PORTUGALIAE MATHEMATICA
Vol. 53 Fasc. 1 – 1996

ON THE RANKS OF CERTAIN FINITE SEMIGROUPS OF ORDER-DECREASING TRANSFORMATIONS

Abdullahi Umar*

Synopsis: Let T_n be the full transformation semigroup on a totally ordered finite set with n elements and let $K^-(n,r) = \{\alpha \in T_n:\ x\alpha \le x \text{ and } |\mathrm{Im}\,\alpha| \le r\}$, be the subsemigroup of T_n consisting of all decreasing maps α, for which $|\mathrm{Im}\,\alpha| \le r$. Similarly, let I_n be the partial one-one transformation semigroup on a totally ordered finite set with n elements and let $L^-(n,r) = \{\alpha \in I_n:\ x\alpha \le x \text{ and } |\mathrm{Im}\,\alpha| \le r\} \cup \{\emptyset\}$, be the subsemigroup of I_n consisting of all decreasing partial one-one maps α (including the empty or zero map), for which $|\mathrm{Im}\,\alpha| \le r$. If we define the rank of a finite semigroup S as the cardinal of a minimal generating set of S, then in this paper it is shown that the Rees quotient semigroups $P_r^- = K^-(n,r)/K^-(n,r-1)$ (for $n \ge 3$ and $r \ge 2$) and $Q_r^- = L^-(n,r)/L^-(n,r-1)$ (for $n \ge 2$ and $r \ge 1$) each admits a unique minimal generating set. Further, it is shown that for $1 \le r \le n-1$, $\mathrm{rank}\,P_r^- = S(n,r)$, the Stirling number of the second kind, and for $1 \le r \le n-1$

$$\mathrm{rank}\,Q_r^- = \binom{n}{r-1} \frac{\Big[(n-r)\,(r+1)+1\Big]}{1} \ .$$

0 – Introduction

The *rank* of a finite semigroup S is usually defined by

$$\mathrm{rank}\,S = \min\Big\{|A|:\ A \subseteq S,\ \langle A\rangle = S\Big\} \ .$$

If S is generated by its set E of idempotents, then the *idempotent rank* of S is defined by

$$\mathrm{idrank}\,S = \min\Big\{|A|:\ A \subseteq E,\ \langle A\rangle = S\Big\} \ .$$

Received: June 28, 1993; *Revised*: July 21, 1995.

* The results of this paper form part of the author's Ph.D. dissertation submitted to the University of St. Andrews.

The questions of the ranks, idempotent ranks and nilpotent ranks of certain finite transformation semigroups have been considered by various authors in recent years and we draw particular attention to Gomes and Howie [6–8], Howie and McFadden [10], Garba [4 & 5] and Umar [12 & 13]. The aim of this paper is to generalize the rank results obtained in [12] and [13] by analogy with [10].

1 – Finite order-decreasing full transformation semigroups

Let X_n be a finite totally ordered set, so that effectively we may identify X_n with the set $\{1, 2, ..., n\}$ of the first n natural numbers. Let T_n be the full transformation semigroup on X_n and for $n > 1$ let

$$\mathrm{Sing}_n = \Big\{\alpha \in T_n \colon\ |\operatorname{Im}\alpha| \le n-1\Big\}$$

be the subsemigroup of all singular self-maps of X_n. Let

$$S_n^- = \Big\{\alpha \in \mathrm{Sing}_n \colon\ (\text{for all}\, x \in X_n)\ x\,\alpha \le x\Big\} \tag{1.1}$$

be the subsemigroup of Sing_n consisting of all decreasing singular self-maps of X_n. For $1 \le r \le n-1$, let

$$K(n,r) = \Big\{\alpha \in T_n \colon\ |\operatorname{Im}\alpha| \le r\Big\}\ , \tag{1.2}$$

$$K^-(n,r) = \Big\{\alpha \in (S_n^-)^1 \colon\ |\operatorname{Im}\alpha| \le r\Big\} \tag{1.3}$$

be the subsemigroups of T_n and $(S_n^-)^1$ consisting of elements α, for which $|\mathrm{Im}\alpha| \le r$ respectively. It is clear that $K(n,r)$ and $K^-(n,r)$ are (two-sided) ideals of T_n and of $(S_n^-)^1$ respectively. Thus, let

$$P_r(n) = K(n,r)/K(n,r-1)\ , \tag{1.4}$$

$$P_r^-(n) = K^-(n,r)/K^-(n,r-1)\ , \tag{1.5}$$

be the Rees quotient semigroups of $K(n,r)$ and $K^-(n,r)$ respectively. As in [15] to avoid excessive use of notation in what follows we will sometimes omit r or n or both if there will be no confusion. Gomes and Howie [7] first showed that the rank and idempotent rank of $K(n, n-1) = \mathrm{Sing}_n$ are both equal to $n(n-1)/2$ a result later generalized by Howie and McFadden [10] who showed that the rank and idempotent rank of $K(n,r)$ and $P_r(n)$ are both equal to $S(n,r)$, the Stirling number of the second kind usually defined as:

$$S(n,1) = S(n,n) = 1 \quad \text{and} \quad S(n,r) = S(n-1,r-1) + r\,S(n-1,r)\ .$$

Ruškuc [11] gave an alternative proof of the result in [10].

It has been shown in [12] that the rank and idempotent rank of $K^-(n, n-1)$ are both equal to $n(n-1)/2$. In this section we are going to show that in fact, we have a similar result for the semigroups $K^-(n,r)$ and $P_r^-(n)$. However, the proof is simpler in this case perhaps because the minimal generating set (as we shall see later) turns out to be unique.

Recall from [3] that on a semigroup S the relation $\mathcal{L}^*(\mathcal{R}^*)$ is defined by the rule that $(a,b) \in \mathcal{L}^*(\mathcal{R}^*)$ if and only if the elements a, b are related by the Green's relation $\mathcal{L}(\mathcal{R})$ in some oversemigroup of S. A semigroup S is *left (right) abundant* if every $\mathcal{L}^*(\mathcal{R}^*)$-class contains an idempotent and it is *abundant* if it is both left and right abundant. The join of the equivalences $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{D}^*$ and their intersection by $\mathcal{H}^*$. To define $\mathcal{J}^*$ we first denote the $\mathcal{L}^*$-class containing the element a of the semigroup S by L_a^*. (The corresponding notation will be used for the classes of the other relations.) Then a *left (right)* $*$*-ideal* of a semigroup S is defined to be a left (right) ideal I of S such that $L_a^* \subseteq I$ ($R_a^* \subseteq I$), for all $a \in I$. A subset I of S is a $*$*-ideal* of S if it is both a left $*$-ideal and a right $*$-ideal. We also recall from [3], that the principal $*$-ideal $J^*(a)$ generated by the element a of S is the intersection of all $*$-ideals of S to which a belongs. The relation $\mathcal{J}^*$ is defined by the rule that $a \, \mathcal{J}^* \, b$ if and only if $J^*(a) = J^*(b)$, where $J^*(a)$ is the principal $*$-ideal generated by a.

We begin our investigation by noting that $K^-(n,r)$ is a $*$-ideal of $(S_n^-)^1$ and hence it is an abundant subsemiband of $(S_n^-)^1$ and for $\alpha, \beta \in K^-(n,r)$

$$\alpha \, \mathcal{L}^* \, \beta \quad \text{if and only if} \quad \operatorname{Im}\alpha = \operatorname{Im}\beta \; ,$$

$$\alpha \, \mathcal{R}^* \, \beta \quad \text{if and only if} \quad \alpha \circ \alpha^{-1} = \beta \circ \beta^{-1} \; ,$$

$$\alpha \, \mathcal{J}^* \, \beta \quad \text{if and only if} \quad |\operatorname{Im}\alpha| = |\operatorname{Im}\beta| \; .$$

Thus $K^-(n,r)$, like T_n itself, is the union of $\mathcal{J}^*$-classes

$$J_1^*, \; J_2^*, \; ..., \; J_r^* \; ,$$

where

$$J_k^* = \Big\{ \alpha \in K^-(n,r) \colon \; |\operatorname{Im}\alpha| = k \Big\} \; .$$

Moreover, $K^-(n,r)$ has $S(n,k)$ (the Stirling number of the second kind) $\mathcal{R}^*$-classes and $\binom{n-1}{k-1}$ $\mathcal{L}^*$-classes in each J_k^*. It follows immediately that P_r^- has $S(n,r)+1$ $\mathcal{R}^*$-classes and $\binom{n-1}{r-1}+1$ $\mathcal{L}^*$-classes. (The term 1 comes from the zero singleton class in each case.)

Next we recall some results and notations from [15] that will be useful in what follows: Let $f(\alpha)$ be the cardinal of

$$F(\alpha) = \Big\{x \in X_n \colon x\,\alpha = x\Big\} ,$$

the set of fixed points of the map α.

Lemma 1.1 [15, Lemma 1.1]**.** *Let $\alpha, \beta \in K^-(n,r)$ or $P_r^-(n)$. Then*

1) *$F(\alpha\beta) = F(\alpha) \cap F(\beta)$;*

2) *$F(\alpha\beta) = F(\beta\alpha)$.*

Theorem 1.2 [15, Theorem 1.3]**.** *Let P_r^- be as defined in (1.5). Then every $\alpha \in P_r^-$ is expressible as a product of idempotents in P_r^-.*

Lemma 1.3 [15, Lemma 2.6]**.** *Every $\mathcal{R}^*$-class of P_r^- contains a unique idempotent.*

Next we establish:

Lemma 1.4. *Let $\varepsilon \in E(K^-(n,r))$. Then ε is expressible as a product of idempotents in $K^-(n,r)$ whose image sets have cardinal r.*

Proof: Suppose that

$$\varepsilon = \begin{pmatrix} A_1 \ A_2 \ \dots \ A_k \\ a_1 \ a_2 \ \dots \ a_k \end{pmatrix} \in K^-(n,r) .$$

Notice that if $k = r$ the result is trivial. Essentially we can either have $|A_i| \geq 2$ and $|A_j| \geq 2$; or $|A_i| \geq 3$ for some $i, j \in \{1, ..., k\}$. In the former case we choose an element $a'_i \neq a_i$ in A_i and an element $a'_j \neq a_j$ in A_j; in the latter case we choose two distinct elements a'_i, a''_i in $A_i \backslash \{a_i\}$. Then in the former case we define

$$a'_i\, f_1 = a'_i\,, \quad x\, f_1 = x\,\varepsilon \ \ (x \neq a'_i) ,$$

$$a'_j\, f_2 = a'_j\,, \quad y\, f_2 = y\,\varepsilon \ \ (y \neq a'_j) ;$$

in the latter we define

$$a'_i\, f_1 = a'_i\,, \quad x\, f_1 = x\,\varepsilon \ \ (x \neq a'_i) ,$$

$$a''_i\, f_2 = a''_i\,, \quad y\, f_2 = y\,\varepsilon \ \ (y \neq a''_i) .$$

In both cases it is clear that f_1, f_2 are idempotents and $\varepsilon = f_1\, f_2$. Moreover, $|\mathrm{Im}\, f_1| = |\mathrm{Im}\, f_2| = k+1$. Hence the result follows by induction. ∎

Thus we deduce from Theorem 1.2 and Lemma 1.4, that the idempotent ranks of $K^-(n,r)$ and P_r^- are the same. Hence it suffices to consider P_r^- only. Now we claim that the set of all non-zero idempotents of P_r^-, i.e. $E(P_r^-\backslash\{0\})$, is the unique minimal generating set for P_r^-. However, notice that sufficiency follows from Theorem 1.2 above and it now remains to show necessity.

Lemma 1.5. *Let $\alpha,\beta \in P_r^-\backslash\{0\}$. Then the following are equivalent:*

1) $\alpha\beta \in E(P_r^-\backslash\{0\})$*;*

2) $\alpha,\beta \in E(P_r^-\backslash\{0\})$ *and* $\alpha\beta = \alpha$.

Proof: **1**)$\Rightarrow$**2**) Suppose that $\alpha\beta \in E(P_r^-\backslash\{0\})$. Then

$$r = f(\alpha\beta) \le f(\alpha) \le |\mathrm{Im}\,\alpha| = r \ ,$$

$$r = f(\alpha\beta) \le f(\beta) \le |\,\mathrm{Im}\,\beta| = r \ ,$$

which implies that

$$F(\alpha) = F(\alpha\beta) = F(\beta) \ ,$$

so that $\alpha,\beta \in E(P_r^-\backslash\{0\})$ and $\alpha\beta = \alpha$.

2)$\Rightarrow$**1**) This is clear. ∎

It is now clear that necessity follows, since the product of a non-idempotent and any other element does not give a non-zero idempotent, by Lemma 1.1. Hence the rank and idempotent rank of P_r^- are the same. Thus we now have the main results of this section:

Theorem 1.6. *Let P_r^- be as defined in (1.5). Then*

$$\mathrm{rank}\,P_r^- = \mathrm{idrank}\,P_r^- = |E(P_r^-\backslash\{0\})| = S(n,r) \ .$$

Proof: It follows from the fact that there are $S(n,r)$ $\mathcal{R}^*$-classes in $P_r^-\backslash\{0\}$ and each $\mathcal{R}^*$-class contains a unique idempotent, by Lemma 1.3. ∎

Theorem 1.7. *Let $K^-(n,r)$ be as defined in (1.3). Then*

$$\mathrm{rank}\,K^-(n,r) = \mathrm{idrank}\,K^-(n,r) = |E(J_r^*)| = S(n,r) \ .$$

Let $(P_n^-)^1$ be the semigroup of all order-decreasing partial transformations (including the empty or zero map). Then

Lemma 1.8 [15, Corollary 3.3]. *Let* $PK^-(n,r) = \{\alpha \in (P_n^-)^1\colon\ |\mathrm{Im}\,\alpha| \le r\}$ *and* $PP_r^-(n) = PK^-(n,r)/PK^-(n,r-1)$. *Then*

1) $PK^-(n,r)$ *and* $K^-(n+1,r+1)$ *are isomorphic;*

2) $PP_r^-(n)$ *and* $P_{r+1}^-(n+1)$ *are isomorphic.*

Theorem 1.9. *Let* $PK^-(n,r) = \{\alpha \in (P_n^-)^1\colon\ |\mathrm{Im}\,\alpha| \le r\}$ *and* $PP_r^-(n) = PK^-(n,r)/PK^-(n,r-1)$. *Then*

1) $\mathrm{rank}\, PK^-(n,r) = \mathrm{idrank}\, PK^-(n,r) = S(n+1,r+1)$;

2) $\mathrm{rank}\, PP_r^-(n) = \mathrm{idrank}\, PP_r^-(n) = S(n+1,r+1)$.

2 – Finite order-decreasing partial one-one transformation semigroups

Let $X_n = \{1,2,...,n\}$ and let I_n be the symmetric inverse semigroup on X_n. Now for $1 \le r \le n-1$, let

$$L(n,r) = \Big\{\alpha \in I_n\colon\ |\mathrm{Im}\,\alpha| \le r\Big\}\ , \tag{2.1}$$

$$L^-(n,r) = \Big\{\alpha \in I_n^-\colon\ |\mathrm{Im}\,\alpha| \le r\Big\}\ , \tag{2.2}$$

$$Q_r(n) = L(n,r)/L(n,r-1)\ , \tag{2.3}$$

$$Q_r^-(n) = L^-(n,r)/L^-(n,r-1)\ . \tag{2.4}$$

Notice that since

$$L^-(n,r) \subseteq PK^-(n,r) \cong K^-(n+1,r+1) \quad \text{and} \quad Q_r^-(n) \subseteq PP_r^-(n) \cong P_{r+1}^-(n+1)\ ,$$

it follows that we can deduce certain 'algebraic' results for $L^-(n,r)$ and $Q_r^-(n)$ from those for $K^-(n,r)$ and $P_r^-(n)$ respectively. In particular we have

Lemma 2.1. *Let* $\alpha,\beta \in L^-(n,r)$ *or* Q_r^-. *Then*

1) $F(\alpha\beta) = F(\alpha) \cap F(\beta)$;

2) $F(\alpha\beta) = F(\beta\alpha)$.

Gomes and Howie [7] showed that the rank (as an inverse semigroup) of $L(n,n)$ $(= I_n)$ is 3 while that of $L(n,n-1)$ is $n+1$, and later Garba [4] generalized their result by showing that the rank of $L(n,r)$ $(r \ge 3)$ is $\binom{n}{r}+1$. It has been shown in [13] that the rank and quasi-idempotent rank of $L^-(n,n-1)$ are both equal to $n(n+1)/2$ and in this section (as in the previous one) we are going to show that the semigroups $L^-(n,r)$ and $Q_r^-(n)$ admit a unique minimal generating set. (A quasi-idempotent is an element α for which $\alpha^4 = \alpha^2$, or equivalently is an element α for which α^2 is an idempotent.)

Before recalling a definition, a result and notations (from [16]) that will be useful in what follows, first let $s(\alpha)$ be the cardinal of

$$S(\alpha) = \Big\{x \in \operatorname{Dom}\alpha \colon\ x\,\alpha \neq x\Big\}\ ,$$

the set of shifting points of the map α. Then an element η in Q_r^- is called *amenable* if $s(\eta) \le 1$ and $A(\eta) \subseteq \operatorname{Dom}\eta$, where

$$\begin{aligned} A(\eta) &= \Big\{y \in X_n \colon\ (\exists\, x \in X_n)\ x\,\eta < y < x\Big\} \\ &= \Big\{y \in X_n \colon\ \ (\exists\, x \in S(\eta))\ x\,\eta < y < x\Big\}\ . \end{aligned}$$

Notice that all amenable elements are quasi-idempotents but not vice-versa.

Lemma 2.2 [16, Lemmas 1.5 and 1.6]**.** *Let α be a non-idempotent element in Q_r^-. Then α is expressible as a product of amenable elements η_i (in Q_r^-) for which $s(\eta_i) = 1$.*

It follows from Lemma 2.2 that Q_r^- is generated by $AQE(Q_r^- \backslash \{0\})$, its set of non-zero amenable elements, whose cardinal is denoted by $q(n,r)$. If we denote by *quaidrank* S the quasi-idempotent rank of S then the following is now immediate:

Corollary 2.3. $\operatorname{quaidrank} Q_r^- \le q(n,r)$.

Now we are going to show that $AQE(Q_r^- \backslash \{0\})$ is the unique minimal generating set for Q_r^-. However, first we establish:

Lemma 2.4. *For $r < n$ let $\eta \in L^-(n,r)$ be an amenable element such that $s(\eta) = 1$. Then η is expressible as a product of amenable elements $\gamma_i \in L^-(n,r)$ for which $|\operatorname{Im}\gamma_i| = r$.*

Proof: Suppose that (for some $1 \le k < r$)

$$\eta = \begin{pmatrix} a_1 \ \dots\ a_t \ \dots\ a_k \\ a_1 \ \dots\ b_t \ \dots\ a_k \end{pmatrix} \qquad (a_t > b_t)\ .$$

Let $Y_r \subseteq X_n$ such that $|Y_r| = r$, $b_t \notin Y_r$ and $\operatorname{Dom}\eta \subseteq Y_r$. Now define γ_1 by

$$x\,\gamma_1 = x\,\eta \ \ (x \in \operatorname{Dom}\eta)\,, \qquad y\,\gamma_1 = y \ \ (y \in Y_r \backslash \operatorname{Dom}\eta)\ .$$

Then clearly $|\mathrm{Im}\,\gamma_1| = r$, $s(\gamma_1) = 1$ and γ_1 is amenable. Moreover, $\eta = \mathrm{id}_{\mathrm{Dom}\,\eta}\,\gamma_1$. However, since $\mathrm{id}_{\mathrm{Dom}\,\eta}$ is expressible as a product of idempotents γ_i for which $|\mathrm{Im}\,\gamma_i| = r$, then the result follows. ∎

Thus we deduce from Lemmas 2.2 and 2.4 that the quasi-idempotent ranks of $L^-(n,r)$ and Q_r^- are the same. Hence it suffices to consider Q_r^- only. However, notice that sufficiency follows from Lemma 2.2 above and it now remains to show necessity.

Lemma 2.5. *Let $\alpha, \beta \in Q_r^- \backslash \{0\}$ such that $\alpha\beta \in Q_r^- \backslash \{0\}$. Then $\alpha\beta$ is an idempotent if and only if $\alpha = \alpha\beta = \beta$.*

Proof: $(\Rightarrow)$ Suppose that $\alpha\beta$ is idempotent. Then

$$r = f(\alpha\beta) \le f(\alpha) \le |\mathrm{Im}\,\alpha| = r \ ,$$

$$r = f(\alpha\beta) \le f(\beta) \le |\mathrm{Im}\,\beta| = r \ ,$$

which implies that

$$\mathrm{Dom}\,\alpha = \mathrm{Im}\,\alpha = F(\alpha) = F(\alpha\beta) = F(\beta) = \mathrm{Im}\,\beta = \mathrm{Dom}\,\beta$$

so that

$$\alpha = \alpha\beta = \beta \ .$$

The converse is clear. ∎

An immediate consequence of Lemma 2.5 is that any generating set for Q_r^- must contain $E(Q_r^-)$. Next we are going to show that if γ, η are two (non-idempotent) amenable elements in $Q_r^- \backslash \{0\}$ such that their product δ is in $Q_r^- \backslash \{0\}$ also, then δ is NOT amenable. Thus, again any generating set for Q_r^- must contain $AQE(Q_r^- \backslash \{0\})$ since idempotents are partial identities in this case.

Lemma 2.6. *Let $\gamma, \eta \in AQE(Q_r^- \backslash \{0\})$ such that $s(\gamma) = s(\eta) = 1$ and $\gamma\eta \in Q_r^- \backslash \{0\}$. Then $\gamma\eta$ is not amenable.*

Proof: Let $\gamma, \eta \in AQE(Q_r^- \backslash \{0\})$ such that $s(\gamma) = s(\eta) = 1$ and $\gamma\eta \in Q_r^- \backslash \{0\}$. First notice that $\mathrm{Im}\,\gamma = \mathrm{Dom}\,\eta$ and $\mathrm{Dom}\,\gamma = \mathrm{Dom}\,\gamma\eta$. Now suppose that $\mathrm{Dom}\,\gamma = W$. Let $g \in W$ be such that $g\,\gamma = h < g$. For all $x \ne g$, we have $x\,\gamma = x$. Then

$$\mathrm{Im}\,\gamma = (W \backslash \{g\}) \cup \{h\} = \mathrm{Dom}\,\eta \ ,$$

and there are two possibilities for η: i.e., $h \in S(\eta)$ or $h \notin S(\eta)$. In the former we have

$$g\,\gamma\,\eta = h\,\eta < h = g\,\gamma < g$$

and $s(\gamma\eta) = 1$. However, $h \notin \operatorname{Dom}\gamma\eta = \operatorname{Dom}\gamma$, since γ if $h \in \operatorname{Dom}\gamma$ then, as $h \neq g$, we would have $h\gamma = h$. Whence $h\gamma = g\gamma$ and so $h = g$ by the injectivity of γ. Thus $\gamma\eta$ is not amenable.

In the latter, since $s(\eta) = 1$, there exists $h' \in S(\eta)$. Now let $g' = h'\,\gamma^{-1}$. Then as $h \neq h'$ we have $g \neq g'$. Also

$$g'\,\gamma\,\eta = h'\,\eta < h' \leq g' \quad \text{and} \quad g\,\gamma\,h = h \neq g \ ,$$

so that $g, g' \in S(\gamma\eta)$. Again $\gamma\eta$ is not amenable. ∎

Thus we have established:

Theorem 2.7. *Let Q_r^- be as defined in (2.5). Then*

$$\operatorname{rank} Q_r^- = \operatorname{quaidrank} Q_r^- = q(n,r) \ .$$

Theorem 2.8. *Let $L^-(n,r)$ be as defined in (2.3). Then*

$$\operatorname{rank} L^-(n,r) = \operatorname{quaidrank} L^-(n,r) = q(n,r) \ .$$

Remark 2.9. The fact that $AQE(Q_r^- \backslash \{0\})$ is the unique minimal generating set for Q_r^- is not a coincidence since Doyen [1] has shown that every periodic $\mathcal{J}$-trivial monoid has a unique minimal generating set.

The next lemma gives an expression for $q(n,r)$, the number of amenable elements in $Q_r^- \backslash \{0\}$.

Lemma 2.10. $q(n,r) = \binom{n}{r} + \sum_{i=1}^{r} (n-i) \binom{n-i-1}{r-i}$ *($r \geq 0$).*

Proof: Clearly there are $\binom{n}{r}$ idempotents in $Q_r^- \backslash \{0\}$. And since for every (non-idempotent) amenable element η, $s(\eta) = 1$, then we may express η as

$$\eta = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad x > y \ ,$$

where $x\,\eta = y$ and $z\,\eta = z$ for all z in $\operatorname{Dom}\eta \backslash \{x\}$. Now notice that there are $(n-i)$ pairs of the type $(x, x+i)$ $(x, x+i \in X_n)$. However since $\{x+1, \ldots, x+i-1\} \subseteq \operatorname{Dom}\eta$ then there are $\binom{n-i-1}{r-i}$ ways of choosing the remaining elements of $\operatorname{Dom}\eta$. Thus the number of amenable elements in $Q_r^- \backslash \{0\}$ is

$$\binom{n}{r} + \sum_{i=1}^{r} (n-i) \binom{n-i-1}{r-i}$$

as required. ∎

However, it is possible to obtain an explicit expression for $q(n,r)$. To do this we require these two certainly known simple results:

Lemma 2.11. $\sum_{i=1}^{r}\binom{n-i}{r-i} = \binom{n}{r-1}$.

Proof: The proof is be repeated application of the Pascal's triangular identity. ∎

Lemma 2.12. $\sum_{i=1}^{r}(n-i)\binom{n-i-1}{r-i} = (n-r)\binom{n}{r-1}$.

Proof:

$$\begin{aligned}\sum_{i=1}^{r}(n-i)\binom{n-i-1}{r-i} &= \sum_{i=1}^{r}\frac{(n-i-1)!\,(n-i)}{(n-r-1)!\,(r-i)!}\\ &= \sum_{i=1}^{r}\frac{(n-i)!\,(n-r)}{(n-r)!\,(r-i)!} = (n-r)\sum_{i=1}^{r}\binom{n-i}{r-i}\\ &= (n-r)\binom{n}{r-1} \quad \text{(by Lemma 2.11)}\ .\ \blacksquare\end{aligned}$$

Hence we have this result

Theorem 2.13. $q(n,r) = \binom{n}{r-1}\frac{\left[(n-r)(r+1)+1\right]}{r}$.

Proof:

$$\begin{aligned}q(n,r) &= \sum_{i=1}^{r}(n-i)\binom{n-i-1}{r-i} + \binom{n}{r} \quad \text{(by Lemma 2.10)}\\ &= (n-r)\binom{n}{r-1} + \binom{n}{r} \quad \text{(by Lemma 2.12)}\\ &= (n-r)\binom{n}{r-1} + \frac{(n-r+1)}{r}\binom{n}{r-1}\\ &= \binom{n}{r-1}\frac{\left[(n-r)(r+1)+1\right]}{r}\ .\ \blacksquare\end{aligned}$$

We conclude with the following tables which record the first six cases of $S(n,r)$

and $q(n,r)$ respectively:

n \ r	1	2	3	4	5	6	$\sum S(n,r) = B_n$
1	1						1
2	1	1					2
3	1	3	1				5
4	1	7	6	1			15
5	1	15	25	10	1		52
6	1	31	90	65	15	1	203

n \ r	1	2	3	4	5	6	$\sum q(n,r)$
1	1						1
2	3	1					4
3	5	6	1				12
4	7	14	10	1			32
5	9	25	30	15	1		80
6	11	39	65	55	21	1	192

ACKNOWLEDGEMENT – I would like to thank my supervisor, Professor J.M. Howie, for his encouragement and invaluable suggestions. I wish also to thank the referee for many helpful comments and suggestions.
Financial support from the Federal Government of Nigeria is gratefully acknowledged.

REFERENCES

[1] Doyen, J. – Equipotence et unicité de systèmes generateurs minimaux dans certains monoides, *Semigroup Forum,* 28 (1984), 341–346.

[2] Fountain, J.B. – Adequate semigroups, *Proc. Edinburgh Math. Soc.,* 22 (1979), 113–125.

[3] Fountain, J.B. – Abundant semigroups, *Proc. London Math. Soc.,* 44(3) (1982), 103–129.

[4] Garba, G.U. – On the nilpotent rank of certain semigroups of transformations, *Glasgow Math. J.* (to appear).

[5] Garba, G.U. – Nilpotents in partial one-one order-preserving transformations, *Semigroup Forum,* 48 (1994), 37–49.

[6] Gomes, G.M.S. and Howie, J.M. – Nilpotents in finite symmetric inverse semigroup, *Proc. Edinburgh Math. Soc.,* 30 (1987), 383–395.

[7] Gomes, G.M.S. and Howie, J.M. – On the ranks of certain finite semigroups of transformations, *Math. Proc. Cambridge Phil. Soc.,* 101 (1987), 395–403.

[8] Gomes, G.M.S. and Howie, J.M. – On the ranks of certain semigroups of order-preserving transformations, *Semigroup Forum,* 45 (1992), 272–282.

[9] Howie, J.M. – *An introduction to semigroup theory*, London: Academic Press, 1976.

[10] Howie, J.M. and McFadden, R.B. – Idempotent rank in finite full transformation semigroups, *Proc. Roy. Soc. Edinburgh Sect. A,* 114 (1990), 161–167.

[11] Ruškuc, N. – On the rank of completely 0-simple semigroups, *Math. Proc. Camb. Phil. Soc.,* 116 (1994), 325–338.

[12] Umar, A. – On the semigroups of order-decreasing finite full transformations, *Proc. Roy. Soc. Edinburgh Sect. A,* 120 (1992), 129–142.

[13] Umar, A. – On the semigroups of partial one-one order-decreasing finite transformations, *Proc. Roy. Soc. Edinburgh Sect. A,* 123 (1993), 355–363.

[14] Umar, A. – *Semigroups of order-decreasing transformations*, Ph.D. Thesis, University of St. Andrews, March 1992.

[15] Umar, A. – *On certain finite semigroups of order-decreasing transformations, I* (submitted).

[16] Umar, A. – *On certain finite semigroups of order-decreasing transformations, II* (submitted).

Abdullahi Umar,
Department of Mathematics, University of Abuja,
Federal Capital Territory – NIGERIA