:\ M \to [-\infty, \infty)$ is said to be plurisubharmonic if for each $i \in I$, the function $f \circ \varphi_i^{-1}$ is plurisubharmonic. We denote by $\mathrm{Ps}(M)$ the set of all plurisubharmonic functions on M. We can define a *submanifold* of the complex manifold M, a *product manifold* of M and another complex manifold, a *holomorphic fibre bundle* over M, a *holomorphic principal bundle* over M and a *holomorphic vector bundle* over M by the same way as in case the dimension of M is finite. If there exists a local biholomorphic mapping φ of a complex manifold ω into the complex manifold M, (ω, φ) is called a *Riemann domain* over M. A *section* of ω is a continuous mapping $\sigma\colon\ A \to \omega$, with $A \subset M$, such that $\varphi \circ \sigma = \mathrm{id}$ on A.

Let (ω, φ) and (ω', φ') be a Riemann domain over a complex manifold M. If a holomorphic mapping λ of ω into ω' satisfies $\varphi = \varphi' \circ \lambda$, the mapping λ is called a *morphism* of (ω, φ) into (ω', φ'). Let (ω, φ) be a Riemann domain over M, and let $\mathcal{F} \subset H(\omega)$. If (ω', φ') is a Riemann domain over M, then a morphism λ of (ω, φ) into (ω', φ') is said to be an *$\mathcal{F}$-extension* of ω if for each $f \in \mathcal{F}$ there exists a unique $f' \in H(\omega')$ such that $f' \circ \lambda = f$. A morphism λ of (ω, φ) into (ω', φ') is said to be a *holomorphic extension* of ω if λ is an *$H(\omega)$-extension* of ω. ω is said to be an *$\mathcal{F}$-domain of holomorphy* if each $\mathcal{F}$-extension of ω is an isomorphism. ω is said to be a *domain of holomorphy* if ω is an $H(\omega)$-domain of holomorphy. ω is said to be a *domain of existence* if there exists $f \in H(\omega)$ such that ω is an $\{f\}$-domain of holomorphy. Let (ω, φ) be a Riemann domain over the complex manifold M and let $\mathcal{F} \subset H(\omega)$. A morphism $\lambda\colon\ \omega \to \omega'$ is called an *$\mathcal{F}$-envelope of holomorphy* of ω if:

(a) λ is $\mathcal{F}$-extension of ω;

(b) If $\mu:\ \omega \to \omega''$ is an $\mathcal{F}$-extension of ω, then there exists a morphism $\nu\colon\ \omega'' \to \omega'$ such that $\nu \circ \mu = \lambda$.

By the same was as Mujica [27, Theorem 56.4] we can prove the following theorem.

Theorem 2.1. *Let (ω, φ) be a Riemann domain over a complex manifold M and let $\mathcal{F} \subset H(\omega)$. Then there exists the $\mathcal{F}$-envelope of holomorphy of ω and the existence of it is unique up to isomorphism.*

For $\mathcal{F} \subset H(\omega)$, we denote by $E_{\mathcal{F}}(\omega)$ the $\mathcal{F}$-envelope of holomorphy of a Riemann domain ω. Then we can prove the following proposition.

Proposition 2.2. *Let (ω, φ) be a Riemann domain over a complex manifold and $\mathcal{F} \subset H(\omega)$.*

(1) *Let $\lambda\colon \omega \to \omega'$ be an $\mathcal{F}$-extension of ω. Then $\omega' = E_{\mathcal{F}}(\omega)$ if and only if ω' is an $\mathcal{F}$-domain of holomorphy.*

(2) *$\omega = E_{\mathcal{F}}(\omega)$ if and only if ω is an $\mathcal{F}$-domain of holomorphy.*

Let M be a complex manifold and S be a subset of M. For a complex valued function f and for a real valued function R on M, we write

$$|f|_S = \sup\Big\{|f(x)|;\ x \in S\Big\}, \qquad R(S) = \inf\Big\{R(x);\ x \in S\Big\}\ .$$

For $\mathcal{F} \subset H(M)$ we write

$$\widehat{S}_{\mathcal{F}} = \Big\{y \in M;\ |f(y)| \le |f|_S \ \text{ for all } f \in \mathcal{F}\Big\}\ .$$

Likewise, for $S \subset M$ and $\mathcal{F} \subset P_S(M)$ we set

$$\widehat{S}_{\mathcal{F}} = \Big\{y \in M;\ f(y) \le \sup_{x\in S} f(x) \ \text{ for all } f \in \mathcal{F}\Big\}\ .$$

Let (ω, φ) be a Riemann domain over M. Let $\mathcal{F} \subset H(\omega)$, ω is said to be *$\mathcal{F}$-separated* if for each pair (x, y) of points of ω satisfying $x \ne y$ there exists a holomorphic function $h \in \mathcal{F}$ such that $h(x) \ne h(y)$. ω is said to be *holomorphically separated* if ω is $H(\omega)$-separated. ω is said to be *$\mathcal{F}$-fibre separated* if for each pair (x, y) of points of ω, satisfying $x \ne y$ and $\varphi(x) = \varphi(y)$, there exists a holomorphic function $h \in \mathcal{F}$ such that $h(x) \ne h(y)$.

We shall collected some properties of Riemann domains over a locally convex space, for which we have use afterwards.

Let E be a locally convex space and (Ω, Φ) be a Riemann domain over E. For $S \subset \Omega$ and for a convex balanced neighbourhood V of 0 in E we write $S + V \subset \Omega$ if for each $x \in S$ there exists a section $\sigma\colon \Phi(x) + V \to \Omega$ such that $\sigma \circ \Phi(x) = x$.

We define the distance functions $d^{\alpha}_{\Omega} : \Omega \to [0, +\infty]$, for $\alpha \in \mathrm{cs}(E)$, and $\delta_\Omega\colon \Omega \times E \to (0, +\infty]$, as follows:

$$d^{\alpha}_{\Omega}(x) = \sup\Big(\Big\{r > 0;\ \text{there is a section } \sigma\colon B^{\alpha}_E(\Phi(x), r) \to \Omega \ \text{ with } \sigma \circ \Phi(x) = x\Big\} \cup \{0\}\Big)$$

and

$$\delta_\Omega(x, a) = \sup\Big\{r > 0;\ \text{there is a section } \sigma\colon D_E(\Phi(x), a, r) \to \Omega \ \text{ with } \sigma \circ \Phi(x) = x\Big\}$$

where for $\xi, a \in E$ and $r > 0$ we write

$$B^{\alpha}_E(\xi, r) = \Big\{\xi + b;\ b \in E,\ \alpha(b) < r\Big\}\ ,$$

$$D_E(\xi, a, r) = \Big\{\xi + \lambda a;\ \lambda \in \mathbf{C},\ |\lambda| < r\Big\}\ .$$

If $d_\Omega^\alpha(x) > 0$ then for each $r \in (0, d_\Omega^\alpha(x)]$ there is a unique set $B_\Omega^\alpha(x,r) \subset \Omega$ containing x such that $\Phi\colon B_\Omega^\alpha(x,r) \to B_E^\alpha(\Phi(x),r)$ is a bijection. Likewise, for each $x \in \Omega$, $a \in E$ and $r \in (0, \delta_\Omega(x,a)]$ there is a unique set $D_\Omega(x,a,r) \subset \Omega$ containing x such that $\varphi\colon D_\Omega(x,a,r) \to D_E(\varphi(x),a,r)$ is bijection. The function d_Ω^α is continuous, and the function δ_Ω is lower semicontinuous. The domain Ω is said to be *pseudoconvex* if the function $-\log\delta_\Omega$ is plurisubharmonic on $\Omega \times E$. The following proposition is on Noverraz [34].

Proposition 2.3. *For a Riemann domain* (Ω,Φ) *over a locally convex space* E*, the following conditions are equivalent:*

(**a**) Ω *is pseudoconvex;*

(**b**) $d_\Omega^\alpha(\widehat{X}_{\mathrm{Ps}(\Omega)}) = d_\Omega^\alpha(X)$ *for every* $X \subset \Omega$ *and* $\alpha \in \mathrm{cs}(E)$*;*

(**c**) *For each compact set* K *of* Ω *there exists* $\alpha \in \mathrm{cs}(E)$ *such that* $d_\Omega^\alpha(\widehat{K}_{\mathrm{Ps}(\Omega)}) > 0$*;*

(**d**) $\Phi^{-1}(F)$ *is a Stein Manifold for each finite dimensional linear subspace* F *of* E*.*

Let E be a Fréchet space. A sequence (e_n) in the Fréchet space E is said to be a *Schauder basis* if every $x \in E$ admits a unique representation as a series $x = \sum_{n=1}^\infty \xi_n(x)\, e_n$ where the series converges in the ordinary sense for the topology of E. Let E_n be the linear span of the set $\{e_1, e_2, ..., e_n\}$ and let $T_n\colon E \to E_n$ be the canonical projection. Then it follows from the open mapping theorem that the sequence (T_n) is equicontinuous and converges to the identity uniformly on compact sets, and that the space E has a fundamental sequence of continuous seminorms α_j which satisfy the conditions $\alpha_j = \sup_n \alpha_j \circ T_n$.

3 – Riemann domains with $\mathbf{C}^*$-action

In this section we investigate properties of Riemann domains with $\mathbf{C}^*$-action over locally convex spaces. Results in this section are useful to investigate some properties of Riemann domains over projective spaces.

A Riemann domain (Ω,Φ) over a locally convex space E is said to be *with* $\mathbf{C}^*$*-action* if (Ω,Φ) satisfies the following conditions.

(**1**) $\mathbf{C}^*$ acts freely on Ω on the left: $(\lambda, x) \in \mathbf{C}^* \times \Omega \to \lambda \cdot x \in \Omega$.

(**2**) The action $(\lambda, x) \in \mathbf{C}^* \times \Omega \to \lambda \cdot x \in \Omega$ is holomorphic.

(**3**) $\Phi(\lambda \cdot x) = \lambda\, \Phi(x)$ for every $(\lambda, x) \in \mathbf{C}^* \times \Omega$.

Let S be a subset of Ω or E. We set

$$\lambda \cdot S = \Big\{\lambda \cdot x;\ x \in S\Big\} \ . \tag{3.1}$$

Then we can prove the following lemma.

Lemma 3.1. *Let E be a locally convex space and (Ω, Φ) be a Riemann domain with $\mathbf{C}^*$-action over E. Then we have*

$$d^\alpha_\Omega(\lambda \cdot x) = |\lambda|\, d^\alpha_\Omega(x) \ , \tag{3.2}$$

$$\delta_\Omega(\lambda \cdot x, a) = |\lambda|\, \delta_\Omega(x, a) = \delta_\Omega(x, \lambda^{-1} \cdot a) \ , \tag{3.3}$$

for any $(\lambda, x) \in \mathbf{C}^* \times \Omega$, $a \in E$ and $\alpha \in \mathrm{cs}(E)$.

Proof: We shall show first the equality (3.3). Since the second equality of (3.3) is trivial, we shall show only the first equality of (3.3). Let (x, a) be a point of $\Omega \times E$. Let r be a real number with $0 < r < \delta_\Omega(x, a)$. There exists a section $\sigma\colon\ D_E(\Phi(x), a, r) \to \Omega$ with $\sigma \circ \Phi(x) = x$. For each $\lambda \in \mathbf{C}^*$, a mapping $z \in \sigma(D_E(\Phi(x), a, r)) \to \lambda \cdot z \in \Omega$ is a biholomorphic mapping of $\sigma(D_E(\Phi(x), a, r))$ onto $\lambda \cdot \sigma(D_E(\Phi(x), a, r))$. Since $\lambda \cdot D_E(\Phi(x), a, r) = D_E(\Phi(\lambda \cdot x), a, |\lambda|\, r)$ and $\Phi(\lambda \cdot \sigma(D_E(\Phi(x), a, r))) = \lambda \cdot D_E(\Phi(x), a, r)$, a mapping $\xi \in D_E(\Phi(\lambda \cdot x), a, |\lambda|\, r) \to \lambda \cdot \sigma(\lambda^{-1}\xi)$ is a section of Ω and satisfies $\lambda \cdot \sigma(\lambda^{-1}\xi) = \lambda \cdot x$ if $\xi = \Phi(\lambda \cdot x)$. Therefore we have

$$\delta_\Omega(\lambda \cdot x, a) \ge |\lambda|\, \delta_\Omega(x, a) \ . \tag{3.4}$$

Since x, a and λ are given arbitrarily, by (3.4) we have

$$\begin{aligned} \delta_\Omega(\lambda \cdot x, a) &= |\lambda|\, |\lambda^{-1}|\, \delta_\Omega(\lambda \cdot x, a) \\ &\le |\lambda|\, \delta_\Omega(\lambda^{-1}\, \lambda \cdot x, a) \\ &= |\lambda|\, \delta_\Omega(x, a) \ . \end{aligned}$$

Thus we obtain the equality (3.3).

The equality (3.2) is obtained from (3.3) and from the equality $d^\alpha_\Omega(x) = \inf\{\delta_\Omega(x, a);\ \alpha(a) = 1\}$. This completes the proof. ∎

Let E be a Fréchet space with a Schauder basis (e_n) and (Ω, Φ) be a pseudoconvex Riemann domain with $\mathbf{C}^*$-action over E.

If U is any open subset of Ω, then we consider the functions $\eta^n_U(x)\colon\ U \to [0, +\infty]$ defined by

$$\eta^n_U(x) = \inf_{k \ge n} \delta_U\Big(x, T_k \circ \Phi(x) - \Phi(x)\Big) \ . \tag{3.5}$$

These functions were introduced by Schottenloher [41], who proved that they are strictly positive and lower semicontinuous on U. Thus the functions $-\log \eta_U^n$ are plurisubharmonic on U whenever U is pseudoconvex. We set

$$A_n = \left\{ x \in \Omega;\ \eta_\Omega^n(x) > 1 \right\} ,$$

$$\tau_n(x) = (\Phi | D_x)^{-1} \circ T_n \circ \Phi(x) \qquad (x \in A_n) ,$$

where $D_x = D_\Omega(x, T_n \circ \Phi(x) - \Phi(x), \eta_\Omega^n(x))$. Then the following lemma can be verified.

Lemma 3.2. *We set $\Omega_n = \Phi^{-1}(E_n)$ for every n. Then there exist a sequence of open sets $A_n \subset \Omega$ and a sequence of holomorphic mappings $\tau_n \colon A_n \to \Omega_n$ with the following properties:*

- **(a)** $\Omega = \bigcup_{n=1}^\infty A_n$, $A_n \subset A_{n+1}$ *and* $\Omega_n \subset A_n$ *for every* n;
- **(b)** $\tau_n = \mathrm{id}$ *on* Ω_n, $\Phi \circ \tau_n = T_n \circ \Phi$ *on* A_n *and* $\tau_n \circ \tau_{n+1} = \tau_{n+1} \circ \tau_n = \tau_n$ *on* A_n *for every* n;
- **(c)** *For each compact subset K of Ω and a balanced open neighbourhood V of 0 in E with $K + V \subset \Omega$ there exists a positive integer n such that $K \subset A_n$ and $\tau_k(x) \in x + V$ for every $x \in K$ and $k \geq n$;*
- **(d)** $\widehat{K}_{\mathrm{Ps}(\Omega)} \subset A_n$ *for every compact subset K of A_n;*
- **(e)** $\lambda \cdot A_n = A_n$ *for every $\lambda \in \mathbf{C}^*$ with $|\lambda| = 1$.*

Proof: The proof of statement (a), (b), (c) and (d) is in Schottenloher [41]. The statement (e) follows from (3.3) and (3.5). ∎

Lemma 3.3. *Assume that the Fréchet space E has a continuous norm and that Ω is connected. Let (α_n) be a fundamental sequence of continuous norms on E with $\alpha_{n+1} \geq 2\alpha_n$ and $\alpha_n = \sup_k \alpha_n \circ T_k$ for every n. Let (A_n) and (τ_n) be two sequences satisfying the conditions in Lemma 3.2. Then there are two sequences of open sets $C_n \subset B_n \subset A_n$ and a sequence (V_n) of balanced convex open neighbourhoods of 0 in E with the following properties:*

- **(a)** $\Omega = \bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty C_n$, $B_n \subset B_{n+1}$ *and* $C_n + V_n \subset C_{n+1}$ *for every* n;
- **(b)** $B_n \cap \Omega_k \subset\subset A_n \cap \Omega_k$ *for every n and k;*
- **(c)** $\tau_k(C_n) \subset B_n \cap \Omega_k$ *whenever* $k > n$;
- **(d)** *The set* $\widehat{(B_n \cap \Omega_k)}_{H(\Omega_k)}$ *is relatively compact in $A_n \cap \Omega_k$ for every n and k;*
- **(e)** $d_\Omega^{\alpha_n}(B_n) \geq 1$ *for every* n;

(f) $\sup\{\alpha_1(\Phi(x));\, x \in B_n\} \le n$ *for every* n*;*

(g) $\lambda \cdot B_n = B_n$, $\lambda \cdot C_n = C_n$ *for every* $\lambda \in \mathbf{C}^*$ *with* $|\lambda| = 1$.

Proof: By Mujica [26, Lemma 2.6] and by an examination of the proof of Mujica [26, Lemma 2.6] there exist two sequences of open sets $C_n \subset B_n \subset A_n$ and a sequence V_n of balanced convex open neighbourhoods of 0 in E satisfying the statements (a), (b), (c), (d), (e) and (f). We replace newly two sets $\bigcup_{|\lambda|=1} \lambda \cdot B_n$ and $\bigcup_{|\lambda|=1} \lambda \cdot C_n$ by B_n and C_n respectively for each n. Then the two sets B_n and C_n are open sets of Ω and satisfy the required conditions. This completes the proof. ∎

A holomorphic function f in Ω is said to be $\mathbf{C}^*$*-invariant* if

$$f(\lambda \cdot x) = f(x)$$

for every $(\lambda, x) \in \mathbf{C}^* \times \Omega$. For each $f \in H(\Omega)$, we set

$$\widetilde{f}(x) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta} \cdot x)\, d\theta \tag{3.6}$$

for every $x \in \Omega$. Then $\widetilde{f}$ is holomorphic on Ω. For each x, a function $\lambda \to \widetilde{f}(\lambda \cdot x)$ ($\lambda \in \mathbf{C}^*$) is holomorphic and constant on $|\lambda| = 1$. Therefore by the identity theorem, the function $\lambda \to \widetilde{f}(\lambda \cdot x)$ ($\lambda \in \mathbf{C}^*$) is constant. Thus the function $\widetilde{f}$ is $\mathbf{C}^*$-invariant.

Lemma 3.4. *With the conditions and the notion of Lemma 3.3, for each* $\mathbf{C}^*$*-invariant function* $f_n \in H(\Omega_n)$ *and for each* $\varepsilon > 0$ *there exists* $\mathbf{C}^*$*-invariant function* $f \in H(\Omega)$ *such that*

(a) $f = f_n$ *on* Ω_n*;*

(b) $|f - f_n \circ \tau_n|_{C_n} \le \varepsilon$*;*

(c) $|f|_{C_j} < +\infty$ *for every* j.

Proof: By Mujica [23, Lemma 2.7], there exists a holomorphic function f satisfying the conditions (a), (b) and (c). Then the function f may not be $\mathbf{C}^*$-invariant. Let $\widetilde{f}$ be a $\mathbf{C}^*$-invariant holomorphic function on Ω_n defined by (3.6). Then it is easy to show that the function $\widetilde{f}$ satisfies the conditions (a), (b) and (c). This completes the proof. ∎

The following lemma is on Dineen [7].

Lemma 3.5. *Let* E *be a Fréchet space with a Schauder basis. Let* α *be a* *continuous seminorm on E satisfying the condition*

$$\alpha(x) = \sup_{m\geq 1} \alpha\Big(\sum_{n=1}^{m} \xi_n(x)\, e_n\Big) \tag{3.7}$$

for every $x \in E$. If we set

$$\begin{aligned} Z^\alpha &= \Big\{n \in \mathbb{N};\ \alpha(e_n) = 0\Big\}\ , \\ E^\alpha &= \Big\{x \in E;\ \xi_n(x) = 0 \ \text{for every}\ n \in Z^\alpha\Big\}\ , \end{aligned} \tag{3.8}$$

then E^α has a Schauder basis and a continuous norm, and E is the topological direct sum of E^α and $\alpha^{-1}(0)$.

Let (Ω, Φ) be a Riemann domain with $\mathbf{C}^*$-action over E and A be a subset of Ω or of E. Then we set

$$\mathbf{C}^* \cdot A = \Big\{\xi \cdot x;\ \xi \in \mathbf{C}^*,\ x \in A\Big\}\ .$$

Lemma 3.6. *Let (Ω, Φ) be a connected pseudoconvex Riemann domain with $\mathbf{C}^*$-action over a Fréchet space E which has a Schauder basis. Let $x_0 \in \Omega$ and let α be a continuous seminorm on E satisfying the condition (3.7) in Lemma 3.5 and $d^\alpha_\Omega(x_0) > 0$. Let $\pi_\alpha\colon E \to E^\alpha$ be the canonical projection and $\Omega^\alpha = \Phi^{-1}(E^\alpha)$. Then:*

- **(a)** *There is a holomorphic mapping $\sigma_\alpha\colon \Omega \to \Omega^\alpha$ such that $\sigma_\alpha = \mathrm{id}$ on Ω^α, $\Phi \circ \sigma_\alpha = \pi_\alpha \circ \Phi$ on Ω and $\sigma_\alpha(\lambda \cdot x) = \lambda \cdot \sigma_\alpha(x)$ for every $(\lambda, x) \in \mathbf{C}^* \times \Omega$;*
- **(b)** *Let U be any connected pseudoconvex open subset of Ω such that $d^\alpha_U(y_0) > 0$ for some $y_0 \in U$. Then $U = \sigma_\alpha^{-1}(U \cap \Omega^\alpha)$ and $f \circ \sigma_\alpha = f$ on U for every $f \in H(U)$ which is bounded on an α-neighbourhood of y_0;*
- **(c)** *For $x, y \in \Omega$ we have $x = y$ if and only if $\Phi(x) = \Phi(y)$ and $\sigma_\alpha(x) = \sigma_\alpha(y)$;*
- **(d)** *For each $a \in E$ and $t \in \Omega^\alpha$ with $\pi_\alpha(a) = \Phi(t)$ there is a unique $x \in \Omega$ such that $\Phi(x) = a$ and $\sigma_\alpha(x) = t$;*
- **(e)** *A net (x_i) in Ω converges in Ω if and only if $(\Phi(x_i))$ converges in E and $(\sigma_\alpha(x_i))$ converges in Ω^α.*

Proof: Any other things except for the equality $\sigma_\alpha(\lambda \cdot x) = \lambda \cdot \sigma_\alpha(x)$ for every $(\lambda, x) \in C^* \times \Omega$ were proved in Mujica [26, Lemma 3.2]. Therefore we shall show only this equality. Let (λ, x) be any element of $\mathbf{C}^* \times \Omega$. We set $z = \sigma_\alpha(x)$ and $w = \sigma_\alpha(\lambda \cdot x)$, and then we have only to show the equality $\lambda \cdot z = w$. We remark $\Phi(\lambda \cdot z) = \Phi(w)$. Since $\Phi(\xi \cdot z) = \xi\, \Phi(z)$ for every $\xi \in \mathbf{C}^*$, a mapping

σ of $\mathbf{C}^* \cdot \{\Phi(z)\}$ into Ω defined by $\sigma(\xi\,\Phi(z)) = \sigma_\alpha(\xi \cdot x)$ for every $\xi \in \mathbf{C}^*$ and a mapping $\sigma'\colon\ \xi \cdot \Phi(z) \in \mathbf{C}^* \cdot \{\Phi(z)\} \to \xi \cdot z$ are sections of Ω with $\sigma \circ \Phi(z) = z$ and $\sigma' \circ \Phi(z) = z$ respectively. Therefore it follows from the uniqueness of existence of a section of Ω that $\xi \cdot z = \sigma_\alpha(\xi \cdot x)$ for every $\xi \in \mathbf{C}^*$. Especially we have $\lambda \cdot z = \sigma_\alpha(\lambda \cdot x) = w$. This completes the proof. ∎

We define a holomorphic mapping $\widetilde{\Phi}$ of the product manifold $\Omega^\alpha \times \alpha^{-1}(0)$ of Ω^α and $\alpha^{-1}(0)$ into E by $\widetilde{\Phi}(x, \xi) = \Phi(x) + \xi$ for every $(x, \xi) \in \Omega^\alpha \times \alpha^{-1}(0)$. Then $(\Omega^\alpha \times \alpha^{-1}(0), \widetilde{\Phi})$ is a Riemann domain over E. Moreover $(\Omega^\alpha \times \alpha^{-1}(0), \widetilde{\Phi})$ is with $\mathbf{C}^*$-action:

$$\mathbf{C}^* \times (\Omega^\alpha \times \alpha^{-1}(0)) \in (\lambda, (x, \xi)) \to (\lambda \cdot x, \lambda \cdot \xi) \in \Omega^\alpha \times \alpha^{-1}(0)\ .$$

We define a morphism μ of Ω into $\Omega^\alpha \times \alpha^{-1}(0)$ by $\mu(x) = (\sigma_\alpha(x), \Phi(x) - \pi_\alpha \cdot \Phi(x))$ for every $x \in \Omega$. Then we have $\mu(\lambda \cdot x) = \lambda \cdot \mu(x)$ for every $(\lambda, x) \in \mathbf{C}^* \times \Omega$ and μ is an isomorphism. Thus we have the following Lemma 3.7.

Lemma 3.7. *Let (Ω, Φ) be a connected pseudoconvex Riemann domain with $\mathbf{C}^*$-action over a Fréchet space E which has a Schauder basis. With the conditions and the notations of Lemma 3.6, (Ω, Φ) is identified with the Riemann domain $(\Omega^\alpha \times \alpha^{-1}(0), \widetilde{\Phi})$ with $\mathbf{C}^*$-action over $E = E^\alpha \oplus \alpha^{-1}(0)$.*

4 – Riemann domains over projective spaces

Let E be a locally convex space. Let z and z' be points in $E - \{0\}$. z and z' are said to be *equivalent* if there exists $\lambda \in \mathbf{C}^*$ such that $z' = \lambda z$. We denote by $\mathbf{P}(E)$ the quotient space of $E - \{0\}$ by this equivalent relation. Then $\mathbf{P}(E)$ is a Hausdorff space. The Hausdorff space $\mathbf{P}(E)$ is called the *complex projective space introduced from E*. We denote by q the quotient map of $E - \{0\}$ onto $\mathbf{P}(E)$. For any $\xi \in E - \{0\}$, we denote by $[\xi]$ the equivalent class of ξ (i.e., $q(\xi) = [\xi]$). Let E' be the complex vector space of all continuous linear functional on E. We set

$$\mathcal{S}_E = \Big\{(f, a) \in E' \times E;\ f(a) \neq 0\Big\}\ . \tag{4.1}$$

For each $f \in E' - \{0\}$, we define a hyperplane $E(f)$ of E and open subset $U(f)$ of $\mathbf{P}(E)$ by

$$E(f) = \Big\{\xi \in E;\ f(\xi) = 0\Big\}\ , \tag{4.2}$$

$$U(f) = \Big\{[\xi] \in \mathbf{P}(E);\ f(\xi) \neq 0\Big\}\ , \tag{4.3}$$

respectively. For every $(f,a) \in \mathcal{S}_E$, we define a homeomorphism $\varphi_{(f,a)}$ of $U(f)$ onto $E(f)$ by

$$\varphi_{(f,a)}([\xi]) = \frac{1}{f(\xi)}\,\xi - \frac{1}{f(a)}\,a \tag{4.4}$$

for every $[\xi] \in U(f)$. Then the family $\{U(f), \varphi_{(f,a)}\}_{(f,a)\in\mathcal{S}_E}$ defines the complex structure of the projective space $\mathbf{P}(E)$.

Let (ω, φ) be a Riemann domain over the complex projective space $\mathbf{P}(E)$ induced from E. We consider the fibre product Ω of ω and $E - \{0\}$ defined by

$$\Omega = \Big\{(z,w) \in \omega \times (E - \{0\});\ \varphi(z) = q(w)\Big\}\ . \tag{4.5}$$

We denote by Φ and Q projections of the fibre product Ω into $E - \{0\}$ and ω respectively. Then (Ω, Φ) is a Riemann domain over E. For each $(z,w) \in \Omega$ and for each $\lambda \in \mathbf{C}^*$, we set

$$\lambda \cdot (z,w) = (z, \lambda w)\ . \tag{4.6}$$

Then points $\lambda \cdot (z,w)$ of $\omega \times (E - \{0\})$ belongs to Ω for all $(z,w) \in \Omega$ and all $\lambda \in \mathbf{C}^*$. The mapping $(\lambda, x) \in \mathbf{C}^* \times \Omega \to \lambda \cdot x$ is holomorphic. By this action, (Ω, Φ) is the Riemann domain with $\mathbf{C}^*$-action over E. The Riemann domain (Ω, Φ) with $\mathbf{C}^*$-action over E is called the *Riemann domain associated with* the Riemann domain (ω, φ). The Riemann domain ω is the quotient space of Ω by this $\mathbf{C}^*$-action and Q is the quotient map of Ω onto ω. Ω is the total space of a holomorphic principal bundle over ω with the complex multiplicative group $\mathbf{C}^*$. We have the following commutative diagram:

$$\begin{array}{ccc} \Omega & \xrightarrow{\ Q\ } & \omega \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ E-\{0\} & \xrightarrow[\ q\ ]{} & \mathbf{P}(E)\ . \end{array} \tag{4.7}$$

Let E be a locally convex space and (ω, φ) be a Riemann domain over the projective space $\mathbf{P}(E)$. Then the Riemann domain ω is said to be *pseudoconvex* if for each $(f,a) \in \mathcal{S}_E$ the Riemann domain $(\varphi^{-1}(U(f)), \varphi_{(f,a)} \circ \varphi|\varphi^{-1}(U(f)))$ over $E(f)$ is pseudoconvex.

Let F be a closed linear subspace of E. we set

$$\Omega_F = \Phi^{-1}(F)\ , \tag{4.8}$$

$$\omega_F = \varphi^{-1}(\mathbf{P}(F))\ . \tag{4.9}$$

Ω_F is a holomorphic principal bundle over ω_F with the complex multiplicative group $\mathbf{C}^*$.

Let (Ω, Φ) be a Riemann domain over a locally convex space E. Let a and b be points of Ω. By a *line segment* $[a, b]$ in Ω we mean a set in Ω containing the points a and b and homeomorphic under Φ to the line segment $[\Phi(a), \Phi(b)]$ in E. By a *polygonal line* $[x_0, x_1, ..., x_n]$ in Ω we mean a finite union of line segments of the form $[x_{j-1}, x_j]$ with $j = 1, ..., n$.

Remark 4.1. Let x and y be two points which belong to a connected component of Ω. Since there exists a polygonal line $[x_0, x_1, ..., x_n]$ with $x_0 = x$ and with $x_n = y$, there exists a finite dimensional linear subspace F of E such that the set $\{x, y\}$ is contained in a connected component of the set $\Phi^{-1}(F)$.

Lemma 4.2. *Let E be a locally convex space and (ω, φ) be a Riemann domain over the complex projective space $\mathbf{P}(E)$. Assume that ω is not homeomorphic to $\mathbf{P}(E)$ through φ. Then for any finite dimensional linear subspace F of E and for any connected component V_F of ω_F, there exists a finite dimensional linear subspace G of E and a connected component V_G of $\omega_G = \varphi^{-1}(\mathbf{P}(G))$ satisfying the following conditions:*

(1) *V_F is a closed complex submanifold of V_G;*

(2) *V_G is not homeomorphism to $\mathbf{P}(G)$ through φ.*

Proof: By Remark 4.1 and the commutative diagram (4.7), there exist a finite dimensional linear subspace F_0 of E and a connected component V_{F_0} of ω_{F_0} such that V_{F_0} is not homeomorphic to $\mathbf{P}(F_0)$ through φ. We take a point z of V_F and a point w of V_{F_0}. By Remark 4.1 and by the commutative diagram (4.7), there exist a finite dimensional subspace F_1 and a connected component V_{F_1} of ω_{F_1} such that V_{F_1} contains the set $\{z, w\}$. Let G be the complex vector space spanned by all elements of the union $F \cup F_0 \cup F_1$. Then both $\mathbf{P}(F)$ and $\mathbf{P}(F_0)$ are closed complex submanifolds of $\mathbf{P}(G)$. We denote by V_G the connected component of ω_G containing the set $\{z, w\}$. Since $(V_G, \varphi|V_G)$ is a Riemann domain over $\mathbf{P}(G)$, both V_F and V_{F_0} are closed complex submanifolds of V_G. Then V_G satisfies the required conditions (1) and (2). This completes the proof. ∎

Lemma 4.3. *In addition to the assumption of Lemma 4.2, we assume that ω is pseudoconvex. Then, for any finite dimensional linear subspace F of E, ω_F is a Stein manifold. Moreover Ω is pseudoconvex.*

Proof: Let F be a finite dimensional linear subspace of E. Let V_F be any component of ω_F. By Lemma 4.2 there exists a finite dimensional linear subspace G of E and a component V_G of ω_G satisfying the conditions (1) and

(2) in Lemma 4.2. Since ω is pseudoconvex, V_G is also pseudoconvex. By Fujita [10], Kiselman [18] and Takeuchi [42], the pseudoconvex Riemann domain V_G over the projective space $\mathbf{P}(G)$ is a Stein manifold. Since V_F is a closed complex submanifold of the Stein manifold V_G, V_F is also a Stein manifold. Therefore ω_F is a Stein manifold.

For every finite dimensional linear subspace F of E, Ω_F is the total space of a holomorphic principal bundle over the Stein manifold ω_F with the complex multiplicative group $\mathbf{C}^*$. Therefore by Matsushima and Morimoto [24] Ω_F is a Stein manifold. Thus it follows from Proposition 2.3 that Ω is pseudoconvex. This completes the proof. ∎

Proposition 4.4. *With the assumption of Lemma 4.2 the following statements are equivalent.*

(1) ω *is pseudoconvex;*

(2) ω_F *is a Stein manifold for every finite dimensional linear subspace* F *of* E*;*

(3) Ω *is pseudoconvex.*

Proof: It follows from Lemma 4.3 that (1) implies (2). An examination of the proof of Lemma 4.3 shows that (2) implies (3).

We shall show that (3) implies (1). For any $(f,a) \in \mathcal{S}_E$ in (4.1), we have only to prove that the Riemann domain $(\varphi^{-1}(U(f)), \varphi_{(f,a)} \circ \varphi | \varphi^{-1}(U(f)))$ over the vector space $E(f)$ is pseudoconvex. Let L be a finite dimensional linear subspace of $E(f)$. Then by Proposition 2.3 we have only to show that $\varphi^{-1} \circ \varphi_{(f,a)}^{-1}(L)$ is a Stein manifold. We set $F = L \oplus \langle a \rangle$ where $\langle a \rangle$ is the linear span of the set $\{a\}$. Since $f = 0$ on L, $\varphi_{(f,a)}^{-1}(L) = q(L+a)$. By the assumption Ω_F is a Stein manifold. Since Ω_F is the total space of a holomorphic principal bundle over the complex manifold ω_F with the complex multiplicative group $\mathbf{C}^*$ and since $\mathbf{C}^*$ is the complexification of the compact group $\{e^{i\theta};\ \theta \in \mathbf{R}\}$, it follows from Matsushima and Morimoto [24] that ω_F is also a Stein manifold. Since $L + a$ is an affine subspace of the finite dimensional space F, the set $\varphi^{-1} \circ \varphi_{(f,a)}^{-1}(L) = \varphi^{-1}(q(L+a))$ is a closed submanifold of the Stein manifold ω_F. Therefore the complex manifold $\varphi^{-1} \circ \varphi_{(f,a)}^{-1}(L)$ is a Stein manifold. This completes the proof. ∎

After this we assume that Riemann domains (ω, φ) over the projective space $\mathbf{P}(E)$ are not homeomorphic to $\mathbf{P}(E)$ through φ.

Let E be a locally convex space, and let α and β be nontrivial continuous seminorms on E with $\alpha \leq \beta$. We set

$$\mathbf{P}(E)_\alpha = \left\{[x] \in \mathbf{P}(E);\ \alpha(x) \neq 0\right\}.$$

We define a pseudodistance $\rho_E^{\alpha,\beta}$ on the open set $\mathbf{P}(E)_\alpha$ by

$$\rho_E^{\alpha,\beta}([x],[y]) = \inf\Big\{\beta\Big(e^{i\theta}\,\frac{x}{\alpha(x)} - e^{i\theta'}\,\frac{y}{\alpha(y)}\Big);\ \theta,\theta' \in \mathbf{R}\Big\} \tag{4.10}$$

for every $[x],[y] \in \mathbf{P}(E)_\alpha$. Let (ω,φ) be a Riemann domain over $\mathbf{P}(E)$ with $\varphi(\omega) \subset \mathbf{P}(E)_\alpha$, that is, (ω,φ) be a Riemann domain over the complex manifold $\mathbf{P}(E)_\alpha$. Let a be a point of $\mathbf{P}(E)_\alpha$ and r be a positive number, we denote by $B_{P(E)}(a,r;\rho_E^{\alpha,\beta})$ the open ball $\{b \in \mathbf{P}(E)_\alpha; \rho_E^{\alpha,\beta}(a,b) < r\}$ with respect to the pseudodistance $\rho_E^{\alpha,\beta}$ with center a and with radius r. We define the *boundary distance function* $\Delta_\omega^{\alpha,\beta}\colon \omega \to [0,\infty)$ for any $\alpha,\beta \in \mathrm{cs}(E)$ with $\beta \geq \alpha$ by

$$\Delta_\omega^{\alpha,\beta}(x) = \sup\Big\{r;\ \text{there is a section } \sigma\colon B_{P(E)}(\varphi(x),r;\rho_E^{\alpha,\beta}) \to \omega \ \text{ with } \sigma\circ\varphi(x) = x\Big\}\,.$$

If $\Delta_\omega^{\alpha,\beta}(x) > 0$, then for each $r \in (0,\Delta_\omega^{\alpha,\beta}(x)]$ there exists a unique subset $B_\omega(x,r;\rho_E^{\alpha,\beta})$ of ω containing x such that a mapping $\varphi:\ B_\omega(x,r;\rho_E^{\alpha,\beta}) \to B_{P(E)}^{\alpha,\beta}(\varphi(x),r;\rho_E^{\alpha,\beta})$ is bijective.

Lemma 4.5. *Let E be a locally convex space and (ω,φ) be a connected pseudoconvex Riemann domain. Then if a continuous seminorm α on E satisfies $d_\Omega^\alpha(a) > 0$ for some points a of Ω, we have $\delta_\Omega(\cdot,\cdot) = \infty$ on $\Omega \times \alpha^{-1}(0)$ and $\varphi(\omega) \subset \mathbf{P}(E)_\alpha$.*

Proof: Since d_Ω^α is continuous, there exists an open neighbourhood $N(a)$ such that $d_\Omega^\alpha > 0$ on $N(a)$. For any $v \in \alpha^{-1}(0)$ and any $x \in N(a)$, we have $\delta(x,v) = \infty$. Since ω is pseudoconvex, it follows from Proposition 4.4 that $\delta_\Omega(\cdot,\cdot) = \infty$ on $\Omega \times \alpha^{-1}(0)$. Thus for any $x \in \Omega$, there exists a section σ of Ω on $\Phi(x) + \alpha^{-1}(0)$. Therefore the set $\Phi(x) + \alpha^{-1}(0)$ is contained in $E - \{0\}$. Thus $\Phi(x)$ is not contained in $\alpha^{-1}(0)$. Thus $\alpha(\Phi(x)) \neq 0$ for every $x \in \Omega$. It follows from the commutative diagram (4.7) that $\varphi(\omega) \subset \mathbf{P}(E)_\alpha$. This completes the proof. ∎

Lemma 4.6. *Let E be a locally convex space. Let (ω,φ) be a Riemann domain over the projective space $\mathbf{P}(E)$. Let S be a subset of ω. Then the mapping $\varphi|S : S \to \mathbf{P}(E)$ is injective if and only if the mapping $\Phi|Q^{-1}(S) : Q^{-1}(S) \to E - \{0\}$ is injective.*

Proof: Assume that the mapping $\varphi|S$ is injective. Let a and b be any different points of $Q^{-1}(S)$. If $Q(a) \neq Q(b)$, it follows from the commutative diagram (4.7) and from the injectivity of $\varphi|S$ that $\Phi(a) \neq \Phi(b)$. If $Q(a) = Q(b)$, there exist different points w_1 and w_2 of $E - \{0\}$ such that $a = (Q(a),w_1)$, $b = (Q(b),w_2)$. Since $w_1 = \Phi(a)$ and $w_2 = \Phi(b)$, $\Phi(a) \neq \Phi(b)$. Therefore $\Phi|Q^{-1}(S)$ is injective.

Assume that the mapping $\Phi|Q^{-1}(S)$ is injective. Let a and b be different points of S. Then there exist points w_1 and w_2 of $E-\{0\}$ such that $(a,w_1),(b,w_2)\in Q^{-1}(S)$. Then $(a,\lambda\cdot w_1),(b,\lambda'\cdot w_2)\in Q^{-1}(S)$ for any $\lambda,\lambda'\in\mathbf{C}^*$. Since $(a,\lambda\cdot w_1)\neq(b,\lambda'\cdot w_2)$ for any $\lambda,\lambda'\in\mathbf{C}^*$ and $\Phi|Q^{-1}(S)$ is injective, $\lambda\cdot w_1\neq\lambda'\cdot w_2$ for any $\lambda,\lambda'\in\mathbf{C}^*$. Therefore $q(w_1)\neq q(w_2)$. Thus it follows from $q\circ\Phi=\varphi\circ Q$ that $\varphi(a)\neq\varphi(b)$. Therefore $\varphi|S$ is injective. This completes the proof. ∎

For a subset S of Ω, we set

$$V(S)=Q^{-1}\circ Q(S)\ .$$

Lemma 4.7. *Let a be a point of Ω. Let α be a continuous seminorm on E with $d^\alpha_\Omega(a)>0$. For every positive number r with $0<r<d^\alpha_\Omega(a)$, the mapping $\Phi|V(B^\alpha_\Omega(a,r))\colon V(B^\alpha_\Omega(a,r))\to E-\{0\}$ is injective.*

Proof: Let (z_1,w_1) and (z_2,w_2) be different points of $V(B^\alpha_\Omega(a,r))\subset\omega\times(E-\{0\})$. We have only to show that $w_1\neq w_2$. We assume that $w_1=w_2$. Since $(z_1,w_1)\neq(z_2,w_2)$, we have $z_1\neq z_2$. Since $\varphi(z_1)=q(w_1)=q(w_2)=\varphi(z_2)$, both z_1 and z_2 belong to $\varphi^{-1}(\varphi(z_1))$. Since both (z_1,w_1) and (z_2,w_2) belong to $V(B^\alpha_\Omega(a,r))$, there exists complex number $\lambda_1,\lambda_2\in\mathbf{C}^*$ such that $(z_1,\lambda_1\cdot w_1),(z_2,\lambda_2\cdot w_2)\in B^\alpha_\Omega(a,r)$. Since $\Phi|B^\alpha_\Omega(a,r)$ is injective, $\lambda_1\cdot w_1\neq\lambda_2\cdot w_2$. Since $B^\alpha_E(\Phi(a),r)$ is convex, the line segment $[\lambda_1\cdot w_1,\lambda_2\cdot w_2]$ is contained in $B^\alpha_E(\Phi(a),r)$. The set $\{(z_1,(1-t)\lambda_1\cdot w_1+t\lambda_2\cdot w_2);\ t\in[0,1]\}$ is homeomorphically mapped by Φ onto $[\lambda_1\cdot w_1,\lambda_2\cdot w_2]$. Since $(z_1,\lambda_1\cdot w_1)\in B^\alpha_\Omega(a,r)$ and $[\lambda_1\cdot w_1,\lambda_2\cdot w_2]\subset B^\alpha_E(\Phi(a),r)$, it is valid that $(z_1,\lambda_2\cdot w_2)\in B^\alpha_\Omega(a,r)$. Then we have $\Phi((z_1,\lambda_2\cdot w_2))=\Phi((z_2,\lambda_2\cdot w_2))$. Since $\Phi|B^\alpha_\Omega(a,r)$ is injective, it follows that $z_1=z_2$. This is a contradiction. This completes the proof. ∎

We obtain the following Lemma 4.8 from Lemma 4.6 and 4.7.

Lemma 4.8. *With the assumption of Lemma 4.7. The mapping $\varphi|Q(B^\alpha_\Omega(a,r))$ is injective.*

Lemma 4.9. *We assume that there exists a nontrivial continuous seminorm α on E such that $\varphi(\omega)\subset\mathbf{P}(E)_\alpha$. Let a a point of Ω. Let β be a continuous seminorm on E with $\beta\geq\alpha$ and with $d^\beta_\Omega(a)>0$. For every positive number r with $0<r<d^\beta_\Omega(a)$, the open set $\varphi\circ Q(B^\beta_\Omega(a,r))$ contains the open set $B_{P(E)}(\varphi\circ Q(a),r/\alpha(\Phi(a));\rho^{\alpha,\beta}_E)$.*

Proof: Let u be a point of $B_{P(E)}(\varphi\circ Q(a),r/\alpha(\Phi(a));\rho^{\alpha,\beta}_E)$. Then there exist a point w of $E-\{0\}$ with $\alpha(w)=1$ and a real number θ such that $u=q(w)$

and

$$\beta\left(e^{i\theta}\,w - \frac{\Phi(a)}{\alpha(\Phi(a))}\right) < \frac{r}{\alpha(\Phi(a))}\ .$$

This implies that $\beta(e^{i\theta}\,\alpha(\Phi(a))w - \Phi(a)) < r$. Therefore $e^{i\theta}\,\alpha(\Phi(a))\,w$ belongs to $B_E^\beta(\Phi(a), r)$. Since the mapping $\Phi :\ B_\Omega^\beta(a, r) \to E - \{0\}$ is injective, there is a unique point z of ω such that $(z, e^{i\theta}\alpha(\Phi(a))w)$ belongs to $B_\Omega^\beta(a, r)$. Then we have $\varphi \circ Q(z, e^{i\theta}\alpha(\Phi(a))w) \in \varphi \circ Q(B_\Omega^\beta(a, r))$ and $\varphi \circ Q(z, e^{i\theta}\alpha(\Phi(a))w) = q \circ \Phi(z, e^{i\theta}\alpha(\Phi(a))w) = q(e^{i\theta}\alpha(\Phi(a))w) = q(w) = u$. Therefore we have $u \in \varphi \circ Q(B_\Omega^\beta(a, r))$. This completes the proof. ∎

Therefore from Lemma 4.8 and 4.9 we obtain the following Lemma 4.10 which plays the important role in section 6.

Lemma 4.10. *Let a be a point of Ω with $d_\Omega^\beta(a) \geq r > 0$. Then we have*

$$\Delta_\omega^{\alpha,\beta}(Q(a)) \geq \frac{r}{\alpha(\Phi(a))}\ .$$

We end this section by proving the following Proposition 4.11.

Proposition 4.11. *Let E be a locally convex space, (ω, φ) be the projective space $\mathbf{P}(E)$ and $\mathcal{F} \subset H(\omega)$. If ω is an $\mathcal{F}$-domain of holomorphy, ω is pseudoconvex.*

Proof: Let a be any point of $E - \{0\}$ and f be any continuous linear functional of E with $f(a) \neq 0$. For the open set $U(f)$ defined by (4.3), we set

$$\omega_f = \varphi^{-1}(U(f))\ . \tag{4.11}$$

We have only to show that the Riemann domain $(\omega_f, \varphi_{(f,a)} \circ \varphi|\omega_f)$ over $E(f)$ is pseudoconvex. We set

$$\mathcal{F}' = \Big\{h|\omega_f;\ h \in \mathcal{F}\Big\} \cup \Big\{g/f;\ g \in E'\Big\}\ .$$

Then ω_f is an $\mathcal{F}'$-domain of holomorphy. Thus by Noverraz [32], ω_f is pseudoconvex. ∎

5 – Cartan–Thullen type theorem

Let E be a locally convex space and (ω, φ) be a Riemann domain over the projective space $\mathbf{P}(E)$.

An increasing sequence $\mathcal{U} = (U_j)_{j=1}^{\infty}$ of open subset of ω is called a *regular cover* of ω if $\omega = \bigcup_{j=1}^{\infty} U_j$ and if there exists an increasing sequence $(\alpha_j)_{j=1}^{\infty}$ of continuous seminorms on E such that

$$\varphi(\omega) \subset \mathbf{P}(E)_{\alpha_1}\,, \qquad \Delta_{U_{j+1}}^{\alpha_1,\alpha_j}(U_j) > 0$$

for every j. We denote by $H^{\infty}(\mathcal{U})$ the Fréchet algebra

$$H^{\infty}(\mathcal{U}) = \Big\{ f \in H(\omega);\ |f|_{U_j} < \infty \ \text{ for avery j } \Big\}$$

endowed with the topology generated by the norms $f \to |f|_{U_j}$.

Let E be a locally convex space, (Ω, Φ) be a Riemann domain and f be a holomorphic function on Ω. For any point a of Ω, there exist continuous n-homogeneous polynomials $P_n\colon\ E \to \mathbf{C}$ and a balanced convex open neighbourhood V of 0 in E such that $a + V \subset \Omega$ and

$$f(a+x) = \sum_{n=0}^{\infty} P_n(x)$$

uniformly for $x \in V$. We denote by $o(f,a)$ the smallest integer n such that P_n are not identically zero in E. We write $o(f,a) = \infty$ if P_n are identically zero in E for all n. We call $o(f,a)$ the *order of zero* of f at a. If functions f and g are holomorphic in a neighbourhood of a point a in E, $o(fg,a) = o(f,a) + o(g,a)$. f is identically zero in a neighbourhood of a if and only if $o(f,a) = \infty$.

Let E be a metrizable locally convex space and $(\alpha_j)_{j=1}^{\infty}$ be a fundamental sequence of continuous seminorms on E. We set

$$\rho_E^{\{\alpha_j\}}(x,y) = \sum_{j=1}^{\infty} 2^{-j}\, \frac{\rho_E^{\alpha_1,\alpha_j}(x,y)}{1 + \rho_E^{\alpha_1,\alpha_j}(x,y)}$$

for every $x, y \in \mathbf{P}(E)_{\alpha_1}$. Then $\rho_E^{\{\alpha_j\}}$ is a continuous distance of $\mathbf{P}(E)_{\alpha_1}$ which defines the same topology as the initial topology of $\mathbf{P}(E)_{\alpha_1}$. We denote by $\Delta_\omega^{\{\alpha_j\}}$ the *boundary distance function* of ω with respect to the distance $\rho_E^{\{\alpha_j\}}$ and by $B_\omega^{\{\alpha_j\}}(x,r)$ the open neighbourhood of x in ω which is homeomorphic to the open set $\{z \in \mathbf{P}(E)_{\alpha_1};\ \rho_E^{\{\alpha_j\}}(\varphi(x), z) < r\}$ through φ for r with $r \le \Delta_\omega^{\{\alpha_j\}}(x)$. we set

$$B_\omega^{\{\alpha_j\}}(x) = B_\omega^{\{\alpha_j\}}(x, \Delta_\omega^{\{\alpha_j\}}(x))\ .$$

Theorem 5.1. *Let E be a separable metrizable locally convex space and (ω, φ) be a connected Riemann domain over the projective space $\mathbf{P}(E)$. Assume* *that there exist a regular cover $\mathcal{U} = (U_i)_{i=1}^{\infty}$ of ω and an increasing sequence $(\alpha_j)_{j=1}^{\infty}$ of continuous seminorms on E such that $\delta_\Omega(\cdot,\cdot) = \infty$ on $\Omega \times \alpha_1^{-1}(0)$, that ω is a $H^\infty(\mathcal{U})$-fibre separated and that $\Delta_\omega^{\alpha_1,\alpha_j}(\widehat{U}_{j\ H(\omega)}) > 0$ for every j. Then ω is a domain of existence.*

Proof: We remark that it follows from an examination of the proof of Lemma 4.5 that $\delta(\cdot,\cdot) = \infty$ on $\Omega \times \alpha_1^{-1}(0)$ implies $\varphi(\omega) \subset \mathbf{P}(E)_{\alpha_1}$. Since the projective space $\mathbf{P}(E)$ is separable, there exists a countable dense subset D of $\mathbf{P}(E)$. we set $A = \varphi^{-1}(D)$. Let (x_k) be a sequence in A with the property that each point of A appears in the sequence (x_k) infinitely many times. We set $V_k = \widehat{U}_{k\ H(\omega)}$ for each $k \geq 1$. By the assumption, we have $\Delta_\omega^{\{\alpha_j\}}(\widehat{U}_k\ H(\omega)) > 0$. Thus $B_\omega^{\{\alpha_j\}}(x)$ is not contained in V_k for each $x \in \omega$ and $k \geq 1$. After replacing a sequence (V_k) by subsequence, if necessary, we can find a sequence (y_k) in ω such that $y_k \in B_\omega^{\{\alpha_j\}}(x_k)$, $y_k \notin V_k$ and $y_k \in V_{k+1}$ for every $k \geq 1$. Hence we can inductively find a sequence (f_k) in $H(\omega)$ such that

$$|f_k|_{V_k} < 2^{-k} \quad \text{and} \quad f_k(y_k) = 1$$

for every $k \geq 1$. Since $\sum_{k=1}^{\infty} \frac{k}{2^k}$ is convergent, the infinite product

$$\prod_{k=1}^{\infty} (1 - f_k)^k$$

converges uniformly on V_k for each k and there it defines a function $f \in H(\omega)$ which is not identically zero in ω. We set $N(f) = \{x \in \omega;\ f(x) = 0\}$ and $A' = A \backslash N(f)$. Then A' is a countable dense subset of ω. We set $B = \{(x,y) \in A' \times A';\ \varphi(x) = \varphi(y)$ and $x \neq y\}$. B is a countable subset of $\omega \times \omega$. Since ω is $H^\infty(\mathcal{U})$-fibre separated, the set $S_{x,y} = \{g \in H^\infty(\mathcal{U});\ \operatorname{Re} g(x) \neq \operatorname{Re} g(y)\}$ is nonvoid for each $(x,y) \in B$. The set $S_{x,y}$ is open in $H^\infty(\mathcal{U})$. We claim the set $S_{x,y}$ is dense in $H^\infty(\mathcal{U})$. Indeed, given $f \in H^\infty(\mathcal{U})$ with $f \notin S_{x,y}$, choose $g \in S_{x,y}$ and set $g_n = f + (1/n)\, g$. Then $g_n \in S_{x,y}$ for every n and the sequence (g_n) converges to f in $H^\infty(\mathcal{U})$. Since $H^\infty(\mathcal{U})$ is a Baire space, the set $S = \bigcap\{S_{x,y};\ (x,y) \in B\}$ is dense in $H^\infty(\mathcal{U})$. Thus there exists a function $g \in H^\infty(\mathcal{U})$ such that $\operatorname{Re} g(x) \neq \operatorname{Re} g(y)$ for every $(x,y) \in B$. Since the set of quotient

$$\frac{\Big(\log|f(x)| - \log|f(y)|\Big)}{\operatorname{Re}\Big(g(x) - g(y)\Big)}$$

with $(x,y) \in B$ is countable, there exists $\theta \in (0,1)$ such that $\log|f(x)| - \log|f(y)| \neq \theta \operatorname{Re}(g(x) - g(y))$ for every $(x,y) \in B$. We set

$$h(x) = f(x) \exp(-\theta\, g(x))$$

for every $x \in \omega$. Then we have $h(x) \neq h(y)$ for every $(x, y) \in B$.

We shall show that ω is the domain of existence of h. Let $\lambda\colon \omega \to \widetilde{\omega}$ be an $\{h\}$-envelope of holomorphy of ω and let $\widetilde{h} \in H(\widetilde{\omega})$ with $\widetilde{h} \circ \lambda = h$. We denote by $\widetilde{\varphi}$ the projection of the Riemann domain $\widetilde{\omega}$ into $\mathbf{P}(E)$. We remark that by Proposition 2.2 and Lemma 4.11 $\widetilde{\omega}$ is pseudoconvex. To prove that λ is injective, we assume that there exist distinct points a and b of ω such that $\lambda(a) = \lambda(b)$. Then there exist an open neighbourhood $U(a)$ of a and an open neighbourhood $U(b)$ of b with $U(a) \cap U(b) = \emptyset$ such that $\lambda|U(a)$, $\lambda|U(b)$, $\varphi|U(a)$ and $\varphi|U(b)$ are homeomorphisms and that $\lambda(U(a)) = \lambda(U(b))$. Then we have $\lambda(x) = \lambda(y)$ if $(x, y) \in U(a) \times U(b)$ and $\varphi(x) = \varphi(y)$. Thus we have $h(x) = \widetilde{h} \circ \lambda(x) = \widetilde{h} \circ \lambda(y) = h(y)$ if $(x, y) \in U(a) \times U(b)$ and $\varphi(x) = \varphi(y)$. We set $W = \varphi(U(a)) = \varphi(U(b))$, $S_1 = \varphi(U(a) \cap N(f))$ and $S_2 = \varphi(U(b) \cap N(f))$. The set $S_1 \cup S_2$ is an analytic subset of the open set W of $\mathbf{P}(E)$. Therefore $W \backslash (S_1 \cup S_2)$ is a dense open subset of W. Therefore we have $D \cap (W \backslash (S_1 \cup S_2)) \neq \emptyset$. Hence there exists a point p of W such that $p \notin S_1 \cup S_2$, $p \in D$. Then there exists a point $(x, y) \in U(a) \times U(b)$ with $\varphi(x) = \varphi(y) = p$. Since $p \notin S_1 \cup S_2$, $\{x, y\} \cap N(f) = \emptyset$. Therefore (x, y) belongs to B. Thus we have $h(x) \neq h(y)$. On the other hand since $\varphi(x) = \varphi(y)$ and $(x, y) \in U(a) \times U(b)$, $h(x) = h(y)$. This is a contradiction. Therefore λ is injective.

To prove that λ is surjective, we assume that $\widetilde{\omega} \neq \lambda(\omega)$. Then there exists a point $z_0 \in (\widetilde{\omega} \backslash \lambda(\omega)) \cap \overline{\lambda(\omega)} \neq \emptyset$ where $\overline{\lambda(\omega)}$ is the topological closure of $\lambda(\omega)$ in $\widetilde{\omega}$. Since $\widetilde{\omega}$ is pseudoconvex, it follows from an examination of the proof of Lemma 4.5 that $\widetilde{\varphi}(\widetilde{\omega}) \subset \mathbf{P}(E)_{\alpha_1}$. We set $a = \widetilde{\varphi}(z_0)$. There exists a continuous linear functional μ on E such that $\mu(a) \neq 0$. Then $\widetilde{\varphi}^{-1}(U(\mu))$ is an open subset of $\widetilde{\omega}$ and contains the subset $\{z_0\}$ of $\widetilde{\omega}$ where $U(\mu)$ is in (4.3). $(\widetilde{\varphi}^{-1}(U(\mu)), \varphi_{(\mu,a)} \circ \widetilde{\varphi})$ is a Riemann domain over the locally convex space $E(\mu)$ where $E(\mu)$ and $\varphi_{(\mu,a)}$ are in (4.3) and in (4.4) respectively. There exists an open neighbourhood V of 0 in $E(\mu)$ such that there exists a section s of the Riemann domain $\widetilde{\varphi}^{-1}(U(\mu))$ on V. Then $\widetilde{h} \circ s$ is holomorphic in V. For any $x \in V$ there exists a sequence of n-homogeneous polynomials $P_x^n\colon E \to \mathbf{C}$ and a convex balanced open neighbourhood U of 0 in E such that $x + U \subset V$ and

$$\widetilde{h} \circ s(\xi) = \widetilde{h} \circ s(x) + \sum_{n=1}^{\infty} P_x^n(\xi)$$

uniformly for $\xi \in U$. Then $P_x^n(\xi)$ is given by

$$P_x^n(\xi) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(x + e^{i\theta} \cdot \xi)\, d\theta$$

for any $\xi \in E$. Since $\widetilde{h} \circ s$ is not identically 0 in V, the order $o(\widetilde{h} \circ s, 0)$ of zero of $\widetilde{h} \circ s$ at 0 is finite. We set $n(0) = o(\widetilde{h} \circ s, 0)$. Then there exists $\xi_0 \in E$

such that $P_0^{n(0)}(\xi_0) \neq 0$. Since $x \to P_x^{n(0)}(\xi_0)$ is continuous, there exists an open neighbourhood $N(0)$ of 0 in V such that $P_x^{n(0)}(\xi_0) \neq 0$ for any $x \in N(0)$. Therefore we have $o(\widetilde{h} \circ s, x) \leq n(0)$ for every $x \in N(0)$. There exists a positive number r such that $2r < \Delta_{\widetilde{\omega}}^{\{\alpha_j\}}(z_0)$ and $\varphi_{(\mu,a)} \circ \widetilde{\varphi}(B_{\widetilde{\omega}}^{\{\alpha_j\}}(z_0, 2r)) \subset N(0)$. We can find $p \in A$ such that $\lambda(p) \in B_{\widetilde{\omega}}^{\{\alpha_j\}}(z_0, r)$. Then we have $\Delta_{\omega}^{\{\alpha_j\}}(p) < r$ and it follows that

$$\lambda(B_{\omega}^{\{\alpha_j\}}(p)) = B_{\widetilde{\omega}}^{\{\alpha_j\}}(\lambda(p), \Delta_{\omega}^{\{\alpha_j\}}(p)) \subset B_{\widetilde{\omega}}^{\{\alpha_j\}}(\lambda(p), r) \subset B_{\widetilde{\omega}}^{\{\alpha_j\}}(z_0, 2r) \ .$$

By the definition of the sequence (x_k) there exists a strictly increasing sequence (k_n) of natural numbers such that $x_{k_n} = p$ for every n. Hence each y_{k_n} belongs to $B_{\omega}^{\{\alpha_j\}}(p)$ and therefore $\lambda(y_{k_n}) \in B_{\widetilde{\omega}}^{\{\alpha_j\}}(z_0, 2r)$. We set $z_{k_n} = \varphi_{(\mu,a)} \circ \widetilde{\varphi}(y_{k_n})$. Then z_{k_n} belong to $N(0)$. On the other hand we have $o(\widetilde{h} \circ s, z_{k_n}) \geq k_n$. Since $o(\widetilde{h} \circ s, x) \leq n(0)$ for every $x \in N(0)$, this is a contradiction. This completes the proof. ∎

6 – Levi problem in a Riemann domain over an infinite dimensional complex projective space

The aim of this section is to prove Theorem 1. Let E be a Fréchet space with a Schauder basis $(e_n)_{n=1}^{\infty}$. We shall first assume that E has a continuous norm. Let (ω, φ) be a connected pseudoconvex Riemann domain over the complex projective space $\mathbf{P}(E)$. Let (Ω, Φ) be the Riemann domain with $\mathbf{C}^*$-action associated with the Riemann domain (ω, φ) over $\mathbf{P}(E)$. We choose a fundamental sequence $(\alpha_n)_{n=1}^{\infty}$ of continuous norms on E with $\alpha_{n+1} \geq 2\alpha_n$ and $\alpha_n = \sup_k \alpha_n \circ T_k$ for every n. With the notations of Lemma 3.2 and Lemma 3.3, we set $\omega_n = \varphi^{-1}(\mathbf{P}(E_n))$, $A_{n,\omega} = Q(A_n)$, $B_{n,\omega} = Q(B_n)$, $C_{n,\omega} = Q(C_n)$ and

$$\tau_{n,\omega}(z) = Q \circ \tau_n \circ (Q|A_n)^{-1}(z) \tag{6.1}$$

for every $z \in A_{n,\omega}$. Then the mapping $\tau_{n,\omega}$ is a holomorphic mapping of $A_{n,\omega}$ into ω_n. By Lemma 3.3 (e), (f) and Lemma 4.10, we have

$$\Delta_{\omega}^{\alpha_1,\alpha_n}(B_{n,\omega}) \geq 1/n \ . \tag{6.2}$$

A sequence $\mathcal{C} = (C_{j,\omega})_{j=1}^{\infty}$ of open sets of ω is a regular cover of ω. In fact, by Lemma 3.3 (a) and Lemma 4.10, there exists an increasing sequence $(\beta_j)_{j=1}^{\infty}$ of continuous norms on E such that $\beta_1 \geq \alpha_1$ and

$$\Delta_{C_{j+1}}^{\alpha_1,\beta_j}(C_{j,\omega}) \geq 1/j \tag{6.3}$$

for every $j \geq 1$.

Lemma 6.1. *For each $f_n \in H(\omega_n)$ and for each $\epsilon > 0$ there exists $f \in H(\omega)$ such that*

(a) $f = f_n$ *on* ω_n*;*

(b) $|f - f_n \circ \tau_{n,\omega}|_{C_{n,\omega}} \leq \epsilon$*;*

(c) $|f|_{C_{j,\omega}} < \infty$ *for every* j*.*

Proof: We define a holomorphic function g_n on $\Omega_n = \Phi^{-1}(E_n)$ by $g_n = f_n \circ (Q|\Omega_n)$ for each n. Each g_n is a $\mathbf{C}^*$-invariant function on Ω_n. By Lemma 3.4, there exists a $\mathbf{C}^*$-invariant function $g \in H(\Omega)$ such that $g = g_n$ on Ω, $|g - g_n \circ \tau_n|_{C_n} \leq \epsilon$, and $|g|_{C_j} < \infty$ for every j. We define a holomorphic function f on ω by

$$f(z) = g \circ Q^{-1}(z) \quad \text{ for every } \ z \in \omega \ .$$

Then f satisfies the required conditions (a), (b) and (c). This completes the proof. ∎

Lemma 6.2. *ω is $H^\infty(\mathcal{C})$-separated.*

Proof: Let a and b be any different points of ω. There exists a positive integer N such that the set $\{a, b, \tau_{N,\omega}(a), \tau_{N,\omega}(b)\}$ is contained in $C_{N,\omega}$ and that $\tau_{N,\omega}(a) \neq \tau_{N,\omega}(b)$. By Proposition 4.4, ω_N is a Stein manifold. By Oka–Cartan theorem, there exists a holomorphic function f_N on ω_N such that $f_N(\tau_{N,\omega}(a)) = 2$ and $f_N(\tau_{N,\omega}(b)) = 0$. By Lemma 6.1, there exists a holomorphic function $f \in H(\omega)$ such that

(a) $f = f_N$ on ω_N;

(b) $|f - f_N \circ \tau_{N,\omega}|_{C_{N,\omega}} \leq 1/2$;

(c) $|f|_{C_{j,\omega}} < \infty$ for every $j \geq 1$.

Since $\{a, b\} \subset C_{N,\omega}$,

$$\begin{aligned} \Big|f(a) - f(b)\Big| &\geq \Big|f_N \circ \tau_{N,\omega}(a) - f_N \circ \tau_{N,\omega}(b)\Big| - \Big|f(a) - f_N \circ \tau_{N,\omega}(a)\Big| \\ &\qquad - \Big|f(b) - f_N \circ \tau_{N,\omega}(b)\Big| \\ &\geq 2 - 1/2 - 1/2 = 1 \ . \end{aligned}$$

Therefore we have $f(a) \neq f(b)$. Thus ω is $H^\infty(\mathcal{C})$-separated. This completes the proof. ∎

Lemma 6.3. *It is valid that*

$$\Delta_\omega^{\alpha_1,\alpha_n}(\widehat{C}_{n,\omega\ H^\infty}(\mathcal{C})) > 0$$

for every n.

Proof: Let n be any positive integer and x_0 be any point of $\widehat{C}_{n,\omega\ H^\infty(\mathcal{C})}$. Then there exists a positive integer n_0 such that $n_0 \geq n$ and $x_0 \in C_{k,\omega}$ for every $k \geq n_0$. By Lemma 6.1, for each $k \geq n_0$, for each $f_k \in H(\omega_k)$ and each $\epsilon > 0$, there exists a function $f \in H^\infty(\mathcal{C})$ such that $|f - f_k \circ \tau_{k,\omega}|_{C_{k,\omega}} \leq \epsilon$. Therefore

$$\begin{aligned} |f_k \circ \tau_{k,\omega}(x_0)| &\leq |f(x_0)| + \epsilon \\ &\leq |f|_{C_{n,\omega}} + \epsilon \\ &\leq |f_k \circ \tau_{k,\omega}|_{C_{n,\omega}} + 2\,\epsilon \\ &\leq |f_k|_{B_{n,\omega} \cap \omega_k} + 2\,\epsilon\ . \end{aligned}$$

Since ϵ is arbitrarily given, $\tau_{k,\omega}(x_0) \in \widehat{(B_{n,\omega} \cap \omega_k)}_{H(\omega_k)}$. We shall show that $\Delta_\omega^{\alpha_1,\alpha_n}(x_0) \geq 1/n$. We assume that $\Delta_\omega^{\alpha_1,\alpha_n}(x_0) < 1/n$. Then there exists an integer $N \geq n_0$ such that

$$\Delta_\omega^{\alpha_1,\alpha_n}(\tau_{m,\omega}(x_0)) < 1/n \tag{6.4}$$

for every $m \geq N$. There exists a point y_0 of Ω such that $Q(y_0) = x_0$. We set

$$V(\tau_m(y_0)) = \left\{\lambda \cdot \tau_m(y_0);\ \lambda \in \mathbf{C}^*\right\}\,,$$

$$S(\tau_m(y_0)) = \left\{e^{i\theta} \cdot \tau_m(y_0);\ \theta \in \mathbf{R}\right\} \quad \text{and}$$

$$K_m = \widehat{(B_n \cap \Omega_m)}_{H(\Omega_m)}$$

for every $m \geq N$. Since Ω is pseudoconvex, since $d_\Omega^{\alpha_n}(B_n \cap \Omega_m) \geq 1$ and $\sup_{B_n} \alpha_1 \leq n$, we have $d_\Omega^{\alpha_n}(K_m) \geq 1$ and $\sup_{K_m} \alpha \leq n$. Therefore by Lemma 4.10, we have

$$\Delta_\omega^{\alpha_1,\alpha_n}(Q(K_m)) \geq \frac{1}{n}\ . \tag{6.5}$$

Since $Q(\tau_m(y_0)) = \tau_{m,\omega}(x_0)$, it follows from (6.4) and (6.5) that $K_m \cap V(\tau_m(y_0)) = \emptyset$ for every $m \geq N$. We set

$$T_m = K_m \cup S(\tau_m(y_0))$$

for every $m \geq N$. Since Ω_m is a Stein manifold, $\widehat{T}_{m\ H(\Omega_m)}$ is compact in Ω_m. We write $\widehat{T}_m = \widehat{T}_{m\ H(\Omega_m)}$. We remark that the set $\widehat{T}_m$ is contained in the set $V(\tau_m(y_0)) \cup K_m$. In fact, let x be a point of $\Omega_m \setminus (V(\tau_m(y_0)) \cup K_m)$. Since Ω_m is a Stein manifold and $V(\tau_m(y_0))$ is a closed submanifold of Ω_m, by Oka–Cartan theorem there exists a holomorphic function s in Ω_m with $s = 0$ on $V(\tau_m(y_0))$

and with $s(x) = 1$. Since K_m is a Runge compact subset of Ω_m, there exists a holomorphic function t in Ω such that $|t(x)| > 1$, $|t|_{K_m} < 1/(|s|_{K_m} + 1)$. Then we have $|s(x)\,t(x)| > 1$ and $|s\,t|_{T_m} < 1$. Therefore x cannot belong to $\widehat{T}_m$. Thus the set $\widehat{T}_m$ is contained in the set $V(\tau_m(y_0)) \cup K_m$. Since $V(\tau_m(y_0)) \cap K_m = \emptyset$, it follows that $\widehat{T}_m \cap V(\tau_m(y_0)) \cap K_m = \emptyset$ and $\widehat{T}_m = (\widehat{T}_m \cap V(\tau_m(y_0))) \cup K_m$. Let V_1 and V_2 be open neighbourhoods of $\widehat{T}_m \cap V(\tau_m(y_0))$ and of K_m, respectively, with $V_1 \cap V_2 = \emptyset$. Let g be a holomorphic function on $V_1 \cup V_2$ defined by $g = 2$ on V_1 and $g = 0$ on V_2. Since $\widehat{T}_m$ is a Runge compact subset of Ω_m, there exists a function $h \in H(\Omega_m)$ such that $|h - g|_{\widehat{T}_m} < 1/2$. Then we have $\operatorname{Re} h \geq 3/2$ on $S(\tau_m(y_0))$ and $|h|_{B_n \cap \Omega_m} \leq 1/2$ where $\operatorname{Re} h$ is the real part of h. Let $\widetilde{h}$ be a holomorphic function on Ω_m defined by (3.6). The holomorphic function $\widetilde{h}$ is constant on $Q^{-1}(z)$ for every $z \in \omega_m$. Thus we can define a holomorphic function h^* on ω_m by $h^*(z) = \widetilde{h} \circ Q^{-1}(z)$ for every $z \in \omega_m$. Then we have $|h^*|_{B_{n,\omega} \cap \omega_m} \leq 1/2$ and $h^*(\tau_{m,\omega}(x_0)) \geq 3/2$. Therefore $\tau_{m,\omega}(x_0)$ does not belong to $\widehat{(B_{n,\omega} \cap \omega_m)}_{H(\omega_m)}$. This is a contradiction. Therefore we have

$$\Delta^{\alpha_1, \alpha_n}(\widehat{C}_{n,\omega\ H^\infty(\mathcal{C})}) \geq \frac{1}{n} > 0\ .$$

This completes the proof. ∎

From Lemma 6.2 and 6.3 we obtain the following Proposition 6.4.

Proposition 6.4. *Let E be a Fréchet space with a Schauder basis and with a continuous norm. Let (ω, φ) be a connected pseudoconvex Riemann domain over the complex projective space $\mathbf{P}(E)$. Then there exist a regular cover $\mathcal{U} = (U_j)_{j=1}^\infty$ of ω and an increasing sequence $(\alpha_j)_{j=1}^\infty$ of continuous norms on E such that ω is $H^\infty(\mathcal{U})$-separated and $\Delta_\omega^{\alpha_1, \alpha_j}(\widehat{U}_{j\ H^\infty(\mathcal{U})}) > 0$ for every j.*

Next we shall assume that the Fréchet space E has not a continuous norm. Let (ω, φ) be a connected pseudoconvex Riemann domain over the complex projective space $\mathbf{P}(E)$. Let (Ω, Φ) be the Riemann domain over E associated with the Riemann domain (ω, φ) over $\mathbf{P}(E)$. Then by Lemma 4.5 there exists a continuous seminorm α_0 on E such that α_0 satisfies the condition (3.7), that $\delta_\omega(\cdot, \cdot) = \infty$ on $\Omega \times \alpha_0^{-1}(0)$ and that $\varphi(\omega) \subset \mathbf{P}(E)_{\alpha_0}$. Let (e_n) be a Schauder basis of E and E^{α_0} be a Fréchet space, defined by (3.8), with a Schauder basis and with a continuous norm. E^{α_0} is a closed subspace of E and E is the topological direct sum of E^{α_0} and of $\alpha_0^{-1}(0)$. We set $\omega^{\alpha_0} = \varphi^{-1}(\mathbf{P}(E^{\alpha_0}))$. We denote by π_{α_0} the canonical projective of E onto E^{α_0}. We set $\Omega^{\alpha_0} = \Phi^{-1}(E^{\alpha_0})$ and $\omega^{\alpha_0} = \varphi^{-1}(\mathbf{P}(E^{\alpha_0}))$. Then $(\Omega^{\alpha_0}, \Phi|\Omega^{\alpha_0})$ and $(\omega^{\alpha_0}, \varphi|\omega^{\alpha_0})$ are Riemann domains over E^{α_0} and $\mathbf{P}(E^{\alpha_0})$ respectively. We set

$$\widetilde{\sigma}_{\alpha_0}(z) = (Q|\Omega^{\alpha_0}) \circ \sigma_{\alpha_0} \circ Q^{-1}(z)$$

for every $z \in \omega$ where σ_{α_0} is a holomorphic mapping of Ω onto Ω^{α_0} in Lemma 3.6 (a). Since by Lemma 3.6 (a) we have $\sigma_{\alpha_0}(\lambda \cdot x) = \lambda \cdot \sigma_{\alpha_0}(x)$ for every $(\lambda, x) \in \mathbf{C}^* \times \Omega$, the mapping $\widetilde{\sigma}_{\alpha_0}$ is well-defined and a holomorphic mapping of ω onto ω^{α_0}. Moreover we have $\widetilde{\sigma}_{\alpha_0} = \mathrm{id}$ on ω^{α_0}. $(\omega^{\alpha_0}, \varphi|\omega^{\alpha_0})$ is the connected pseudoconvex Riemann domain over the projective space $\mathbf{P}(E^{\alpha_0})$ and E^{α_0} has a Schauder basis and a continuous norm. Thus, by Proposition 6.4, there exist a regular cover $\mathcal{U} = (U_j)_{j=1}^{\infty}$ of ω^{α_0} and an increasing sequence $(\alpha_j)_{j=1}^{\infty}$ with $\alpha_1 \geq \alpha_0|E^{\alpha_0}$ of continuous norms on E^{α_0} such that ω^{α_0} is $H^\infty(\mathcal{U})$-separated and $\Delta_{\omega}^{\alpha_1,\alpha_j}(\widehat{U}_{j\ H^\infty(\mathcal{U})}) > 0$ for every j. We set $V_j = \widetilde{\sigma}_{\alpha_0}^{-1}(U_j)$ for every j and $\mathcal{V} = (V_j)_{j=1}^{\infty}$. For each continuous norm β on E^{α_0}, we define a continuous seminorm $\widetilde{\beta}$ on E by $\widetilde{\beta}(x) = \beta(\pi_{\alpha_0}(x))$ for every $x \in E$. If a continuous norm β on E^{α_0} with $\beta \geq \alpha_1$ satisfies $\Delta_{U_{j+1}}^{\alpha_1,\beta}(U_j) > 0$, it is valid that $\Delta_{V_{j+1}}^{\widetilde{\alpha}_1,\widetilde{\beta}}(V_j) = \Delta_{U_{j+1}}^{\alpha_1,\beta}(U_j) > 0$. Since $\widehat{V}_{j\ H(\omega)} \subset \widehat{V}_{j\ H^\infty(\mathcal{V})}$,

$$\begin{aligned}\Delta_{\omega}^{\widetilde{\alpha}_1,\widetilde{\alpha}_j}(\widehat{V}_{j\ H(\omega)}) &\geq \Delta_{\omega}^{\widetilde{\alpha}_1,\widetilde{\alpha}_j}(\widehat{V}_{j\ H^\infty(\mathcal{V})}) \\ &\geq \Delta_{\omega}^{\widetilde{\alpha}_1,\widetilde{\alpha}_j}(\widetilde{\sigma}_{\alpha_0}^{-1}(\widehat{U}_{j\ H^\infty(\mathcal{U})})) \\ &= \Delta_{\omega^{\alpha_0}}^{\alpha_1,\alpha_j}(\widehat{U}_{j\ H^\infty(\mathcal{U})}) > 0\ .\end{aligned}$$

Let w be a point of $\varphi(\omega)$. Let a and b be different points in the set $\varphi^{-1}(w)$. Then there exist points x and y of Ω such that $Q(x) = a$ and $Q(y) = b$. Then we have $\Phi(x) = \Phi(y)$ and $x \neq y$. By Lemma 3.6 (c), $\sigma_\alpha(x) \neq \sigma_\alpha(y)$. Thus $\widetilde{\sigma}_{\alpha_0}(a) \neq \widetilde{\sigma}_{\alpha_0}(b)$. Since ω^{α_0} is $H^\infty(\mathcal{U})$-separated, there exists $f \in H^\infty(\mathcal{U})$ such that $f(\widetilde{\sigma}_{\alpha_0}(a)) = f(\widetilde{\sigma}_{\alpha_0}(b))$. Since $f \circ \widetilde{\sigma}_0 \in H^\infty(\mathcal{V})$, ω is $H^\infty(\mathcal{V})$-fibre separated. Thus we can obtain the following Proposition 6.5.

Proposition 6.5. *Let E be a Fréchet space with a Schauder basis and (ω, φ) be a connected pseudoconvex Riemann domain over the complex projective space $\mathbf{P}(E)$. Then there exist a regular cover $\mathcal{U} = (U_j)_{j=1}^{\infty}$ of ω and an increasing sequence $(\alpha_j)_{j=0}^{\infty}$ of continuous seminorms on E such that $\delta_\Omega(\cdot,\cdot) = \infty$ on $\Omega \times \alpha_0^{-1}(0)$, that ω is $H^\infty(\mathcal{U})$-fibre separated and that $\Delta_{\omega}^{\alpha_1,\alpha_j}(\widehat{U}_{j\ H(\omega)}) > 0$ for every j.*

A separable Fréchet space is said to have the *bounded approximation property* if there is a sequence of continuous linear operator of finite rank which converges pointwise to the identity. Pelczynski [38] has shown that every separable Fréchet space with the bounded approximation property is topologically isomorphic to a complement subspace of a Fréchet space with a Schauder basis.

Proposition 6.6. *Let E be a separable Fréchet space with the bounded approximation property or DFN-space, and (ω, φ) be a connected pseudoconvex*

Riemann domain over the complex projective space $\mathbf{P}(E)$*. Then there exist a regular cover* $\mathcal{U} = (U_j)_{j=1}^{\infty}$ *of* ω *and an increasing sequence* $(\alpha_j)_{j=1}^{\infty}$ *of continuous seminorms on* E *such that* $\delta_\Omega(\cdot,\cdot) = \infty$ *on* $\Omega \times \alpha_1^{-1}(0)$*, that* ω *is* $H^\infty(\mathcal{U})$*-fibre separated and that* $\Delta_\omega^{\alpha_1,\alpha_j}(\widehat{U}_{j\ H(\omega)}) > 0$ *for every* j*.*

Proof: If E is a separable Fréchet space with the bounded approximation property, by Pelczynski [38] there exist complex Fréchet spaces F and G such that G is the topological direct sum of E and F and that G has a Schauder basis. Let $\widetilde{\Omega}$ be the product space of the Riemann domain Ω associated with the Riemann domain (ω,φ) and the space F and $\widetilde{\Phi}$ be a mapping of $\widetilde{\Omega}$ into the space G defined by $\widetilde{\Phi}(x,y) = \Phi(x) + y$ for every $(x,y) \in \widetilde{\Omega} = \Omega \times F$. We set

$$\lambda \cdot (x,y) = (\lambda \cdot x, \lambda \cdot y) \ .$$

Then $(\widetilde{\Omega},\widetilde{\Phi})$ is a connected pseudoconvex Riemann domain with $\mathbf{C}^*$-action over $G = E \oplus F$. Let $\widetilde{\omega}$ be the quotient space by this $\mathbf{C}^*$-action. Let $\widetilde{Q}$ be the quotient map of $\widetilde{\Omega}$ onto $\widetilde{\omega}$. We denote by $\widetilde{q}$ the quotient map $G - \{0\}$ of the projective space $\mathbf{P}(G)$. Let $\widetilde{\varphi}$ be the mapping of $\widetilde{\omega}$ into $\mathbf{P}(G)$ defined by $\widetilde{\varphi}(x) = \widetilde{q} \circ \widetilde{\Phi} \circ \widetilde{Q}^{-1}(x)$ for every $x \in \widetilde{\omega}$. Then $\widetilde{\varphi}$ is well-defined and $(\widetilde{\omega},\widetilde{\varphi})$ is a connected Riemann domain over $\mathbf{P}(G)$. Moreover the Riemann domain $(\widetilde{\Omega},\widetilde{\Phi})$ is that associated with the Riemann domain $(\widetilde{\omega},\widetilde{\varphi})$. Since $\widetilde{\Omega}$ is pseudoconvex, it follows from Proposition 4.4 that $\widetilde{\omega}$ is also pseudoconvex. By Proposition 6.5, there exist a regular cover $\mathcal{U} = (U_j)_{j=1}^{\infty}$ of $\widetilde{\omega}$ and an increasing sequence $(\alpha_j)_{j=1}^{\infty}$ of continuous seminorms on G such that $\delta_{\widetilde{\Omega}}(\cdot,\cdot) = \infty$ on $\widetilde{\Omega} \times \alpha_1^{-1}(0)$, that $\widetilde{\omega}$ is $H^\infty(\mathcal{U})$-fibre separated and that $\Delta_{\widetilde{\omega}}^{\alpha_1,\alpha_j}(\widehat{U}_{j\ H(\widetilde{\omega})}) > 0$ for every j. Riemann domains Ω and ω are identified with a closed submanifold $\Omega \times \{0\}$ of $\widetilde{\Omega}$ and with a closed submanifold $\widetilde{Q}(\Omega \times \{0\})$ of $\widetilde{\omega}$ respectively. We set

$$V_j = U_j \cap \omega \quad \text{ and } \quad \beta_j = \alpha_j | E \ .$$

Then a sequence $\mathcal{V} = (V_j)_{j=1}^{\infty}$ of open sets in ω and a sequence $(\alpha_j)_{j=1}^{\infty}$ of continuous seminorms of E satisfy the required condition.

If E is a DFN-space, by Colombeau and Mujica [5] and Nachbin [28] and Paques and Zaine [37] there exists a Hilbert norm α on E such that (Ω,Φ) is a Riemann domain over the separable pre-Hilbert space (E,α). Thus by the same way as the proof of Proposition 6.4, we can obtain a regular cover $(U_j)_{j=1}^{\infty}$ of ω satisfying the required conditions. This completes the proof. ∎

Proposition 6.7. *With the condition of Proposition 6.5,* ω *is holomorphically separated.*

Proof: Let E be a separated Fréchet space with the bounded approximation property. By an examination of the proof of Proposition 6.6, there exist a Fréchet space G with a Schauder basis and a pseudoconvex Riemann domain $(\widetilde{\omega}, \widetilde{\varphi})$ over the projective space $\mathbf{P}(G)$ such that the space E is a complement subspace of G and that ω is a closed submanifold of $\widetilde{\omega}$. Therefore we have only to prove this proposition in case E has a Schauder basis. Let (Ω, Φ) be the Riemann domain over E associated with the Riemann domain (ω, φ) over $\mathbf{P}(E)$. Let a and b be different points of ω. There exist points x and y of Ω such that $Q(x) = a$, $Q(y) = b$. We choose a Schauder basis $(e_n)_{n=1}^{\infty}$ of E such that the linear span of the set $\{e_1, e_2\}$ contains the set $\{\Phi(x), \Phi(y)\}$. By an examination of the proof of Proposition 6.4, there exist a continuous seminorm α, a complement subspace E^α of E and a holomorphic mapping $\widetilde{\sigma}_\alpha : \omega \to \omega^\alpha = \varphi^{-1}(\mathbf{P}(E^\alpha))$ such that E^α has a Schauder basis and a continuous norm, that $\{e_1, e_2\} \subset E^\alpha$ and that $\widetilde{\sigma}_\alpha = \mathrm{id}$ on ω^α. By Lemma 6.2, ω^α is $H(\omega^\alpha)$-separated. Since $\{a, b\} \subset \omega^\alpha$, there exists a function $h \in H(\omega^\alpha)$ such that $h(a) \neq h(b)$. We define a holomorphic function f on ω by $f = h \circ \widetilde{\sigma}_\alpha$. Since $f|\omega^\alpha = h$, we have $f(a) \neq f(b)$. Thus ω is $H(\omega)$-separated.

Let E be a DFN-space. Then by Colombeau and Mujica [5], Nachbin [28] and Paque and Zaine [37], there exists a Hilbert norm α of E such that (Ω, Φ) is a Riemann domain over the separable pre-Hilbert space (E, α). Thus by the same way as the proof of Lemma 6.2, we can show that ω is $H(\omega)$-separated. This completes the proof. ∎

Proof of Theorem 1: Without loss of generality we may assume that ω is connected. We remark that (Ω, Φ) can be regarded as a Riemann domain over a separable pre-Hilbert space by Colombeau and Mujica [5], Nachbin [28] and Paque and Zaine [35] if E is a DFN-space. The proof of this theorem is completed by Proposition 6.6, Proposition 6.7, Theorem 5.1, an examination of the proof of Theorem 5.1 and Proposition 4.11. ∎

7 – The indicator of entire functions of exponential type

The aim of this section is to prove Theorem 2 and Corollary 3. Let E be a locally convex space. An entire function $f \in H(E)$ is said to be *exponential type* if

$$\limsup_{\mathbf{C} \ni \zeta \to \infty} \frac{\log |f(\zeta z)|}{|\zeta|} < \infty \ . \tag{7.1}$$

for every $z \in E$. If f is an entire function of exponential type on E, by Hervé [13] there exists a continuous seminorm α on E such that

$$|f(z)| \leq |f(0)| + \exp \alpha(z) - 1 \tag{7.2}$$

for every $z \in E$. We denote by $\mathrm{EXP}(E)$ the space of all entire functions of exponential type on the space E. A function p on E is said to be *positively homogeneous of order* σ if $p(\lambda z) = \lambda^\sigma p(z)$ for every $\lambda > 0$ and every $x \in E$. For an entire function f of exponential type on E, the indicator I_f of the function f on E is plurisubharmonic and positively homogeneous of order 1.

Lemma 7.1. *Let E be a separable Fréchet space with the bounded approximation property or a DFN-space and Ω be a pseudoconvex domain of E. Let F be a finite dimensional linear subspace of E. Then the restriction mapping of $H(\Omega)$ into $H(\Omega \cap F)$ is surjective.*

Proof: If E is a DFN-space, it follows from Colombeau–Mujica [5, Lemma 4.2] that this restriction mapping is surjective. Therefore it is sufficient to show this lemma in case E is a separable Fréchet space with the bounded approximation property. Let E be a separable Fréchet space with the bounded approximation property. By Pelczynski [38] there exist a Fréchet space E_1 with a Schauder basis and a Fréchet space E_2 such that E_1 is the topological direct sum of E and E_2. Let $\{e_1, e_2, ..., e_n\}$ be a basis of F where n is the dimension of the space F. We choose a sequence $(e_j)_{j=1}^\infty$ of E_1 so that $(e_j)_{j=1}^\infty$ is a Schauder basis of E_1. There exists uniquely a sequence $(\xi_n)_{n=1}^\infty$ of a continuous linear functionals on E such that $x = \sum_{n=1}^\infty \xi_n(x)\, e_n$ for each $x \in E$. We denote by π the canonical projection of E_1 onto E. Since the restriction mapping $H(\pi^{-1}(\Omega)) \to H(\Omega)$ is surjective, we have only to show that the restriction mapping $H(\pi^{-1}(\Omega)) \to H(\Omega \cap F)$ is surjective. If E_1 has a continuous norm, it follows from Mujica [26, Lemma 2.7] that the restriction mapping is surjective. We assume that E_1 has not a continuous norm. Let x_0 be a point of $\pi^{-1}(\Omega)$. Then there exists a continuous seminorm α on E_1 such that $d^\alpha_{\pi^{-1}(\Omega)}(x_0) > 0$, $\alpha(e_j) \neq 0$ for every j with $1 \leq j \leq n$ and $\alpha(x) = \sup\{\alpha(\sum_{n=1}^m \xi_n(x)\, e_n);\ m \geq 1\}$. Since $\pi^{-1}(\Omega)$ is pseudoconvex, by Lemma 4.5 the continuous seminorm α on E_1 satisfies that $\delta_{\pi^{-1}(\Omega)} = \infty$ on $\pi^{-1}(\Omega) \times \alpha^{-1}(0)$. By Lemma 3.5, there exists a Fréchet space E_1^α with a Schauder basis and with a continuous norm such that the space E_1 is the topological direct sum of E_1^α and $\alpha^{-1}(0)$. Then $\pi^{-1}(\Omega) = \{x + y \in E_1;\ x \in \pi^{-1}(\Omega) \cap E_1^\alpha,\ y \in \alpha^{-1}(0)\}$. Since $\pi^{-1}(\Omega) \cap E_1^\alpha$ is a pseudoconvex domain of E_1^α and since F is a subspace of E_1^α, the restriction mapping $H(\pi^{-1}(\Omega) \cap E_1^\alpha) \to H(\Omega \cap F)$ is surjective. Since the restriction mapping $H(\pi^{-1}(\Omega)) \to H(\pi^{-1}(\Omega) \cap E_1^\alpha)$ is surjective, the restriction mapping $H(\pi^{-1}(\Omega)) \to H(\Omega \cap F)$ is surjective. This completes the proof. ∎

Lemma 7.2. *Let E be a separable Fréchet space with the bounded approximation property or a DFN-space. Let Ω be a pseudoconvex domain with $\mathbf{C}^*$-action of the product space $E \times \mathbf{C}$ with $0 \notin \Omega$ and $\Omega \cap (E \times \{0\}) \neq \emptyset$. We denote by q the quotient mapping of $(E \times \mathbf{C}) - \{0\}$ onto the projective space*

$\mathbf{P}(E\times\mathbf{C})$. *We set* $\omega = q(\Omega)$. *Then if the domain* ω *of* $\mathbf{P}(E\times\mathbf{C})$ *is pseudoconvex, there exists a holomorphic function* f *on* ω *such that* f *is not identically zero and* $f\circ q(x,0) = 0$ *for every* $(x,0)\in\Omega$.

Proof: There exists a finite dimensional linear subspace F of E such that $(F\times\mathbf{C})\cap\Omega \neq \emptyset$. We set $\omega_{F\times\mathbf{C}} = \omega\cap\mathbf{P}(F\times\mathbf{C})$. By Oka–Cartan theorem, there exists a holomorphic function h on $\omega_{F\times\mathbf{C}}$ such that h is not identically zero and $h\circ q(x,0) = 0$ for every $x\in F$ with $(x,0)\in\Omega$. Then a function

$$(x,\zeta)\to\frac{h\circ q(x,\zeta)}{\zeta}\qquad\Big((x,\zeta)\in(F\times\mathbf{C})\cap\Omega\Big)$$

is holomorphic in $(F\times\mathbf{C})\cap\Omega$. By Lemma 7.1, there exists a holomorphic function $\widetilde{h}$ on Ω such that $\widetilde{h}(x,\zeta) = \frac{h\circ q(x,\zeta)}{\zeta}$ for every $(x,\zeta)\in\Omega\cap(F\times\mathbf{C})$. We define a holomorphic function g on Ω by $g(x,\zeta) = \zeta\,\widetilde{h}(x,\zeta)$ for every $(x,\zeta)\in\Omega\subset E\times\mathbf{C}$. We define a $\mathbf{C}^*$-invariant holomorphic function $\widetilde{g}$ on Ω by

$$\widetilde{g}(x,\zeta) = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta}x, e^{i\theta}\zeta)\,d\theta$$

for every $(x,\zeta)\in\Omega\subset E\times\mathbf{C}$. Then we have $\widetilde{g}|\Omega\cap(F\times\mathbf{C}) = h\circ q$. We define a holomorphic function f on ω by $f(z) = \widetilde{g}\circ q^{-1}(z)$ for every $z\in\omega$. Then we have $f\circ q(x,0) = 0$ for every $(x,0)\in\Omega\subset E\times\mathbf{C}$. Since $f|\omega_{F\times\mathbf{C}} = h$, f is not identically zero. This completes the proof. ∎

We recall that the Borel transform of an entire function F of exponential type in one complex variable is given for large $|t|$ by

$$H(t) = \sum_{j=1}^{\infty} A_j\,t^{-j-1}\quad\text{ if }\ F(\tau) = \sum_{j=1}^{\infty} A_j\,\frac{\tau^j}{j!}\ .$$

The corresponding integral representation is

$$H(t) = \int_0^{\infty} F(s\,\tau)\,e^{-st\tau}\,\tau\,ds\ ,$$

where $\tau\in\mathbf{C}$ has to chosen suitable for every t. It follows from this formula that H can be holomorphically continued into the complement of the convex compact set

$$K = \left\{t\in\mathbf{C};\ \forall\tau\in\mathbf{C},\ \limsup_{s\to\infty}\frac{1}{s}\log|F(st)|\geq \operatorname{Re} t\,\tau\right\}\ .$$

Conversely we have

$$F(\tau) = \frac{1}{2\pi i}\int_{\Gamma} H(t)\,e^{t\tau}\,dt\ ,$$

where Γ is some large circle. This integral representation of F shows immediately that for all $\epsilon > 0$ we have

$$|F(\tau)| \le C_\epsilon \exp\Bigl(\sup_{t\in L} \operatorname{Re} t\tau + \epsilon|\tau|\Bigr)$$

if H is holomorphic outside a compact convex set $L \subset \mathbf{C}$.

Let E be a locally convex space and p be a positively homogeneous plurisubharmonic function of order 1 on E with values in $[-\infty, \infty)$. Then we set

$$D_p = \bigcup_{t\in\mathbf{C}} \Bigl\{ z \in E;\ p(tz) < \operatorname{Re} t \Bigr\}\ , \tag{7.3}$$

$$\Omega_p = \bigcup_{t\in\mathbf{C}} \Bigl\{ (z,\zeta) \in E \times \mathbf{C};\ p(tz) < \operatorname{Re} t\,\zeta \Bigr\}\ , \tag{7.4}$$

$$\omega_p = q(\Omega_p)\ . \tag{7.5}$$

Then by Kiselman [18, Theorem 3.1 and 3.3] and by Proposition 2.3, we have the following Lemma 7.3.

Lemma 7.3. *The open set D_p of E is connected and pseudoconvex. The open set ω_p of the projective space $\mathbf{P}(E \times \mathbf{C})$ is connected, proper and pseudoconvex if and only if p is not identically $-\infty$. Moreover ω_p determines F uniquely: if plurisubharmonic functions p and r on E are positively homogeneous of order* 1, *we have $r \le p$ if and only if $\omega_p \subset \omega_r$.*

Proof of Theorem 2: If $p = -\infty$, we take $f = 0$. We assume that p is not identically $-\infty$. Then we consider the open sets Ω_p and ω_p defined by (7.4) and (7.5), of $(E \times \mathbf{C}) - \{0\}$ and of the projective space $\mathbf{P}(E \times \mathbf{C})$ respectively. By Lemma 7.3 the open set ω_p of the projective space $\mathbf{P}(E \times \mathbf{C})$ is pseudoconvex. Therefore by Theorem 1 there exists a non-constant holomorphic function f_1 of ω_p such that for every connected open neighbourhood V of an arbitrary point on the boundary of ω_p, each component of $V \cap \omega_p$ contains zero of f_1 of arbitrary high order. By Lemma 7.2 there exists a function $f_2 \in H(\omega_p)$ such that f_2 is not identically zero and $f_2 \circ q|\omega \cap (E \times \{0\}) = 0$. We set $f_3 = f_1\, f_2$. Then the domain ω_p is the domain of existence of a holomorphic function f_3 on ω_p. We define a holomorphic function g on D_p by $g(z) = f_3 \circ q(z, 1)$ for every $z \in D_p$. The open set D_p of the space E is connected and contains the origin 0 in E. Thus there exist continuous n-homogeneous polynomials $p_n :\ E \to \mathbf{C}$ and a balanced open neighbourhood W of 0 in E such that $W \subset D_p$ and that the expansion

$$g(z) = \sum_{n=0}^{\infty} p_n(z)$$

uniformly for $z \in W$. We define an entire function on E by

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} p_n(z)$$

for every $z \in E$. Then by Boas [2, Theorem 5.3.1] and by Hervé [13, Theorem 3.3.9] the function f is an entire function of exponential type on E. We consider a function r on E which is positively homogeneous of order 1 defined by

$$r(z) = \limsup_{s \to \infty} \frac{\log |f(sz)|}{s}$$

for every $z \in E$. We set $h_z(t) = g(z/t)/t = f_3 \circ q(z,t)/t$ for some fixed $z \in E$. Then h_z is the Borel transform of $\tau \to f(\tau\, z)$ so that

$$f(\tau\, z) = \frac{1}{2\pi i} \int_\Gamma h_z(t)\, e^{t\tau}\, dt \ .$$

In view of our choice of $f_3 \circ q$, h_z can be holomorphically continued to every point t such that $(z,t) \in \Omega_p$; in particular there is no singularity at the origin if $(z,0) \in \Omega_p$. We can therefore choose Γ in any neighbourhood of the convex set

$$\Big\{ t \in \mathbf{C};\ \forall \tau \in \mathbf{C},\ p(\tau\, z) \geq \operatorname{Re} t\, \tau \Big\}\ .$$

Thus for every $\epsilon > 0$ there exists a positive constant C_ϵ such that

$$|f(\tau z)| \leq C_\epsilon \exp\Big(p(\tau z) + \epsilon |\tau| \Big)$$

for every $\tau \in \mathbf{C}$ (z is fixed). Hence $r(z) \leq p(z)$ and since z is arbitrary, $r^* \leq p$ where we denote by r^* the upper regularized of the function r on E.

On the other hand, the integral

$$h_z(t) = \int_0^\infty f(s\,\tau\, z)\, e^{-st\tau}\, \tau\, ds$$

converges absolutely and uniformly for all (z,t) satisfying $r(\tau z) \leq \operatorname{Re} t\tau - \epsilon$. It follows that $h_z(\zeta)$ is a holomorphic function of (z,ζ) in Ω_{r^*}, in particular $\zeta h_z(\zeta) = f_3 \circ q(z,\zeta)$ can be holomorphic continued to a function in $H(\Omega_{r^*})$. Since $\Omega_{r^*} = q^{-1}(\omega_{r^*})$ and the function $(z,\zeta) \to \zeta h_z(\zeta) = f_3 \circ q(z,\zeta)$ is $\mathbf{C}^*$-invariant, the function f_3 is continued holomorphically to ω_{r^*}. Since $\omega_{r^*} \cap \omega_p \neq \emptyset$, since ω_p is the domain of existence of f_3 and since by Lemma 7.3 ω_{r^*} is connected, we have $\omega_{r^*} \subset \omega_p$. Thus by Lemma 7.3, $p \leq r^*$. This completes the proof. ∎

Let E be a locally convex space. We induce the compact open topology in the space $H(E)$ of all entire functions in E. A continuous linear functional μ on

$H(E)$ or, in other words, an element of the dual space $H(E)'$ of the space $H(E)$, is called an *analytic functional* in E. Let $\mu \in H(E)'$. The (generalized) Laplace transform $\widehat{\mu}$ of μ is defined by $\widehat{\mu}(\varphi) = \mu(e^{\varphi})$, $\varphi \in H(E)$. Then the restriction of $\widehat{\mu}$ on the dual space E' of E is the Fourier–Borel transformation of μ, which is an entire function of exponential type on E'.

Proof of Corollary 3: If E is a Fréchet nuclear space or a DFN-space, the correspondence $H(E')' \ni \mu \to \widehat{\mu} \in \mathrm{EXP}(E)$ by the Fourier–Borel transformation is bijective (cf. Colombeau [4] and Dineen [8]). Thus by Theorem 2 the proof of Corollary 3 is completed. ∎

REFERENCES

[**1**] Aurich, V. – The spectrum as envelope of holomorphy of a domain over an arbitrary product of lines, Proceedings of Infinite Dimensional Holomorphy, *Lecture Notes in Math.*, Springer, 364 (1974), 19–30.

[**2**] Boas, R.P. – *Entire Functions*, Academic Press, New York, 1954.

[**3**] Bremermann, H.J. – Über die Äquivalenz der pseudoconvexen Gebiete und der Holomorphiegebiete im Raum von n komplexen Veränderlichen, *Math. Ann.*, 128 (1954), 63–91.

[**4**] Colombeau, J.F. – *Differential Calculus and Holomorphy*, North-Holland Math. Studies, **64**, 1982.

[**5**] Colombeau, J.F. and Mujica, J. – The Levi problem in nuclear Silva spaces, *Arkiv für Math.*, 18 (1980), 117–123.

[**6**] Dineen, S. – Sheaves of holomorphic functions on infinite dimensional vector spaces, *Math. Ann.*, 202 (1973), 337–345.

[**7**] Dineen, S. – Surjective limits of locally convex spaces and their applications to infinite dimensional holomorphy, *Bull. Soc. Math. France*, 103 (1975), 441–509.

[**8**] Dineen, S. – *Complex Analysis in Locally Convex Spaces*, North-Holland Math. Studies, **57**, 1981.

[**9**] Dineen, S., Noverraz, Ph. and Schottenloher, M. – Le problème de Levi dans certains espaces vectoriels topologiques localement convexes, *Bull. Soc. Math. France*, 104 (1976), 87–97.

[**10**] Fujita, R. – Domaines sans point critique intérieur sur l'espace projectif complexe, *J. Math. Soc. Japan*, 15 (1963), 443–473.

[**11**] Gruman, L. – The Levi problem in certain infinite dimensional vector spaces, *Illinois J. Math.*, 18 (1974), 20–26.

[**12**] Gruman, L. and Kiselman, C.O. – Le problème de Levi dans les espaces de Banach à base, *C.R. Acad. Sc. Paris*, 274 (1972), 1296–1299.

[**13**] Hervé, M. – *Analyticity in Infinite Dimensional Spaces*, De Gruyter Studies in Math., **10**, 1989.

[**14**] Hervier, Y. – Sur le problème de Levi pour les espaces étalés banachiques, *C.R. Acad. Sc. Paris*, 275 (1972), 821–824.

[15] Hirschowitz, A. – Remarques sur les ouverts d'holomorphie d'un produit denombrable de droites, *Ann. Inst. Fourier,* 19 (1969), 219–229.

[16] Josefson, B. – A counterexample in the Levi problem, Proceedings on Infinite Dimensional Holomorphy, *Lecture Notes in Math.*, Springer*,* 364 (1974), 168–177.

[17] Kajiwara, J. – Les espaces projectifs complexes de dimension infinie, *Mem. Fac. Sci. Kyushu Univ.,* 30 (1976), 123–133.

[18] Kiselman, C.O. – On entire functions of exponential type and indicators of analytic functionals, *Acta Math.,* 23 (1967), 1–35.

[19] Lelong, P. – Fonctions entières (n variables) et fonctions plurisousharmoniques de type exponentiel, *C.R. Acad. Sc. Paris,* 260 (1965), 1063–1066.

[20] Lelong, P. – *Fonctions entieres et fonctionnelles analytiques*, Cours professé à Montréal, Presse de Montréal, 1968.

[21] Lelong, P. and Gruman, L. – *Entire Functions in Several Complex Variables*, Grundl. d. Math. Wiss, Springer, **282**, 1986.

[22] Martineau, A. – Indicatrices des croissances des fonctions entiéres de N-variables, *Invent. Math.,* 2 (1966), 81–86.

[23] Matos, M. – The envelope of holomorphy of Riemann domains over a countable product of complex planes, *Trans. Amer. Math. Soc.,* 167 (1972), 379–387.

[24] Matsushima, Y. and Morimoto, A. – Sur certains espaces fibrés holomorphes sur une variété de Stein, *Bull. Soc. Math. France,* 88 (1960), 137–155.

[25] Mujica, J. – Domains of holomorphy in DCF spaces, Functional Analysis, Holomorphy and Approximation Theory, *Lecture Notes in Math.*, Springer*,* 843 (1981), 500–533.

[26] Mujica, J. – Holomorphic approximation in infinite-dimensional Riemann domains, *Studia Math.,* 82 (1985), 107–134.

[27] Mujica, J. – *Complex Analysis in Banach Spaces*, North-Holland Math. Studies, **120**, 1986.

[28] Nachbin, L. – On pure uniform holomorphy in spaces of holomorphic germs, *Results in Math.,* 8 (1985), 117–122.

[29] Nachbin, L. – Some aspects and problems in holomorphy, *Extracta Math.,* 1 (1986), 57–72.

[30] Nachbin, L. – On the weighted approximation of continuously differentiable functions, *Proc. Amer. Math. Soc.,* 111 (1991), 481–485.

[31] Nishihara, M. – On a pseudoconvex domain spread over a complex projective space induced from a complex Banach space with a Schauder basis, *J. Math. Soc. Japan,* 39 (1987), 701–717.

[32] Norguet, F. – Sur les domaines d'holomorphie des fonctions uniformes de plurieurs variables complexes, *Bull. Soc. Math. France,* 82 (1954), 137–159.

[33] Noverraz, Ph. – Sur le Théorème de Cartan–Thullen–Oka en dimension infinie, *Lecture Notes in Math. Springer,* 332 (1973), 59–68.

[34] Noverraz, Ph. – *Pseudo-Convexite, Convexite Polynomiale et Domaines d'Holomorphie en Dimension Infinie*, North-Holland Math. Studies, **3**, 1973.

[35] Oka, K. – Sur les fonctions analytiques de plusieurs variables complex, VI. Domaines pseudoconvexes, *Tôhoku Math. J.,* 49 (1942), 15–52.

[36] Oka, K. – Sur les fonctions analytiques de plusieurs variables complexes, IX. Domaines finis sans point critique intérieur, *Japan J. Math.,* 23 (1953), 97–155.

[37] Paques, O.W. and Zaine, M.C.F. – Uniformly holomorphic contination in locally convex spaces, *J. Math. Anal. and Appl.,* 123 (1987), 448–454.

[38] Pelczynski, A. – Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with a basis, *Studia Math.,* 40 (1971), 149–243.

[39] Pomes, R. – Solution du problème de Levi dans les espaces de silva à base, *C.R. Acad. Sc. Paris,* 278 (1974), 707–710.

[40] Popa, N. – Sur le problème de Levi dans les espaces de Silva à base, *C.R. Acad. Sc. Paris,* 277 (1973), 211–214.

[41] Schottenloher, M. – The Levi problem for domains spread over locally convex spaces with a finite dimensional Schauder decomposition, *Ann. Inst. Fourier,* 26 (1976), 207–237.

[42] Takeuchi, A. – Domaines pseudoconvexes in finis et la metrique riemanniene dans un espace projectif, *J. Math. Soc. Japan,* 16 (1964), 159–181.

[43] Ueda, T. – Pseudoconvex domains over Grassmann manifolds, *J. Math. Kyoto Univ.,* 20 (1980), 391–394.

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