PORTUGALIAE MATHEMATICA
Vol. 52 Fasc. 4 – 1995

THE LINEAR CAUCHY PROBLEM FOR A CLASS OF DIFFERENTIAL EQUATIONS WITH DISTRIBUTIONAL COEFFICIENTS

C.O.R. Sarrico

Abstract: We consider the problem $X^{(n)} = \sum_{i=1}^{n} U_i X^{(n-i)} + V$, $X^{(n-i)}(t_0) = a_i$ in dimension 1 ($X \in \mathcal{D}'$ is unknown, n is a positive integer, $V \in \mathcal{D}'$, $U_1, ..., U_n \in C^\infty \oplus \mathcal{D}'^p_m$, $\mathcal{D}'^p_m = \mathcal{D}'^p \cap \mathcal{D}'_m$, $\mathcal{D}'^p$ is the space of distributions of order $\leq p$ in the sense of Schwartz, $\mathcal{D}'_m$ is the space of distributions with nowhere-dense support, $a_1, ..., a_n \in \mathbb{C}$ and $t_0 \in \mathbb{R}$).

Necessary and sufficient conditions for existence and uniqueness of this problem in $C^q \oplus \mathcal{D}'_m$ where $q = \max(n, n-1+p)$ are given and also the way of getting an explicit solution when it exists.

The solutions are considered in a generalized sense defined with the help of the distributional product we introduced in [2] and they are consistent with the usual solutions.

As an example we take $X'(t) = i\,g\,\delta'(t)\,X(t)$, $X(t_0) = 1$ for a certain $t_0 < 0$ ($i = \sqrt{-1}$, $g \in \mathbb{R}$ and δ is the Dirac measure) and we prove that in our sense, its unique solution in $C^1 \oplus \mathcal{D}'_m$ is $X(t) = 1 + i\,g\,\delta(t)$ (Colombeau [1] also considers this problem with another approach). More examples are presented.

0 – Introduction

Let $\mathcal{D}$ be the space of indefinitely differentiable complex functions on $\mathbb{R}^N$ with compact support, $\mathcal{D}'$ the space of distributions, $L(\mathcal{D})$ the continuous linear maps $\mathcal{D} \to \mathcal{D}$. The basic idea of [2] is to define products of distributions by employing the algebraic structure of $L(\mathcal{D})$, given by the composition product. First we define a product $T\phi \in \mathcal{D}'$ for $T \in \mathcal{D}'$, $\phi \in L(\mathcal{D})$, by $\langle T\phi, x\rangle = \langle T, \phi(x)\rangle$ for $x \in \mathcal{D}$.

Received: September 21, 1993; *Revised*: September 7, 1994.
AMS Subject Classification: Primary 34A30; Secondary 46F10.
Keywords: Ordinary differential equations, Products of distributions, Distributions, Generalized functions.

Second, we define an epimorphism $\widetilde{\zeta}\colon\ L(\mathcal{D}) \to \mathcal{D}'$ given by $\langle \widetilde{\zeta}(\phi), x\rangle = \int \phi(x)$. Finally given $\alpha \in \mathcal{D}$ with $\int \alpha = 1$, a projection $s_\alpha\colon\ L(\mathcal{D}) \to L(\mathcal{D})$ is defined in such a way that for $T, S \in \mathcal{D}'$, $T \underset{\alpha}{\cdot} S := T(s_\alpha\, \phi)$ does not depend on the choice of $\phi \in L(\mathcal{D})$ with $\widetilde{\zeta}(\phi) = S$. The operator s_α is given by

$$\Big[(s_\alpha \phi)(x)\Big](y) = \int \phi_t \Big[\alpha(y-t)\, x(t)\Big]\, dt, \qquad \text{for} \ \ y \in \mathbb{R}^N \ .$$

Here, ϕ_t denotes the operator ϕ when it acts on functions of $t \in \mathbb{R}^N$.

In order to maintain consistency with the classical product, we single out a subspace $\mathcal{H}_\alpha \subset L(\mathcal{D})$ such that $\zeta_\alpha = \widetilde{\zeta} \mid \mathcal{H}_\alpha\colon\ \mathcal{H}_\alpha \to C^\infty \oplus \mathcal{D}'_m$ is an isomorphism, where $\mathcal{D}'_m$ denotes the space of distributions with nowhere dense support (in [2] we denote $\mathcal{D}'_m$ by $\mathcal{D}'_n$). Then, given $\alpha \in \mathcal{D}$ with $\int \alpha = 1$, the product $T \in \mathcal{D}'$ with $S = \beta + f \in C^\infty \oplus \mathcal{D}'_m$ turns out to be

$$T \underset{\alpha}{\cdot} S = T\beta + (T * \check{\alpha})\, f \ ,$$

where $\check{\alpha} \in \mathcal{D}$ is defined by $\check{\alpha}(t) = \alpha(-t)$, and the products on the right-hand side are the classical ones.

The product on $\mathcal{D}' \times (C^\infty \oplus \mathcal{D}'_m)$ thus defined depends on α, is distributive, satisfies the Leibnitz rule, is invariant for translations and is also invariant for a group G of unimodular transformations (linear transformations $h\colon\ \mathbb{R}^N \to \mathbb{R}^N$ with $|\det h| = 1$), if α is so invariant. It is neither commutative nor associative. Commutativity may be recovered after integration if both factors are in $\mathcal{D}'_m$, if one of them has compact support and if the map $t \to -t$ belongs to G. We also give a sufficient condition for associativity.

In the following examples we take $\alpha \in \mathcal{D}$ with $\int \alpha = 1$, invariant for the group of orthogonal transformations G in $\mathbb{R}^N$ (we always do the same in non relativistic applications). Thus, if $N = 1$, α is an even function. In the following δ denotes the Dirac distribution concentrated on $0 \in \mathbb{R}^N$ and H denotes the Heaviside distribution.

Examples:

1) With $N = 1$,

$$\delta \underset{\alpha}{\cdot} \delta = \delta \cdot 0 + (\delta * \check{\alpha})\, \delta = (\delta * \alpha)\, \delta = \alpha\, \delta = \alpha(0)\, \delta \ .$$

Sometimes the product does not depend of the α-function, as examples 2 and 3 show.

2) With $N = 1$,

$$H \underset{\alpha}{\cdot} \delta = H \cdot 0 + (H \ast \check{\alpha})\,\delta = (H \ast \alpha)\,\delta = \Big[\int_0^{+\infty} \alpha(u-t)\,dt\Big]\delta$$
$$= \Big[\int_0^{+\infty} \alpha(-t)\,dt\Big]\delta = \frac{1}{2}\,\delta\ ,$$

because α is an even function. In dimension N we have $H \underset{\alpha}{\cdot} \delta = \frac{1}{2^N}\,\delta$.

3) With $N = 1$ and $\beta \in C^\infty$,

$$\delta' \underset{\alpha}{\cdot} (\beta + \delta) = \delta'\,\beta + (\delta' \ast \check{\alpha})\,\delta = \beta(0)\,\delta' - \beta'(0)\,\delta + \alpha'\,\delta =$$
$$= \beta(0)\,\delta' - \beta'(0)\,\delta + \alpha'(0)\,\delta = \beta(0)\,\delta' - \beta'(0)\,\delta\ ,$$

because $\alpha'(0) = 0$.
The consistency with the classical product can be obtained if we put the C^∞-function β in the right-hand side factor;

4) With $N = 1$, $\delta \underset{\alpha}{\cdot} \beta = \delta\,\beta + (\delta \ast \check{\alpha}) \cdot 0 = \delta\,\beta = \beta(0)\,\delta$. On the other hand,

$$\beta \underset{\alpha}{\cdot} \delta = \beta \cdot 0 + (\beta \ast \check{\alpha})\,\delta = (\beta \ast \alpha)\,\delta = (\beta \ast \alpha)(0)\,\delta\ .$$

For details, we refer the reader to [2].

Let $\mathcal{D}'^p$, $p \in \{0, 1, 2, ..., \infty\}$, be the space of distributions of order $\leq p$ in the sense of Schwartz. We can naturally extend our definition of product.

0.1 Definition. Let $T \in \mathcal{D}'^p$, $S = \beta + f \in C^p \oplus \mathcal{D}'_m$ and let G be a group of unimodular transformations of $\mathbb{R}^N$. We define the (G,α)-product $T \underset{\alpha}{\cdot} S$ by putting

$$T \underset{\alpha}{\cdot} S = T\beta + T \underset{\alpha}{\cdot} f\ ,$$

where $T\beta$ is interpreted in the classical sense.

In the following we always take as G the orthogonal group in dimension 1.
We always employ this product with $N = 1$ in problems like the following:

$$P_a^V \equiv \begin{cases} X' = UX + V, \\ X(t_0) = a\ , \end{cases}$$

where $U = \gamma + T \in C^\infty \oplus \mathcal{D}'_m$, $a \in \mathbb{C}$ and $t_0 \in \mathbb{R}$. In this problem, we know that there are sometimes distributions X such that P_a^V is satisfied with the product considered in the classical sense: such solutions will be called "classical solutions".

We also define new solutions, called "w_α-solutions", as follows. First we associate to the problem P_a^V the problem Q_a^V defined by

$$Q_a^V \equiv \begin{cases} X' = X\,\gamma + T \underset{\alpha}{\cdot} X + V, \\ X(t_0) = a\ . \end{cases}$$

We will say that $X \in \mathcal{D}'$ is a w_α-solution of P_a^V when there is an open set $\Omega \subset \mathbb{R}$, with $t_0 \in \Omega$, such that the restriction X_Ω of X to Ω is a continuous function and X satisfies Q_a^V. It is important to note that in general $X\,\gamma + T \underset{\alpha}{\cdot} X \neq U \underset{\alpha}{\cdot} X$ and $X\,\gamma + T \underset{\alpha}{\cdot} X \neq X \underset{\alpha}{\cdot} U$ (the map $P_a^V \to Q_a^V$ takes advantage of the non-commutativity of the product). Clearly all classical solutions are w_α-solutions. We shall see that P_a^V may have no classical solutions and have a w_α-solution which can be independent of α (Example 5.1). We will prove that if there is a w_α-solution of P_a^V in a certain space this solution is unique, we give conditions for the existence of a w_α-solution and a way of getting an explicit solution when it exists. We present solved problems such that

a) For any α chosen, there is a w_α-solution of P_a^V and this solution is independent of α.

b) The existence of a w_α-solution of P_a^V depends on α, but for all α for which the w_α-solution exists, the w_α-solution does not depend explicitly on α.

c) The w_α-solution of P_a^V exists for a certain set of α's and depends explicitly on α.

In the following, the n order Cauchy problem is considered.

1 – The classical solutions of the linear Cauchy problem P_a^V

Let us consider the linear Cauchy problem

$$P_a^V \equiv \begin{cases} X^{(n)} = \displaystyle\sum_{i=1}^{n} U_i X^{(n-i)} + V, \\ X^{(n-i)}(t_0) = a_i, \quad i = 1, 2, ..., n\ , \end{cases}$$

where n is a positive integer, $U_1, ..., U_n \in C^\infty \oplus \mathcal{D}'^p_m$, $\mathcal{D}'^p_m = \mathcal{D}'^p \cap \mathcal{D}'_m$, $V \in \mathcal{D}'$, $a = (a_1, ..., a_n) \in \mathbb{C}^n$ and $t_0 \in \mathbb{R}$.

If we ask for a solution $X \in \mathcal{D}'(\mathbb{R})$ which shall be a C^{n-1} function in some neighbourhood of t_0, the problem is sometimes possible if we interpret the products in classical sense, that is, products of $\mathcal{D}'^p$-distributions by C^p-functions. We call these solutions, classical solutions. Thus, we must ask for them in the space C^{n-1+p}.

2 – The w_α-solutions of the linear Cauchy problem P_a^V

Now, let us associate to the problem P_a^V the problem

$$Q_a^V \equiv \begin{cases} X^{(n)} = \sum_{i=1}^{n} \Big(X^{(n-i)}\,\gamma_i + T_i \underset{\alpha}{\cdot} X^{(n-i)} \Big) + V, \\ X^{(n-i)}(t_0) = a_i, \quad i = 1, 2, ..., n \;, \end{cases}$$

where γ_i and T_i are such that $\gamma_i + T_i = U_i \in C^\infty \oplus \mathcal{D}'^p_m$.

2.1 Definition. We say that $X \in \mathcal{D}'$ is a w_α-solution of P_a^V when there is an open set Ω of $\mathbb{R}$ containing t_0 such that the restriction X_Ω of X to Ω is a $C^{n-1}(\Omega)$-function and X is solution of Q_a^V.

It is an immediate consequence of the definitions 0.1 and 2.1 that

2.2 Proposition. *For all even functions $\alpha \in \mathcal{D}$ with $\int \alpha = 1$, if $X \in C^{n-1+p}$ is a classical solution of P_a^V then X is a w_α-solution of P_a^V.*

We shall see that P_a^V may have no classical solutions in C^{n-1+p} and have a w_α-solution in $C^{n-1+p} \oplus \mathcal{D}'_m$, which obviously is, in a generalized sense, a new solution of the problem P_a^V. In some cases, this solution does not even depend on the α-function.

3 – The uniqueness of the w_α-solution of P_a^V in $C^q \oplus \mathcal{D}'_m$ with $q = \max(n, n-1+p)$

3.1 Proposition. *If there exists a w_α-solution of P_a^V in $C^q \oplus \mathcal{D}'_m$, with $q = \max(n, n-1+p)$, then this solution is unique.*

Proof: We shall give the proof only in the case $n = 1$. The general case is similar. Note also that it is sufficient to prove that if X is a w_α-solution of P_a^V, with $a = 0$ and $V = 0$, then $X = 0$.

By assumption there is an open set Ω of $\mathbb{R}$ containing t_0 such that $X_\Omega \in C^0(\Omega)$ and $X = \beta + f \in C^q \oplus \mathcal{D}'_m$ is a solution of

$$Q_0^0 \equiv \begin{cases} X' = X\,\gamma_1 + T_1 \underset{\alpha}{\cdot} X, \\ X(t_0) = 0 \ , \end{cases}$$

with $\gamma_1 \in C^\infty$ and $T_1 \in \mathcal{D}'^p_m$. Then, $\beta' + f' = \beta\,\gamma_1 + f\,\gamma_1 + T_1\,\beta + f(\alpha * T_1)$ and $\beta(t_0) = 0$, which is equivalent to

$$\begin{cases} \beta' - \beta\,\gamma = -f' + f\,\gamma_1 + T_1\,\beta + f(\alpha * T_1), \\ \beta(t_0) = 0 \ . \end{cases}$$

Noting that $\beta' - \beta\gamma \in C^{q-1}$ and $-f' + f\,\gamma_1 + T_1\,\beta + f(\alpha * T_1) \in \mathcal{D}'_m$, we have

a) $\beta' - \beta\,\gamma = 0$;

b) $-f' + f\,\gamma_1 + T_1\,\beta + f(\alpha * T_1) = 0$;

c) $\beta(t_0) = 0$.

From a) and c) it follows that $\beta = 0$. Thus, b) is equivalent to

$$f' - f\Big[\gamma_1 + (\alpha * T_1)\Big] = 0 \ ,$$

which is a differential equation with C^∞ coefficients. We know that the solutions of this equation in $\mathcal{D}'$ are distributions corresponding to C^∞-functions and so $f = 0$ because $f \in \mathcal{D}'_m$. Finally $X = \beta + f = 0$. ∎

4 – The existence of a w_α-solution of P_a^V in $C^q \oplus \mathcal{D}'_m$

Let us consider the problem P_a^0.

4.1 Proposition. *$X = \beta_1 + f \in C^q \oplus \mathcal{D}'_m$ is a w_α-solution of P_a^0 with $q = \max\{n, n-1+p\}$ if and only if the following conditions are satisfied with $U_i = \gamma_i + T_i$*

a) *$\beta_1 \in C^q$ is the solution of the Cauchy problem*

$$\text{(4.1.1)} \qquad \begin{cases} \beta_1^{(n)} = \sum_{i=1}^n \beta_1^{(n-i)}\,\gamma_i, \\ \beta_1^{(n-i)}(t_0) = a_i, \quad i = 1, ..., n \ . \end{cases}$$

b) *$f \in \mathcal{D}'_m$ is a solution of the differential equation*

$$\text{(4.1.2)} \qquad f^{(n)} - \sum_{i=1}^n f^{(n-i)}\Big[\gamma_i + (\alpha * T_i)\Big] = \sum_{i=1}^n T_i\,\beta_1^{(n-i)} \ .$$

c) *There is an open set* Ω *containing* t_0 *and such that* $f_\Omega = 0$.

Proof: We only consider the case $n = 1$. The general case is similar. First, let us assume that $X = \beta_1 + f$ is a w_α-solution of $P^0_{a_1}$ in $C^q \oplus \mathcal{D}'_m$ with $q = \max(1, p)$. By 2.1 there is an open set Ω containing t_0 such that $X_\Omega \in C^0(\Omega)$ and X is a solution of

$$Q^0_a \begin{cases} X' = X\,\gamma_1 + T_1 \underset{\alpha}{\cdot} X, \\ X(t_0) = a_1 \ , \end{cases}$$

in $C^q \oplus \mathcal{D}'_m$, with $q = \max\{1, p\}$. Thus, as in the proof of 3.1, we have

a$'$) $\beta_1' - \beta_1\,\gamma_1 = 0$;

b$'$) $f' - f[\gamma_1 + (\alpha * T_1)] = T_1\,\beta_1$;

c$'$) $\beta_1(t_0) = a_1$.

Hence, conditions a) and b) are satisfied. Condition c) follows immediately from $X_\Omega = (\beta_1 + f)_\Omega = \beta_{1_\Omega} + f_\Omega \in C^0(\Omega)$ and $f \in \mathcal{D}'_m$.

Now suppose that a), b) and c) are satisfied. Then, $X = \beta_1 + f$ is a w_α-solution of P^0_a because

$$\begin{aligned} X' &= \beta_1' + f' = \beta_1\,\gamma_1 + f\Big[\gamma_1 + (\alpha * T_1)\Big] + T_1\,\beta_1 = \beta_1\,\gamma_1 + f\,\gamma_1 + T_1 \underset{\alpha}{\cdot} f + T_1\,\beta_1 \\ &= (\beta_1 + f)\,\gamma_1 + T_1 \underset{\alpha}{\cdot} (f + \beta_1) = X\,\gamma_1 + T_1 \underset{\alpha}{\cdot} X \end{aligned}$$

and also because $X_\Omega = (\beta_1 + f)_\Omega = \beta_{1_\Omega} + f_\Omega = \beta_{1_\Omega} \in C^0(\Omega)$ and $t_0 \in \Omega$. ∎

Sometimes, the following note can be useful when we are looking for a solution of 4.1.2.

4.2 Note. If $\beta_1 \in C^q$ is a solution of the Cauchy problem 4.1.1 and there exists $S \in \mathcal{D}'_m$ such that $S^{(n)} = \sum_{i=1}^n T_i\,\beta_1^{(n-i)}$ and $\sum_{i=1}^n S^{(n-i)}[\gamma_i + (\alpha * T_i)] = 0$ then S is a solution of 4.1.2 in $\mathcal{D}'_m$.

Finally we can verify the proposition which allows us to determine the w_α-solution of the P^V_a problem.

4.3 Proposition. *If*

I) $g \in \mathcal{D}'$ *is a particular* w_α*-solution of* $X^{(n)} = \sum_{i=1}^n U_i\,X^{(n-i)} + V$, *that is,* g *is a solution of*

$$X^{(n)} = \sum_{i=1}^{n} \Big(X^{(n-i)}\,\gamma_i + T_i \underset{\alpha}{\cdot} X^{(n-i)} \Big) + V$$

and

II) *There exists* $c = (c_1, ..., c_n)$ *such that*

a) *Y_c is a w_α-solution of*

$$P^0_c \equiv \begin{cases} X^{(n)} = \sum_{i=1}^{n} U_i\, X^{(n-i)}, \\ X^{(n-i)}(t_0) = c_i, \quad i = 1, ..., n \ ; \end{cases}$$

b) *$(Y_c + g)^{(n-i)}(t_0) = a_i$ in the sense that there exists an open set Ω of $\mathbb{R}$ such that $t_0 \in \Omega$, $(Y_c + g)_\Omega \in C^{n-1}(\Omega)$ and $(Y_c + g)^{(n-i)}_\Omega(t_0) = a_i$, $i = 1, 2, ..., n$,*

then

$$X = Y_c + g \ \text{ is the } w_\alpha\text{-solution of } P^V_a \text{ problem} .$$

5 – Examples

5.1. Let us consider the problem

$$P^0_a = Q^0_a \equiv \begin{cases} X' = i\, g\, \delta'\, X, & (5.1.1) \\ X(t_0) = a\ , & (5.1.2) \end{cases}$$

where $i = \sqrt{-1}$, δ' is the derivative of Dirac measure, $g, t_0, a \in \mathbb{R}$, $t_0 < 0$ and $g \neq 0$.

C^1 is the space of classical solutions X because $\delta' \in \mathcal{D}'^1$. P^0_a has no classical solutions unless $a = 0$. In fact, $X' \in C^0$ and $i\, g\, \delta' X \in \mathcal{D}'_m$ which implies $X' = i\, g\, \delta' X = 0$. This is possible only in the case $X = 0$ which is not compatible with 5.1.2 unless $a = 0$. Hence, if $a = 0$, P^0_a has only the solution $X = 0$ in C^1. If $a \neq 0$, P^0_a has no classical solutions. We will prove that for all $a \in \mathbb{R}$, P^0_a always has the w_α-solution $X = a(1 + ig\delta)$ in $C^1 \oplus \mathcal{D}'_m$, which does not depend of the choice of α and coincides with the classical solution $X = 0$ if $a = 0$. In fact, by applying 4.1 we have the following:

a) The Cauchy problem

$$\begin{cases} \beta_1' = 0, \\ \beta_1(t_0) = a\ , \end{cases}$$

has the unique solution $\beta_1(t) = a$.

b) By 4.2 the equation $S' = i\, g\, \delta' a$ has the solution $S = i\, g\, a\, \delta \in \mathcal{D}'_m$, and $i\, g\, a\, \delta[0 + (\alpha * i\, g\, \delta')] = 0$ for all α. Thus, $f = i\, g\, a\, \delta$ is a solution of 4.1.2 in $\mathcal{D}'_m$.

c) There is an open set Ω of $\mathbb{R}$ containing t_0 such that $f_\Omega = (i\, g\, a\, \delta)_\Omega = 0$ because $t_0 < 0$.

We conclude that $X = a + i\,g\,a\,\delta = a(1 + i\,g\,\delta)$ is a w_α-solution of P_a^0 in $C^1 \oplus \mathcal{D}'_m$. The uniqueness of this solution in $C^1 \oplus \mathcal{D}'_m$ follows by 3.1.

Colombeau [1], p. 69, asserts that the "scattering operator" can be heuristically defined from the Cauchy problem

$$\begin{cases} S'(t) = -i\,g\,H(t)\,S(t), \\ S(t_0) = I \ , \end{cases}$$

where $g \in \mathbb{R}$, $H(t)$ is the Hamiltonean interaction (distribution operator valued) and I the identity operator on the Fock space. Thus, if we denote by $S_{t_0}(t)$ the *formal solution* of this problem, the scattering operator will be defined by $S_{-\infty}(+\infty)$.

A drastic simplification which consists in taking $\mathbb{C}$ as a Fock space and $H(t) = -\delta'(t)$ leads Colombeau to consider the problem P_a^0 with $a = 1$. Thus, the scattering operator, a complex number in this case, can be computed.

$$S_{-\infty}(+\infty) = 1 \ .$$

This result is in agreement with example 2 page 75 of Colombeau [1].

Remark. Problem P_1^0 has the solution $e^{ig\delta(t)}$ in the sense of Colombeau, but this solution is not a distribution and it is not true that

$$e^{ig\delta(t)} = \sum_{n=0}^{\infty} \frac{[i\,g\,\delta(t)]^n}{n!}$$

as it is usually supposed in heuristic computations, on account of the divergence of this series in G (see [1]). If we consider the distributional product [2] this series is always convergent in $\mathcal{D}'$ and its α-sum can be computed:

$$\text{(5.1.3)} \qquad e^{ig\delta(t)} = \sum_{n=0}^{\infty} \frac{[i\,g\,\delta(t)]^n}{n!} = \begin{cases} 1 + \dfrac{e^{ig\alpha(0)} - 1}{\alpha(0)}\,\delta(t), & \text{if } \alpha(0) \neq 0, \\ 1 + i\,g\,\delta(t), & \text{if } \alpha(0) = 0 \ . \end{cases}$$

However, only in the case $\alpha(0) = 0$ does the series 5.1.3 converge to the solution of the problem P_1^0. Thus, in this case, it is possible in $\mathcal{D}'$ to make consistent the heuristic solution $e^{ig\delta(t)}$ with the solution $1 + i\,g\,\delta(t)$ and write

$$e^{ig\delta(t)} = \sum_{n=0}^{\infty} \frac{[i\,g\,\delta(t)]^n}{n!} = 1 + i\,g\,\delta(t) \ .$$

5.2. Let us consider the problem

$$P_a^V \equiv \begin{cases} X' + (1 + \delta')\,X = \sin t, \\ X(-\pi) = a \ , \end{cases}$$

with $V = \sin t$. We can prove that if $a = \frac{1}{2}(e^\pi + 1)$ this problem has only the classical solution $X(t) = \frac{1}{2}e^{-t} + \frac{1}{2}(\sin t - \cos t)$ in C^1 and has no classical solutions if $a \neq \frac{1}{2}(e^\pi + 1)$.

Now we will prove that for all $a \in \mathbb{R}$ the problem P_a^V has always one and only one w_α-solution in $C^1 \oplus \mathcal{D}'_m$, and this solution does not depend of the choice of the α-function. This solution is

$$X(t) = \left(a - \frac{1}{2}\right)e^{-(t+\pi)} + \frac{1}{2}(\sin t - \cos t) + e^{-\pi}\left[\frac{1}{2}(e^\pi + 1) - a\right]\delta(t)$$

and it coincides with the classical solution when $a = \frac{1}{2}(e^\pi + 1)$. In fact, if we consider the problem P_c^0 and the associated

$$Q_c^0 \equiv \begin{cases} X' = -X - \delta' \underset{\alpha}{\cdot} X, \\ X(-\pi) = c \ , \end{cases}$$

we have, by applying 4.1:

a) $\beta_1(t) = c\, e^{-(t+\pi)} \in C^1$ is the unique solution of the problem

$$\begin{cases} \beta_1' = -\beta_1, \\ \beta_1(-\pi) = c \ . \end{cases}$$

b) $f = -c\, e^{-\pi}\, \delta \in \mathcal{D}'_m$ is a solution of $f' - f[(-1) + \alpha * (-\delta')] = -\delta'\, c\, e^{-(t+\pi)}$ for any α chosen (now we cannot apply 4.2 because there does not exist $S' \in \mathcal{D}'_m$ such that $S' = -\delta'\, c\, e^{-(t+\pi)} = -c\, e^{-\pi}\, \delta' - c\, e^{-\pi}\, \delta$);

c) There is an open set Ω of $\mathbb{R}$ such that $-\pi \in \Omega$ and $f_\Omega = (-c\, e^{-\pi}\, \delta)_\Omega = 0$. Hence, for any α chosen, $X(t) = c\, e^{-(t+\pi)} - c\, e^{-\pi}\, \delta(t)$ is a w_α-solution of P_c^0. Also, by applying 4.3, it is easy to see that

I) $g(t) = \frac{1}{2}(\sin t - \cos t) + \frac{1}{2}\, \delta(t) \in \mathcal{D}'$ is a solution of $X' = -X - \delta' \underset{\alpha}{\cdot} X + \sin t$

and

II) There exists c such that $Y_c(t) = c\, e^{-(t+\pi)} - c\, e^{-\pi}\, \delta(t)$ is a w_α-solution of P_c^0 and $(Y_c + g)(-\pi) = a$. In fact, $Y_c(-\pi) + g(-\pi) = c + \frac{1}{2}$ and $c + \frac{1}{2} = a$ implies $c = a - \frac{1}{2}$.

Hence,

$$\begin{aligned} X(t) &= \left(a - \frac{1}{2}\right)e^{-(t+\pi)} - \left(a - \frac{1}{2}\right)e^{-\pi}\, \delta(t) + \frac{1}{2}(\sin t - \cos t) + \frac{1}{2}\, \delta(t) \\ &= \left(a - \frac{1}{2}\right)e^{-(t+\pi)} + \frac{1}{2}(\sin t - \cos t) + e^{-\pi}\left[\frac{1}{2}(e^\pi + 1) - a\right]\delta(t) \ , \end{aligned}$$

is the unique solution of P_a^V in $C^1 \oplus \mathcal{D}'_m$.

5.3. In the examples presented the w_α-solution does not depend on the α function chosen. This does not happen in general although in this example the α function does not appear explicitly in the solution.

Let us consider the problem

$$P_1^V \equiv \begin{cases} X' - \delta'\, X = \delta'', \\ X(-1) = 1 \ . \end{cases}$$

The associated problem

$$P_c^0 \equiv Q_c^0 \equiv \begin{cases} X' - \delta'\, X = 0, \\ X(-1) = c \ , \end{cases}$$

can be seen as a particular case of 5.1 with $g = -i$, $a = c$ and $t_0 = -1$ although g was real in that case. Thus, there is one and only one w_α-solution $Y_c = c(1+\delta)$ of P_c^0 in $C^1 \oplus \mathcal{D}'_m$ for any α chosen. Also $X = \delta'$ is a solution of $X' = \delta \underset{\alpha}{\cdot} X + \delta''$ for all α such that $\alpha''(0) = 0$ and we can compute c because $(Y_c + g)(-1) = 1$ and $c = 1$ follows. Hence, $X = 1 + \delta + \delta'$ is the unique w_α-solution of P_1^V in $C^1 \oplus \mathcal{D}'_m$ if we choose α such that $\alpha''(0) = 0$.

5.4. A little modification of the last example allows us to understand that the solution can depend explicitly on the α-function. It is what happens in the following problem

$$P_1^1 \equiv \begin{cases} X' - \delta'\, X = 1, \\ X(-1) = 1 \ . \end{cases}$$

It is easy to see that for each α the w_α-solution of P_1^1 in $C^1 \oplus \mathcal{D}'_m$ is

$$X(t) = \frac{1}{1 + e^{\alpha(-1)}} \Big(1 + e^{\alpha(t)} + \delta(t)\Big) \ .$$

ACKNOWLEDGEMENT – Some years ago, after reading my paper [2], Prof. Vaz Ferreira, of Bologna University, wrote me a letter where the present problem was raised. I am very grateful for his kind suggestions.

I am grateful to Prof. Michael Oberguggenberger for his beautiful and concise description of my product in [3]. I follow his treatment in the introduction.

I would also like to thank the referee for his helpful suggestions and Prof. Owen Brison for assistance with the English.

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C.O.R. Sarrico,
Centro de Matemática e Aplicações Fundamentais,
Av. Prof. Gama Pinto, 2, 1699 Lisboa Codex – PORTUGAL