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SUBDIRECT PRODUCTS OF A BAND AND A SEMIGROUP*

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Abstract: Subdirect products of a band and a semigroup have been studied in various special cases by a number of authors. In the present paper, using the constructions and the methods from our earlier papers, we give characterizations of all subdirect products of a band and a semigroup.

Introduction and preliminaries

Subdirect products of a band and a semigroup have been studied in various special cases by a number of authors. A characterization of all subdirect products of a rectangular band and a semigroup was given by J.L. Chrislock and T. Tamura [3]. Subdirect products connected with sturdy bands of semigroups were investigated by the authors in [4], and in the semilattice case by M. Petrich [9, 10]. Spined products of a band and a semigroup, predominantly with respect to the greatest semilattice homomorphic image of this band, were also considered many times. More information about these can be found in [6]. A characterization of all subdirect products of a band and a semilattice of semigroups contained in their spined product were given by the authors in [6]. A band composition used in this paper, which is an extension of Petrich's construction from [9], has been also explored by the authors in [4–7].

In the present paper we consider such compositions in which all members of the related system of homomorphisms are one-to-one, and using this, by Theorem 1 we describe all subdirect products of a band and a semigroup. In Theorem 2 we give an alternative construction of such products, similar to the ones of J.L.

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Chrislock and T. Tamura [3] and H. Mitsch [8]. Theorem 3 shows that all subdirect products of a given semigroup and a band can be obtained from the subdirect product of this semigroup and of the greatest semilattice homomorphic image of this band, using spined products. In Theorem 4 we give a characterization of subdirect products of a band and a semilattice of semigroups. Finally, Section 3 is devoted to the study of subdirect products of a band and a group. The results obtained there are generalizations of some results of M. Petrich [9, 10], H. Mitsch [6] and of the authors [4].

Let B be a band. By $\leq$ we will denote the natural partial order on B, i.e. a relation on B defined by: $j \leq i \Leftrightarrow ij = ji = j$ $(i, j \in B)$, and $\preceq$ will denote a quasi-order on B defined by: $j \preceq i \Leftrightarrow j = jij$ $(i, j \in B)$. Clearly, $\leq$ and $\preceq$ coincide if and only if B is a semilattice. Further, for $i \in B$, $[i]$ will denote the class of i with respect to the smallest semilattice congruence on B. It is easy to verify that $j \preceq i \Leftrightarrow [j] \leq [i]$, for all $i, j \in B$.

Let B be a band. To each $i \in B$ we associate a semigroup S_i and an oversemigroup D_i of S_i such that $D_i \cap D_j = \emptyset$, if $i \neq j$. For $i, j \in B$, $i \succeq j$, let $\phi_{i,j}$ be a mapping of S_i into D_j and suppose that the family of $\phi_{i,j}$ satisfies the following conditions:

(**1**) $\phi_{i,i}$ is the identity mapping on S_i, for each $i \in B$;

(**2**) $(S_i\,\phi_{i,ij})\,(S_j\,\phi_{j,ij}) \subseteq S_{ij}$, for all $i, j \in B$;

(**3**) $[(a\,\phi_{i,ij})\,(b\,\phi_{j,ij})]\,\phi_{ij,k} = (a\,\phi_{i,k}\,(b\,\phi_{j,k}))$, for $a \in S_i$, $b \in S_j$, $ij \succeq k$, $i, j, k \in B$.

Define a multiplication $*$ on $S = \bigcup_{i\in B} S_i$ by: $a * b = (a\,\phi_{i,ij})\,(b\,\phi_{j,ij})$, for $a \in S_i$, $b \in S_j$. Then S is a band B of semigroups S_i, $i \in B$, in notation $S = (B; S_i, \phi_{i,j}, D_i)$ [6]. If we assume $i = j$ in (3), then we obtain that $\phi_{i,k}$ is a homomorphism, for all $i, k \in B$, $i \succeq k$. If all $\phi_{i,j}$ are one-to-one, then we write $S = \langle B; S_i, \phi_{i,j}, D_i\rangle$.

Further, if $D_i = S_i$, for each $i \in B$, then we write $S = (B; S_i, \phi_{i,j})$. Here the condition (2) can be omitted. If $S = (B; S_i, \phi_{i,j})$ and if $\{\phi_{i,j} \mid i, j \in B,\ i \succeq j\}$ is a *transitive system of homomorphisms*, i.e. if $\phi_{i,j}\,\phi_{j,k} = \phi_{i,k}$, for $i \succeq j \succeq k$, then we will write $S = [B; S_i, \phi_{i,j}]$, and we will say that S is a *strong band* B of semigroups S_i. If $S = [B; S_i, \phi_{i,j}]$ and all $\phi_{i,j}$ are one-to-one, then we will write $S = \langle B; S_i, \phi_{i,j}\rangle$ and we will say that S is a *sturdy band* B of semigroups S_i. In the case when B is a semilattice, we obtain a *strong* (*sturdy*) *semilattice* of semigroups.

For undefined notions and notations we refer to [9] and [10].

It is easy to prove the following

Lemma 1. *Let $S = (B; S_i, \phi_{i,j}, D_i)$ and let T be a subsemigroup of S. Then $B' = \{i \in B \mid S_i \cap T \neq \emptyset\}$ is a subsemigroup of B and if $T_i = T \cap S_i$, $i \in B'$, and for $i, j \in B'$, $i \succeq j$, $\psi_{i,j}$ is the restriction of $\phi_{i,j}$ onto T_i, then $T = (B'; T_i, \psi_{i,j}, D_i)$.*

2 – The main results

In this section we will give various characterizations of subdirect products of a band and a semigroup, in the general case. The following is the main theorem of this paper:

Theorem 1. *Let $S = (B; S_i, \phi_{i,j}, D_i)$ and let ξ be a relation on S defined by:*

(**4**) *$a\,\xi\,b$ if and only if $a \in S_i$, $b \in S_j$, $i, j \in B$, and there exists $k \in B$ such that $k \preceq i, j$, and $a\,\phi_{i,l} = b\,\phi_{j,l}$, for every $l \in B$, $l \preceq k$.*

Then ξ is a congruence on S. Furthermore, if $S = \langle B; S_i, \phi_{i,j}, D_i\rangle$, then S is a subdirect product of B and S/ξ.

Conversely, if a semigroup S is a subdirect product of a band B and a semigroup T, then $S = \langle B; S_i, \phi_{i,j}, D_i\rangle$, where for each $i \in B$, S_i is isomorphic to some subsemigroup of T.

Proof: Clearly, ξ is reflexive and symmetric. Assume $a, b, c \in S$ such that $a\,\xi\,b$ and $b\,\xi\,c$. Let $a \in S_i$, $b \in S_j$, $c \in S_k$, $i, j, k \in B$. Then there exists $m_1, m_2 \in B$ such that $m_1 \preceq i, j$ and $m_2 \preceq j, k$, and $a\,\phi_{i,l_1} = b\,\phi_{j,l_1}$, $b\,\phi_{j,l_2} = c\,\phi_{k,l_2}$, for all $l_1, l_2 \in B$, $l_1 \preceq m_1$ and $l_2 \preceq m_2$. Clearly, there exists $m \in B$ such that $m \preceq m_1, m_2$, and for every $l \in B$, $l \preceq m$, we obtain that $l \preceq m_1, m_2$, whence $a\,\phi_{i,l} = b\,\phi_{j,l} = c\,\phi_{k,l}$. Therefore, $a\,\xi\,c$, so ξ is transitive.

Let $a, b, c \in S$, $a\,\xi\,b$. Assume that $a \in S_i$, $b \in S_j$, $c \in S_k$, $i, j, k \in B$. Then there exists $m_0 \in B$, $m_0 \preceq i, j$, such that $a\,\phi_{i,l} = b\,\phi_{j,l}$, for every $l \in B$, $l \preceq m_0$. Assume that $m \in B$ is such that $m \preceq m_0, ik, jk$, and that $l \in B$, $l \preceq m$. Then $l \preceq m_0$, whence

$$(a * c)\,\phi_{ik,l} = (a\,\phi_{i,l})\,(c\,\phi_{k,l}) = (b\,\phi_{j,l})\,(c\,\phi_{k,l}) = (b * c)\,\phi_{jk,l}\ .$$

Thus, $a * c\,\xi\,b * c$. Similarly we prove that $c * a\,\xi\,c * b$. Hence, ξ is a congruence on S.

Let $S = \langle B; S_i, \phi_{i,j}, D_i\rangle$. Assume that $(a, b) \in \xi \cap \eta$, where η is a band congruence on S such that $S/\eta \cong B$. Then $a, b \in S_i$, for some $i \in B$, and there exists $k \in B$, $k \preceq i$, such that $a\,\phi_{i,k} = b\,\phi_{i,k}$, whence $a = b$, since $\phi_{i,k}$ is one-to-one. Therefore, $\xi \cap \eta = \varepsilon$, where ε is the equality relation. Thus, S is a subdirect product of B and S/ξ.

Conversely, let $S \subseteq T \times B$ be a subdirect product of a semigroup T and a band B. For $i \in B$, let $S_i = (T \times \{i\}) \cap S$. Clearly, $S_i \neq \emptyset$ and it is isomorphic to a subsemigroup of T, for each $i \in B$, and S is a band B of semigroups S_i, $i \in B$. Let $D_i = T \times \{i\}$, $i \in B$, and for $i, j \in B$, $i \succeq j$, let $\phi_{i,j} \colon S_i \to D_j$ be a mapping defined by:

$$(a,i)\,\phi_{i,j} = (a,j) \qquad ((a,i) \in S_i)\ .$$

Now it is easy to verify that $S = \langle B; S_i, \phi_{i,j}, D_i \rangle$. ∎

Remark. Note that if $S = [B; S_i, \phi_{i,j}]$ and ξ is a congruence on S defined as in (4), then S/ξ is the well-known *direct limit* of the family S_i, $i \in B$, carried by B.

Considering the mappings of a band B into the set $\mathcal{G}(T)$ of *all subsemigroups of a semigroup* T, satisfying some suitable conditions, we give another characterization of subdirect products of B and T, similar to the ones of J.L. Chrislock and T. Tamura [3] and H. Mitsch [8].

Theorem 2. *Let B be a band, let T be a semigroup and let $\mu \colon B \to \mathcal{G}(T)$ be a mapping satisfying the following conditions:*

i) $\bigcup_{i \in B} i\mu = T$*;*

ii) $(i\mu) \cdot (j\mu) \subseteq (ij)\,\mu$*, for all* $i, j \in B$*.*

Then $S = \{(i,a) \in B \times T \mid a \in i\mu\}$ is a subdirect product of B and T, in notation $S = (B; \mu, T)$.

Conversely, any subdirect product of B and T can be obtained in this way.

Proof: The proof is similar to the proofs of Theorem 1 [3] and Theorem 7 [8]. ∎

Let B be a band, let T be a semigroup, let $\mu : B \to \mathcal{G}(T)$ be a mapping satisfying i) of the previous theorem and let μ be *antitone*, i.e. let for all $i, j \in B$, $i \succeq j$ implies $i\mu \subseteq j\mu$. Then clearly μ satisfies ii). A semigroup S constructed by such a mapping as in the previous theorem will be denoted by $S = [B; \mu; T]$.

By Theorem 2 we obtain the following two corollaries. The first of them is in fact Proposition 1 [4], and the first part of the second corollary is the result of M. Petrich [10, p. 87–88], [9, p. 98].

Corollary 1. *If S is a sturdy band B of semigroups, then $S = [B; \mu; S/\xi]$, where ξ is a relation defined as in (4).*

Conversely, if $S = [B; \mu; T]$, then S is a sturdy band B of semigroups $S_i = i\mu$, $i \in B$.

Corollary 2. *If S is a sturdy semilattice Y of semigroups, then $S = [Y; \mu; S/\xi]$, where ξ is a relation defined as in (4).*

Conversely, if $S = [Y; \mu; T]$, where Y is a semilattice, then S is a sturdy band B of semigroups $S_i = i\mu$, $i \in B$.

If P and Q are two semigroups with a common homomorphic image Y, then the *spined product of P and Q with respect to Y* is $S = \{(a, b) \in P \times Q \mid a\varphi = b\psi\}$, where $\varphi\colon P \to Y$ and $\psi\colon Q \to Y$ are homomorphisms onto Y. If $P_\alpha = \alpha\varphi^{-1}$, $Q_\alpha = \alpha\psi^{-1}$, $\alpha \in Y$, then $S = \bigcup_{\alpha \in Y}(P_\alpha \times Q_\alpha)$. Clearly, spined products are easier for construction than other subdirect products, so it is of interest the following result that reduces the problem of construction of subdirect products of a given semigroup and a band to the problem of construction of subdirect products of this semigroup and of the greatest semilattice homomorphic image of this band.

Theorem 3. *Let B be a band, let Y be its greatest semilattice homomorphic image and let T be a semigroup. Then a semigroup S is a subdirect product of B and T if and only if it is a spined product, with respect to Y, of B and of a subdirect product of Y and T.*

Proof: Let B be a semilattice Y of rectangular bands B_α, $\alpha \in Y$.

Let $S \subseteq B \times T$ be a subdirect product of B and T. Define a mapping φ of S into $Y \times T$ by:

$$(i, a)\varphi = ([i], a) \qquad ((i, a) \in S)\ .$$

By a routine verification we obtain that φ is a homomorphism. Let us prove that $P = S\varphi$ is a subdirect product of Y and T. Indeed, for $\alpha \in Y$, $\alpha = [i]$ for some $i \in B$, and $(i, a) \in S$ for some $a \in T$; hence $(\alpha, a) = ([i], a) = (i, a)\varphi \in P$. Similarly we prove that for $a \in T$ there exists $\alpha \in Y$ such that $(\alpha, a) \in P$. Therefore, P is a subdirect product of Y and T.

For $\alpha \in Y$, let $P_\alpha = (\{\alpha\} \times T) \cap P$. Clearly, P is a semilattice Y of semigroups P_α, $\alpha \in Y$. Define a mapping ψ of S into $B \times P$ by:

$$(i, a)\psi = (i, ([i], a)) \qquad ((i, a) \in S)\ .$$

It is not hard to verify that ψ is an embedding of S into $B\times T$. Assume $(i,a)\in S$. Then $i\in B_\alpha$, for some $\alpha\in Y$, whence

$$(i,a)\,\psi=(i,([i],a))=(i,(\alpha,a))\in B_\alpha\times P_\alpha\ .$$

Thus, $S\,\psi\subseteq\bigcup_{\alpha\in Y}(B_\alpha\times P_\alpha)$. On the other hand, if $\alpha\in Y$ and $(i,(\alpha,a))\in B_\alpha\times P_\alpha$, then $i\in B_\alpha$, so

$$(i,(\alpha,a))=(i,a)\,\psi\in S\,\psi\ .$$

Therefore, $S\,\psi=\bigcup_{\alpha\in Y}(B_\alpha\times P_\alpha)$, so S is a spined product of B and P with respect to Y.

Conversely, let $S\subseteq B\times P$ be a spined product of B and P, with respect to Y, where P is a subdirect product of Y and T, i.e. let $S=\bigcup_{\alpha\in Y}(B_\alpha\times P_\alpha)$, where $P_\alpha=(\{\alpha\}\times T)\cap P$, $\alpha\in Y$. Define a mapping ϕ of S into $B\times T$ by:

$$(i,(\alpha,a))\,\phi=(i,a)\qquad((i,(\alpha,a))\in S)\ .$$

Then ϕ is an embedding of S into $B\times T$. It remains to prove that $Q=S\,\phi$ is a subdirect product of B and T. Indeed, for $i\in B$, $i\in B_\alpha$, for some $\alpha\in Y$, and there exists $a\in T$ such that $(\alpha,a)\in P$, since P is a subdirect product of Y and T, whence $(i,(\alpha,a))\in S$ and $(i,a)=(i,(\alpha,a))\,\phi\in Q$. Similarly we prove that for any $a\in T$ there exists $i\in B$ such that $(i,a)\in Q$. Therefore, Q is a subdirect product of B and T. ∎

An element of a semigroup is *π-regular* if some of its power is regular, and a semigroup is *π-regular* if each of its element is π-regular.

Corollary 3. *The following conditions on a semigroup S are equivalent:*

i) *S is π-regular and a subdirect product of a band and a semilattice of groups;*

ii) *S is regular and a subdirect product of a band and a semilattice of groups;*

iii) *S is a spined product of a band and a semilattice of groups.*

Proof: The authors in [1] proved that if a semigroup is a subdirect product of semilattices of groups, then it is a semilattice of groups if and only if it is π-regular. By this and by Theorem 3 we obtain i)⇔iii). The equivalence ii)⇔iii) was proved by M. Petrich [11]. ∎

By the well-known Tamura's result [12], any semigroup can be represented as a semilattice of semilattice indecomposable semigroups. Also, M. Petrich in

Theorem III 7.2 [9] proved that every semilattice of semigroups can be composed as $(Y;S_\alpha,\phi_{\alpha,\beta},D_\alpha)$. Therefore, every semigroup S can be represented as $S=(Y;S_\alpha,\phi_{\alpha,\beta},D_\alpha)$, where Y is a semilattice, so it is of interest to consider subdirect products of a band and a semilattice of semigroups. This we will do in the next theorem.

Let B be a band and let Y be a semilattice. Assume that P is a subdirect product of B and Y an let π and ϖ be projection homomorphisms of P onto B and Y, respectively. It is easy to verify that for $i,j\in P$, $i\preceq j$ in P if and only if $i\pi\preceq j\pi$ in B and $i\varpi\le j\varpi$ in Y. Define a quasi-order $\trianglelefteq$ on P by:

$$i \trianglelefteq j \iff i\,\pi \preceq j\,\pi \ \text{ and } \ i\,\varpi = j\,\varpi \quad (i,j\in P)\ .$$

If $S=(P;S_i,\phi_{i,j},D_i)$ and if $\phi_{i,j}$ is one-to-one for all $i,j\in P$ such that $i\trianglerighteq j$, then we will write $S=(B,Y,P;S_i,\phi_{i,j},D_i)$.

Theorem 4. *Let B be a band and let Y be a semilattice.*

Let P be a subdirect product of B and Y, let $S=(B,Y,P;S_i,\phi_{i,j},D_i)$ and define relations η and ξ on S by:

(**5**) *$a\,\eta\,b$ if and only if $a\in S_i$, $b\in S_j$, $i,j\in P$, and $i\pi=j\pi$;*

(**6**) *$a\,\xi\,b$ if and only if $a\in S_i$, $b\in S_j$, $i,j\in P$, $i\varpi=j\varpi$, and there exists $k\in P$, $k\trianglelefteq i,j$, such that $a\,\phi_{i,l}=b\,\phi_{j,l}$, for each $l\in P$, $l\preceq k$.*

Then η and ξ are congruences on S, S/η is isomorphic to B, S/ξ is a semilattice Y of semigroups, and S is a subdirect product of S/η and S/ξ.

Conversely, every subdirect product of B and a semigroup that is a semilattice Y of semigroups can be obtained in this way.

Proof: Clearly, η is a congruence on S, S/η is isomorphic to B and ξ is reflexive and symmetric.

Assume that $a,b,c\in S$ are such that $a\,\xi\,b$ and $b\,\xi\,c$. Let $a\in S_i$, $b\in S_j$, $c\in S_k$, $i,j,k\in P$, $i\varpi=j\varpi=k\varpi$. By the hypothesis, there exist $m_1,m_2\in P$ such that $m_1\trianglelefteq i,j$ and $m_2\trianglelefteq j,k$, and $a\,\phi_{i,l_1}=b\,\phi_{j,l_1}$, $b\,\phi_{j,l_2}=c\,\phi_{k,l_2}$, for all $l_1,l_2\in P$ such that $l_1\preceq m_1$, $l_2\preceq m_2$. Now for $m=m_1m_2$, $m\trianglelefteq m_1,m_2$, so for any $l\in P$, $l\preceq m$, we obtain that $a\,\phi_{i,l}=c\,\phi_{k,l}$. Therefore, $a\,\xi\,c$, so ξ is transitive.

Assume that $a,b,c\in S$ are such that $a\,\xi\,b$. Let $a\in S_i$, $b\in S_j$, $c\in S_k$, $i,j,k\in P$. By the hypothesis, $i\varpi=j\varpi$, whence $(ik)\varpi=(jk)\varpi$, since ϖ is a homomorphism. Also, there exists $m_0\in P$ such that $m_0\trianglelefteq i,j$ and $a\,\phi_{i,l}=b\,\phi_{j,l}$, for each $l\in P$, $l\preceq m_0$. Let $m=m_0k$. Then $m\trianglelefteq ik,jk$ and for any $l\in P$, $l\preceq m$ we have

$$(a*c)\,\phi_{ik,l}=(a\,\phi_{i,l})\,(c\,\phi_{k,l})=(b\,\phi_{j,l})\,(c\,\phi_{k,l})=(b*c)\,\phi_{jk,l}\ ,$$

since $l \preceq m_0$. Therefore, $a*c\,\xi\, b*c$, and similarly $c*a\,\xi\, c*b$, so ξ is a congruence on S.

Assume that $(a,b) \in \eta \cap \xi$. Then $a \in S_i$, $b \in S_j$, $i,j \in P$, and $i\varpi = j\varpi$, whence $i = j$. Also, there exists $k \in P$, $k \trianglelefteq i$, such that $a\,\phi_{i,k} = b\,\phi_{i,k}$, whence $a = b$, since $\phi_{i,k}$ is one-to-one. Therefore, $\eta \cap \xi = \varepsilon$, so S is a subdirect product of S/η and S/ξ. Clearly, S/ξ is a semilattice Y of semigroups $T_\alpha = S_\alpha\,\xi^\natural$, $\alpha \in Y$, where $S_\alpha = \bigcup_{i\in P_\alpha} S_i$ and $P_\alpha = \{i \in P \mid i\pi = \alpha\}$, $\alpha \in Y$.

Conversely, let $S \subseteq B \times T$ be a subdirect product of B and a semigroup T that is a semilattice Y of semigroups T_α, $\alpha \in Y$. Let $P = \{(i,\alpha) \in B \times Y \mid (\{i\} \times T_\alpha) \cap S \neq \emptyset\}$. It is easy to check that P is a subdirect product of B and Y. Let π and ϖ denote the projection homomorphisms of P onto B and Y, respectively, and for $i \in P$, let $S_i = (\{i\pi\} \times T_{i\varpi}) \cap S$. Clearly, S is a band P of semigroups S_i, $i \in P$. By Theorem III 7.2 [9], $T = (Y; T_\alpha, \phi_{\alpha,\beta}, D_\alpha)$. Now, for $i \in P$, let $D_i = \{i\pi\} \times D_{i\varpi}$ and for $i,j \in P$, $i \succeq j$, define a mapping $\phi_{i,j}$ of S_i into S_j by:

$$(i\pi, a)\,\phi_{i,j} = (j\pi, a\,\phi_{i\varpi,j\varpi}) \quad (a \in T_{i\varpi})\ .$$

Now it is easy to show that $S = (B, Y, P; S_i, \phi_{i,j}, D_i)$. ∎

3 – Subdirect products of a band and a group

Subdirect products of a band and a group were considered in various special cases by M. Petrich [9-11], H. Mitsch [8] and the authors [4]. In this section we will characterize such products in the general case.

Let $E(S)$ denote the set of all idempotents of a semigroup S. An element a of a semigroup S is *E-inversive* if there exists $x \in S$ such that $a\,x \in E(S)$, or equivalently, if there exists $x \in S$ such that $x = x\,a\,x$ [2]. A semigroup S is *E-inversive* if each of its elements is E-inversive. For more informations about such semigroups we refer to [2] and [8].

Lemma 2. *Let S be a subdirect product of a band B and an E-inversive semigroup T. Then S is also E-inversive.*

Proof: Let $S \subseteq B \times T$, $(i,a) \in S$. For $a \in T$ there exists $x \in T$ such that $ax \in E(T)$ and there exists $j \in B$ such that $(j,x) \in S$. Therefore, $(i,a)(j,x) = (ij, ax) \in E(S)$, so S is E-inversive. ∎

Note that if $S = (B; S_i, \phi_{i,j}, D_i)$, then $D = \bigcup_{i\in B} D_i$ need not be a semigroup. One very interesting case when the multiplication on S can be extended to a multiplication on D will be considered in the following

Theorem 5. *Let $S = (B; S_i, \phi_{i,j}, D_i)$, where D_i, $i \in B$, are cancellative semigroups and $D_k = \{a\,\phi_{i,k} \mid a \in S_i,\ i \succeq k\}$, for each $k \in B$. Then*

i) *For all $i, j \in B$, $i \succeq j$, $\phi_{i,j}$ can be extended up to a homomorphism $\varphi_{i,j}$ of D_i into D_j such that there exists a composition $D = [B; D_i, \varphi_{i,j}]$;*

ii) *If $S = \langle B; S_i, \phi_{i,j}, D_i\rangle$, then $D = \langle B; D_i, \varphi_{i,j}\rangle$;*

iii) *If S is E-inversive, then D is also E-inversive.*

Proof: **i)** Assume that $k, l \in B$ are such that $k \succeq l$. For $a \in D_k$, by the hypothesis, $a = x\,\phi_{i,k}$, for $x \in S_i$, $i \in B$, $i \succeq k$, and we define a mapping $\varphi_{i,j}$ of D_k into D_l by

$$a\,\varphi_{k,l} = x\,\phi_{i,l}\ .$$

To prove that φ is well-defined, it is necessary and sufficient to prove that for $x \in S_i$, $y \in S_j$, $i, j \succeq k \succeq l$, $x\,\phi_{i,k} = y\,\phi_{j,k}$ implies $x\,\phi_{i,l} = y\,\phi_{j,l}$. Indeed, by $x\,\phi_{i,k} = y\,\phi_{j,k}$, for arbitrary $u, v \in S_k$,

$$\begin{aligned}(u\,\phi_{k,l})\,(x\,\phi_{i,l})\,(v\,\phi_{k,l}) &= (u * x * v)\,\phi_{k,l} = \Big[u(x\,\phi_{i,k})\,v\Big]\,\phi_{k,l} = \Big[u(y\,\phi_{j,k})\,v\Big]\,\phi_{k,l}\\ &= (u * y * v)\,\phi_{k,l} = (u\,\phi_{k,l})\,(y\,\phi_{j,l})\,(v\,\phi_{k,l})\ ,\end{aligned}$$

so by the cancellativity in D_l, $x\,\phi_{i,l} = y\,\phi_{k,l}$. Hence, $\varphi_{k,l}$ is well-defined and clearly, it is an extension of $\phi_{k,l}$.

Assume that $a \in D_k$, $b \in D_l$, $a = x\,\phi_{i,k}$, $b = y\,\phi_{j,l}$, $x \in S_i$, $y \in S_j$, $i, j, k, l \in B$, $i \succeq k$, $j \succeq l$, and assume that $m \in B$, $m \preceq k, l$. Then by (3) and by the definition of mappings $\varphi_{i,j}$ we obtain

$$\begin{aligned}&\Big[(a\,\varphi_{k,kl})\,(b\,\varphi_{l,kl})\Big]\,\varphi_{kl,m} = \Big[(x\,\phi_{i,kl})\,(y\,\phi_{j,kl})\Big]\,\varphi_{kl,m} =\\ &\quad = \Big[\Big((x\,\phi_{i,ij})\,(y\,\phi_{j,ij})\Big)\,\phi_{ij,kl}\Big]\,\varphi_{kl,m} = \Big[(x * y)\,\phi_{ij,kl}\Big]\,\varphi_{kl,m} = (x * y)\,\phi_{ij,m}\\ &\quad = \Big[(x\,\phi_{i,ij})\,(y\,\phi_{j,ij})\Big]\,\phi_{ij,m} = (x\,\phi_{i,m})\,(y\,\phi_{j,m}) = (a\,\varphi_{k,m})\,(b\,\varphi_{l,m})\ .\end{aligned}$$

Therefore, there exists a composition $D = (B; D_i, \varphi_{i,j})$. Since D_i, $i \in B$, are cancellative, then $D = [B; D_i, \varphi_{i,j}]$.

ii) Let all $\phi_{i,j}$ be one-to-one. Assume that $a\,\varphi_{k,l} = b\,\varphi_{k,l}$, for $a, b \in D_k$, $k, l \in B$, $k \succeq l$. Then $a = x\,\phi_{i,k}$, $b = y\,\phi_{j,k}$, $x \in S_i$, $y \in S_j$, $i, j \in B$, $i, j \succeq k$. Let $u, v \in S_k$ be arbitrary. By $a\,\varphi_{k,l} = b\,\varphi_{k,l}$, it follows that $x\,\phi_{i,l} = y\,\phi_{j,l}$, whence

$$(u * x * v)\,\phi_{k,l} = (u\,\phi_{k,l})\,(x\,\phi_{i,l})\,(v\,\phi_{k,l}) = (u\,\phi_{k,l})\,(y\,\phi_{j,l})\,(v\,\phi_{k,l}) = (u * y * v)\,\phi_{k,l}\ .$$

Since $\phi_{k,l}$ is one-to-one, then $u * x * v = u * y * v$, whence

$$u(x\,\phi_{i,k})\,v = u * x * v = u * y * v = u(y\,\phi_{j,k})\,v\ .$$

Now, by the cancellativity in D_k, $x\,\phi_{i,k} = y\,\phi_{j,k}$, i.e. $a = b$. Therefore, $\varphi_{k,l}$ is one-to-one.

iii) Assume that $a \in D$. Then $a \in D_k$, $k \in B$, and $a = x\,\phi_{i,k}$, $x \in S_i$, $i \in B$, $i \succeq k$. Now, $x * y \in E(S)$, for some $y \in S_j$, $j \in B$, so

$$\begin{aligned} a * y &= (a\,\varphi_{k,kj})\,(u\,\varphi_{j,kj}) = (x\,\phi_{i,kj})\,(y\,\phi_{j,kj}) \\ &= \Big[(x\,\phi_{i,ij})\,(y\,\phi_{ju,ij})\Big]\,\phi_{ij,kj} = (x * y)\,\phi_{ij,kj} \in E(D)\ . \end{aligned}$$

Thus, D is also E-inversive. ∎

A semigroup containing exactly one idempotent will be called a *unipotent semigroup*, and a semigroup without idempotents will be called an *idempotent-free semigroup*. Now we go to the main theorem of this section.

Theorem 6. *The following conditions on a semigroup S are equivalent:*

i) *S is a subdirect product of a band and a group;*

ii) *S is E-inversive, $S = \langle B; S_i, \phi_{i,j}, D_i\rangle$, and for every $i \in B$, D_i is cancellative;*

iii) *S is E-inversive, $S = \langle B; S_i, \phi_{i,j}, D_i\rangle$, and for every $i \in B$, D_i is either a unipotent monoid or an idempotent-free semigroup;*

iv) *S is E-inversive and it can be embedded into a sturdy band of cancellative semigroups;*

v) *S is E-inversive and it can be embedded into a sturdy band of unipotent monoids and idempotent-free semigroups;*

vi) *S is E-inversive and it can be embedded into a spined product of a band and a sturdy semilattice of cancellative semigroups;*

vii) *S is E-inversive and it can be embedded into a spined product of a band and a sturdy semilattice of unipotent monoids and idempotent-free semigroups.*

Proof: i)⇒ii) Let $S \subseteq B \times G$ be a subdirect product of a band B and a group G. For $i \in B$, let $D_i = \{i\} \times G$, $S_i = S \cap D_i$. Clearly, $S_i \neq \emptyset$ and D_i is a cancellative semigroup, for each $i \in B$. If for $i, j \in B$, $i \succeq j$, we define a mapping $\phi_{i,j} : S_i \to D_j$ by $(i,a)\,\phi_{i,j} = (j,a)$, then it is easy to verify that $S = \langle B; S_i, \phi_{ij}, D_i\rangle$ and by Lemma 2, S is E-inversive.

ii)$\Rightarrow$**v)** Let ii) hold. Without loss of generality we can assume that $D_k = \{a\,\phi_{i,k} \mid i \in B,\ i \succeq k,\ a \in S_i\}$, for each $k \in B$. By Theorem 5, S can be embedded into $D = \langle B; D_i, \varphi_{i,j}\rangle$ and D is E-inversive.

Let $i \in B$ be such that $E(D_i) \neq \emptyset$. Assume that $a \in D_i$, $e \in E(D_i)$. Since D is E-inversive, then $x = x{*}e{*}a{*}x$, for some $x \in D$. If $x \in D_j$, $j \in B$, then clearly $i \succeq j$ and $(e * a * x)\,\varphi_{ij,j}, e\varphi_{i,j} \in E(D_j)$, since $e * a * x \in E(D_{ij})$, $e \in E(D_i)$. By the cancellativity in D_j, $|E(D_j)| = 1$, whence $e\,\varphi_{i,j} = (e * a * x)\,\varphi_{ij,j} = (e\,\varphi_{i,j})\,(a\,\varphi_{i,j})\,x$. Now, by the cancellativity in D_j, $e\,\varphi_{i,j} = (a\,\varphi_{i,j})\,x$, whence

$$\Big[(e * a)\,\varphi_{i,j}\Big]\,x = (e * a * x)\,\varphi_{ij,j} = e\,\varphi_{i,j} = (a\,\varphi_{i,j})\,x\ ,$$

and again by the cancellativity in D_j, $(e * a)\,\varphi_{i,j} = a\,\varphi_{i,j}$. Therefore, $e * a = a$, since $\varphi_{i,j}$ is one-to-one. Similarly we prove that $a{*}e = a$. Hence, D_j is a monoid. Since D_j is cancellative, then it is unipotent.

v)$\Rightarrow$**iii)** This follows immediately.

iii)$\Rightarrow$**i)** Let iii) hold. By Theorem 1, S is a subdirect product of B and a semigroup S/ξ, where ξ is a congruence defined as in (4). Clearly, $e\,\xi\,f$, for all $e,f \in E(S)$. Let $u = e\,\xi^{\natural}$, $e \in E(S)$. Assume $v \in S/\xi$. Then $v = a\,\xi^{\natural}$, for some $a{\in}S$. Since S is E-inversive, then $x = x * a * x$, for some $x{\in}S$. If $a{\in}S_i$, $x{\in}S_j$, $i,j \in B$, then $i \succeq j$, $x * a = e \in E(S_{ji})$ and $a * e \in S_{iji}$. Assume $k \in B$, $k \preceq i, iji$. Then

$$(a * e)\,\phi_{iji,k} = (a\,\phi_{i,k})\,(e\,\phi_{ji,k}) = (a\,\phi_{i,k})\ ,$$

since $e\,\phi_{ji,k}$ is the identity of D_k. Thus, $a * e\,\xi\,a$, whence $v = a\,\xi^{\natural} = (a * e)\,\xi^{\natural} = (a\,\xi^{\natural})\,(e\,\xi^{\natural}) = v\,u$, and similarly $v = u\,v$. On the other hand, $u = e\,\xi^{\natural} = (x{*}a)\,\xi^{\natural} = (x\,\xi^{\natural})\,(a\,\xi^{\natural}) = (x\,\xi^{\natural})\,v$, and similarly $u = v(x\,\xi^{\natural})$. Hence, S/ξ is a group.

ii)$\Leftrightarrow$**iv)** This follows by Theorem 5 and Lemma 1.

iv)$\Leftrightarrow$**vi)** and **v)**$\Leftrightarrow$**vii)** This follows by Theorem 3 [6]. ∎

Similarly we can prove the following

Corollary 4. *The following conditions on a semigroup S are equivalent:*

i) *$S = [B, \mu, G]$, where B is a band and G is a group;*

ii) *S is E-inversive and a sturdy band of cancellative semigroups;*

iii) *S is E-inversive and a sturdy band of unipotent monoids and idempotent-free semigroups;*

iv) *S is E-inversive and a spined product of a band and a sturdy semilattice of cancellative semigroups;*

v) *S is E-inversive and a spined product of a band and a sturdy semilattice of unipotent monoids and idempotent-free semigroups.*

Corollary 5. *[4] A semigroup S is a sturdy band of groups if and only if it is regular and a subdirect product of a band and a group.*

Corollary 5. *[9, 10] A semigroup S is a sturdy semilattice of groups if and only if it is regular and a subdirect product of a semilattice and a group.*

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