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MEASURES OF WEAK NONCOMPACTNESS IN BANACH SEQUENCE SPACES

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Abstract: Based on a criterion for weak compactness in the ℓ^p product of the sequence of Banach spaces E_i, $i = 1, 2, \ldots$, we construct a measure of weak noncompactness in this space. It is shown that this measure is regular but not equivalent to the De Blasi measure of weak noncompactness provided the spaces E_i have the Schur property. Apart from this a formula for the De Blasi measure in the sequence space $c_0(E_i)$ is also derived.

1 – Introduction

The notion of a measure of weak noncompactness was introduces by De Blasi [5] and was subsequently used in numerous branches of functional analysis and the theory of differential and integral equations (cf. [1, 2, 3, 7, 8, 11], for instance).

In order to recall this notion denote by E a Banach space with the norm $\|\ \|$ and the zero element θ. Let $B(x_0, r)$ stand for the closed ball centered at x_0 and with radius r and let $B = B(\theta, 1)$.

Next, denote by $\operatorname{Conv} X$ the closed convex hull of the set X, $X \subset E$. Moreover, let M_E denote the family of all nonempty and bounded subsets of E and W_E its subfamily consisting of all relatively weakly compact sets.

The measure of weak noncompactness of De Blasi [5] is defined in the following way:

$$\beta(X) = \inf\Big\{\varepsilon > 0\colon \text{ there exists a set } Y \in W_E \ \text{ such that } X \subset Y + \varepsilon B_E\Big\} ,$$

where $X \in M_E$. This function possesses several useful properties [5] (see also below). For example, $\beta(B_E) = 1$ whenever E is nonreflexive and $\beta(B_E) = 0$ otherwise.

There exists also an axiomatic approach in defining of measures of noncompactness [4]. Let us recollect this definition.

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Definition. A function $\mu\colon M_E \to R_+ = [0,\infty)$ is said to be a measure of weak noncompactness in E if it satisfies the following conditions:

(1) $\mu(X) = 0 \Leftrightarrow X \in W_E$;

(2) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$;

(3) $\mu(\operatorname{Conv} X) = \mu(X)$;

(4) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$;

(5) $\mu(X+Y) \le \mu(X) + \mu(Y)$;

(6) $\mu(c\,X) = |c|\,\mu(X),\ \ c \in R$.

Let us mention that in the paper [4] a measure of weak noncompactness in the above sense is called to be *regular*.

Notice that De Blasi measure β is a measure of weak noncompactness in this sense and has also some additional properties [5]. However, for any measure μ the following inequality holds [4]

$$\mu(X) \le \mu(B_E)\,\beta(X)\ . \tag{1}$$

Finally, let us recall [4] that each measure of weak noncompactness satisfies also the Cantor intersection condition.

2 – Main results

At the beginning let us establish some notation. Assume that $(E_i, \|\ \|_i)$, $i = 1, 2, \ldots$, is a given sequence of Banach spaces. Fix a number p, $1 \le p < \infty$ and consider the set of the sequences $x = (x_i)$ such that $x_i \in E_i$ for any $i = 1, 2, \ldots$ and $\sum_{i=1}^{\infty} \|x_i\|_i^p < \infty$. Denote this set by $\ell^p(E_1, E_2, \ldots)$ or shortly by $\ell^p(E_i)$. If we normed it by

$$\|x\| = \|(x_i)\| = \Big(\sum_{i=1}^{\infty} \|x_i\|_i^p\Big)^{1/p}$$

then it becomes a Banach space [10, 12].

Similarly, let $c_0(E_i)$ denote the space of all sequences $x = (x_i)$, $x_i \in E_i$, with the property $\|x_i\|_i \to 0$ as $i \to \infty$ and endowed by the norm

$$\|x\| = \|(x_i)\| = \max\Big\{\|x_i\|_i\colon\ i = 1, 2, \ldots\Big\}\ .$$

Further, let e_k denote the canonical projection of one of the spaces $\ell^p(E_i)$, $c_0(E_i)$ or $\ell^p(E_1, E_2, \ldots, E_n)$ onto the space E_k, i.e. $e_k(x_1, x_2, \ldots) = x_k$. Observe that $e_k(B_p) = e_k(B_0) = B_{E_k}$, where $B_p = B_{\ell^p(E_i)}$ and $B_0 = B_{c_0(E_i)}$.

In what follows we shall need the following theorem.

Theorem 1. *A subset X of the space $\ell^p(E_i)$, $1 < p < \infty$, is relatively weakly compact if and only if*

(a) *X is bounded;*

(b) *the set $e_k(X)$ is relatively weakly compact in E_k for any $k = 1, 2, \ldots$.*

This theorem comes from [12], where the case $E_k = E$, $k = 1, 2, \ldots$, was investigated. Reapeting step by step the reasoning from [12] we can easily obtain the proof of Theorem 1.

In order to define measures of weak noncompactness in the space $\ell^p(E_i)$ let us assume that β_i is De Blasi measure in the space E_i, $i = 1, 2, \ldots$ and let β_p denote De Blasi measure in $\ell^p(E_i)$. Further, for $X \in M_{\ell^p(E_i)}$ let us put

$$\mu(X) = \sup\Big\{\beta_n(e_n(X))\colon\, n = 1, 2, \ldots\Big\} . \tag{2}$$

Then we have the following theorem.

Theorem 2. *The function μ is a measure of weak noncompactness in the space $\ell^p(E_i)$, $1 < p < \infty$, such that $\mu(X) \le \beta_p(X)$ for any $X \in M_{\ell^p(E_i)}$.*

Proof: Notice first that when all the spaces E_i are reflexive then $\ell^p(E_i)$ is also reflexive [10], so in view of Theorem 1, we have that $\mu(X) = 0$ for any $X \in M_{\ell^p(E_i)}$.

Let us suppose that at least one of the space E_i is nonreflexive. Then taking into account the properties of the function β we can easily infer that the function μ satisfies all the conditions of our Definition (in fact, the condition (1) is a consequence of Theorem 1).

Finally, let us notice that $\beta_k(e_k(B_k)) = \beta_k(B_{E_k}) = 1$ at least for one natural number k. Thus we deduce that $\mu(B_p) = 1$ and by (1) we obtain that $\mu(X) \le \beta_p(X)$. This complete the proof. ∎

In the sequel we are going to show that the measure of weak noncompactness defined by (2) has not to be equivalent to De Blasi measure β_p.

First, let us recall that a Banach space E is said to have *Schur property* if weakly convergent sequences in E are norm convergent. For example, the classical space ℓ^1 has this property [6].

In what follows we shall need the following two lemmas.

Lemma 1. *Let E be a Banach space having Schur property. Then a set $X \subset E$ is weakly compact if and only if X is compact.*

Lemma 2. *Let $E_1, E_2, \ldots, E_n$ be Banach spaces with Schur property. Then the space $\ell^p(E_1, E_2, \ldots, E_n)$ has also Schur property for $1 \le p < \infty$.*

We omit trivial proofs of the lemmas.

Starting from now on let us assume that $(E_i, \| \|_i)$ is a sequence of Banach spaces being nonreflexive and such that every space E_i has Schur property. Then we have the following theorem.

Theorem 3. *Under the above assumptions the measure of weak noncompactness μ in the space $\ell^p(E_i)$ defined by (2) is not equivalent to De Blasi measure β_p $(1 < p < \infty)$.*

Proof: Suppose the contrary. Then there exists a constant $c > 0$ such that

$$c\,\beta_p(X) \le \mu(X) \tag{3}$$

for any $X \in M_{\ell^p(E_i)}$.

Now, consider the sequence (X_n) of subsets of $\ell^p(E_i)$ having the form

$$X_n = \Big\{x = (x_1, x_2, \ldots, x_n, \theta, \theta, \ldots)\colon\, x_1 \in B_{E_1},\ \ldots,\ x_n \in B_{E_n}\Big\}\ ,$$

for $n = 1, 2, \ldots$. Obviously we can write

$$X_n = B_{E_1} \times B_{E_2} \times \ldots \times B_{E_n} \times \{\theta\} \times \{\theta\} \times \ldots$$

which implies that we can treat $X_n \subset \ell^p(E_1, E_2, \ldots, E_n)$. Particularly we have that $e_i(X_n) = B_{E_i}$ $(i = 1, 2, \ldots, n)$ and consequently

$$\mu(X_n) = 1$$

for $n = 1, 2, \ldots$. Thus, in virtue of (3) we get

$$\beta_p(X_n) \le 1/c \tag{4}$$

for $n = 1, 2, \ldots$.

Further, let us choose an integer n such that $n^{1/p} - (2/c) > 0$ and take $\varepsilon > 0$ such that $n^{1/p} - 2\left(\frac{1}{c} + \varepsilon\right) > 0$. By (4) we can find a relatively weakly compact set W_n in the space $\ell^p(E_i)$ such that

$$X_n \subset W_n + \left(\frac{1}{c} + \varepsilon\right) B_{\ell^p(E_i)}\ .$$

In view of the remark made before, instead of the above inclusion we may write

$$X_n \subset W_n + \left(\frac{1}{c} + \varepsilon\right) B_{\ell^p(E_1, E_2, \ldots, E_n)}\ , \tag{5}$$

where W_n is treated as a relatively weakly compact set in the space $\ell^p(E_1, E_2, \ldots, E_n)$.

Now, fix arbitrarily i, $1 \le i \le n$. In view of generalized version of Riesz lemma [9] we can select a sequence $(x_k^i) \subset B_{E_i}$ such that

$$\|x_k^i - x_m^i\| > 1 \tag{6}$$

for $k \neq m$, $k, m = 1, 2, \ldots$ and for every $i = 1, 2, \ldots, n$.

Next, consider the sequence $(y_k)_{k \in \mathbb{N}}$ of points from X_n of the form

$$y_n = (x_k^1, x_k^2, \ldots, x_k^n, \theta, \theta, \ldots) \ ,$$

$k = 1, 2, \ldots$. Taking $k \neq m$ and keeping in mind (6) we derive

$$\|y_k - y_m\| = \|y_k - y_m\|_{\ell^p(E_1, E_2, \ldots, E_n)} = \Big(\sum_{i=1}^{n} \|x_k^i - x_m^i\|_i^p\Big)^{1/p} > \ n^{1/p} \ .$$

On the other hand in view of (5) we can find $w_k \in W_k$ and $z_k \in B_{\ell^p(E_1, E_2, \ldots, E_n)}$ (for any $k = 1, 2, \ldots$) such that

$$y_k = w_k + \Big(\frac{1}{c} + \varepsilon\Big) z_k \ .$$

Hence, taking $k \neq m$ we obtain

$$\begin{aligned}
\|w_k - w_m\|_{\ell^p(E_1, \ldots, E_n)} &= \Big\|(y_k - y_m) - \Big(\frac{1}{c} + \varepsilon\Big)(z_k - z_m)\Big\|_{\ell^p(E_1, \ldots, E_n)} \\
&\ge \|y_k - y_m\|_{\ell^p(E_1, \ldots, E_n)} - \Big(\frac{1}{c} + \varepsilon\Big)\|z_k - z_m\|_{\ell^p(E_1, \ldots, E_n)} \\
&> n^{1/p} - \Big(\frac{1}{c} + \varepsilon\Big)\|z_k - z_m)\|_{\ell^p(E_1, \ldots, E_n)} \ .
\end{aligned}$$

Consequently

$$\|w_k - w_m\|_{\ell^p(E_1, \ldots, E_n)} > n^{1/p} - 2\Big(\frac{1}{c} + \varepsilon\Big) > 0$$

for $k, m = 1, 2, \ldots$, $k \neq m$.

Thus we lead to a contradiction because in view of Lemmas 1 and 2 the set W_k is relatively compact in the space $\ell^p(E_1, E_2, \ldots, E_n)$. This complete the proof. ∎

In the sequel we shall deal with a measure of weak noncompactness in the space $c_0(E_i)$. Similarly as before let β_k denote De Blasi measure in E_k $(k = 1, 2, \ldots)$

and β_0 stand for this measure in the space $c_0(E_i)$. For further purposes denote by d_k the operator acting from $c_0(E_i)$ into itself defined by

$$d_k(x) = d_k(x_1, x_2, \ldots) = (\theta, \theta, \ldots, \theta, x_k, x_{k+1}, \ldots) .$$

Finally, define for $X \in M_{c_0(E_i)}$:

$$a(X) = \sup\Big\{\beta_n(e_n(X))\colon\ n = 1, 2, \ldots\Big\} ,$$

$$b(X) = \inf\Big\{\beta_0(d_n(X))\colon\ n = 1, 2, \ldots\Big\} ,$$

$$\gamma(X) = \max\Big\{a(X), b(X)\Big\} .$$

Then we have the following theorem.

Theorem 4. $\beta_0(X) = \gamma(X)$.

Proof: Let us take an arbitrary number $r > \gamma(X)$. Then there exists a positive integer n such that

$$\beta_0(d_n(X)) < r$$

which implies that we can choose a subset $W \in W_{c_0(E_i)}$ with the property

$$d_n(X) \subset W + rB_0 \ . \tag{7}$$

Without loss of generality we can assume that $W = d_n(W)$.

On the other hand $\beta_k(e_k(X)) < r$ for any $k = 1, 2, \ldots, n-1$ which allows us to deduce that there is $W_k = W_{E_k}$ such that

$$e_k(X) \subset W_k + rB_{E_k} \tag{8}$$

for $k = 1, 2, \ldots, n-1$.

Now, keeping in mind (7) and (8) we infer that

$$X \subset \Big((W_1 + rB_{E_1}) \times \cdots \times (W_{n-1} + rB_{E_{n-1}}) \times \{\theta\} \times \cdots\Big) + W + rB_0$$

and consequently

$$x \subset \Big(W_1 \times W_2 \times \cdots \times W_{n-1} \times \{\theta\} \times \cdots\Big) + W + rB_0 \ .$$

Hence, by the properties of De Blasi measure we have

$$\beta_0(X) \le r$$

which means that

$$\beta_0(X) \le \gamma(X) \ .$$

In order to show the converse inequality take $r > \beta_0(X)$. Then we can find a set $W \in W_{c_0(E_i)}$ such that $X \subset W + rB_0$. Hence we have

$$\beta_n(e_n(X)) \leq \beta_n(e_n(W)) + r\,\beta_n(e_n(B_0)) \leq r$$

for $n = 1, 2, \ldots$. Consequently

$$a(X) \leq r\,, \qquad b(X) \leq r\,,$$

which gives the desired inequality and ends the proof. ∎

Let us notice that $d_n(X) \supset d_k(X)$ for $n \leq k$ which implies that

$$b(X) = \lim_{n\to\infty} \beta_0(d_n(X))\ .$$

Finally observe that from Theorem 4 we obtain the following criterion for weak compactness in the space $c_0(E_i)$.

Corollary 1. *A subset X of the space $c_0(E_i)$ is relatively weakly compact if and only if*

- **(i)** *X is bounded,*
- **(ii)** *the set $e_k(X)$ is relatively weakly compact in E_k for any $k = 1, 2, \ldots$, and*
- **(iii)** *for any $\varepsilon > 0$ there exists a positive integer n_0 such that $\beta_0(d_n(X)) \leq \varepsilon$ for $n \geq n_0$.*

Corollary 2. *Let X be a subset of the space $c_0(E_i)$ satisfying the conditions (i), (ii) of Corollary 1 and instead of (iii) the following one*

(iv) $\lim_{n\to\infty}\Big[\sup_{x\in X}\Big[\max\{\|x_k\|_k\colon\ k \geq n\}\Big]\Big] = 0.$

Then X is relatively weakly compact.

Indeed, notice that

$$\sup_{x\in X}\Big[\max\Big\{\|x_k\|_k\colon\ k \geq n\Big\}\Big] = \|d_n(X)\|\ .$$

Thus in view of the inequality

$$\beta_0(d_n(X)) \leq \|d_n(X)\|$$

we infer that X satisfies the condition (iii) of Corollary 1.

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