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BOUNDARIES IN INDUCTIVE LIMIT TOPOLOGICAL ALGEBRAS

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Abstract: We study several kinds of boundaries of an inductive limit topological algebra in terms of those of the factor algebras.

1 – Introduction

When considering an inductive limit topological algebra, we know that the spectrum of this algebra is homeomorphic to the projective limit of the spectra of the factor algebras [8: p. 156, Theorem 3.1]. The aim of this paper is to provide sufficient conditions, under which the aforementioned homeomorphism is "boundary preserving", in the sense that it preserves the Šilov, Bishop and Choquet boundaries. Thus, generalizing a result of G.M. Leibowitz [7: p. 214, Theorem 11], we show that, if the connecting maps between the factor algebras are onto, along with their transpose maps between the respective spectra, then the family of the Šilov boundaries constitutes a projective system of topological spaces. In the same context, the corresponding projective limit of Šilov boundaries is homeomorphic to the Šilov boundary of the inductive limit topological algebra concerned (Lemma 3.1, Theorem 3.1).

As far as the Bishop boundaries are concerned, these form a projective system of topological spaces, in the context of an inductive system of suitable Urysohn algebras (Lemma 4.2). Furthermore, the respective projective limit of them coincides, under the appropriate conditions, homeomorphically, with the Bishop boundary of the inductive limit topological algebra involved (Theorem 4.1), due

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to the fact that the latter becomes an Urysohn algebra, when the factor algebras are of this type (Lemma 4.1).

On the other hand, we observe that the family of the Choquet boundaries of the factor algebras of the inductive system at issue, as before, always forms a projective system of topological spaces (Lemma 5.1), in the corresponding (projective) limit of which the Choquet boundary of the inductive limit topological algebra involved is imbedded, within a homeomorphism. This becomes surjective under suitable conditions for the algebras concerned (Theorem 5.1).

2 – Preliminaries

In all that follows by a *topological algebra* we mean a $\boldsymbol{C}$-algebra with (non-empty) *spectrum* $\mathfrak{M}(E)$ endowed with the *Gel'fand topology*. The *Gel'fand map of* E is given by

$$\begin{aligned}\mathcal{G}\colon\ E \to \mathcal{C}(\mathfrak{M}(E))\colon\ x \mapsto \mathcal{G}(x) \equiv \widehat{x}\colon\ \mathfrak{M}(E) \to \boldsymbol{C}\\ \colon\ f \mapsto \widehat{x}(f) := f(x)\ .\end{aligned} \tag{2.1}$$

The image of $\mathcal{G}$, denoted by $E^\wedge$, is called the *Gel'fand transform algebra* of E and is topologized as a *locally m-convex algebra* by

$$E^\wedge \subseteq \mathcal{C}_c(\mathfrak{M}(E))\ , \tag{2.2}$$

where the algebra $\mathcal{C}(\mathfrak{M}(E))$ carries the topology "c" of compact convergence [8: p. 19, Example 3.1]. Now, given an inductive system of topological algebras $(E_\alpha, f_{\beta\alpha})$, with respect to a directed index set I, let

$$E = \varinjlim_{\alpha}(E_\alpha, f_{\beta\alpha}) = \bigcup_{\alpha\in I} f_\alpha(E_\alpha) \tag{2.3}$$

be the corresponding *inductive limit topological algebra*, where f_α stands for the canonical map [8: p. 113 ff]. Considering the spectra $\mathfrak{M}(E_\alpha)$ of the topological algebras involved, for any $\alpha \leq \beta$ in I, the transpose map ${}^{\mathrm{t}}f_{\beta\alpha}$ of $f_{\beta\alpha}$ preserves the respective spectra iff

$$f_{\beta\alpha}(E_\alpha) \cap (\mathrm{Ker}(u_\beta))^c \neq \emptyset\ , \tag{2.4}$$

for every $u_\beta \in \mathfrak{M}(E_\beta)$ [8: p. 155, (3.24)]. Thus, the family

$$\Big(\mathfrak{M}(E_\alpha),\ {}^{\mathrm{t}}f_{\beta\alpha}\Big) \tag{2.5}$$

constitutes a *projective system of topological spaces*.

On the other hand, the transpose maps ${}^{\mathrm{t}}f_\alpha$, $\alpha \in I$, of the canonical maps f_α, are well defined iff

$$f_\alpha(E_\alpha) \cap (\mathrm{Ker}(u))^c \neq \emptyset \ , \tag{2.6}$$

for every $u \in \mathfrak{M}(E)$ and $\alpha \in I$, and hence yield a uniquely defined map

$$j \equiv \varprojlim_\alpha {}^{\mathrm{t}}f_\alpha \colon \mathfrak{M}(E) \to \varprojlim_\alpha \mathfrak{M}(E_\alpha) \ , \tag{2.7}$$

in such a manner that

$${}^{\mathrm{t}}f_\alpha = \rho_\alpha \circ j \ , \tag{2.8}$$

for every $\alpha \in I$, where $\rho_\alpha \colon \varprojlim_\alpha \mathfrak{M}(E_\alpha) \to \mathfrak{M}(E_\alpha)$ stands for the canonical map.

The above map j, under the previous conditions (2.4) and (2.6), becomes a *homeomorphism* of the respective topological spaces [8: p. 156, Theorem 3.1].

Remark 3.1. In this context, we also note that if the topological algebras E_α, $\alpha \in I$, have identity elements, hence $E = \varinjlim_\alpha E_\alpha$ has one as well, both of the above conditions (2.4) and (2.6) are satisfied [8].

Now considering the Gel'fand maps $\mathcal{G}_\alpha$, $\alpha \in I$, of the topological algebras considered, we first have the following relation

$$\mathcal{G}(x) = \mathcal{G}_\alpha(x_\alpha) \circ {}^{\mathrm{t}}f_\alpha \, , \qquad \alpha \in I \ , \tag{2.9}$$

such that $f_\alpha(x_\alpha) = x \in E$, with $\mathcal{G} \colon E \to E^\wedge$ the Gel'fand map of E. Furthermore, *the map* ${}^{\mathrm{t}}({}^{\mathrm{t}}f_{\beta\alpha}) \equiv h_{\beta\alpha}$ *preserves the Gel'fand transform algebras*, so that the family

$$\Big(E_\alpha^\wedge \equiv \mathcal{G}_\alpha(E_\alpha), \ h_{\beta\alpha} \equiv {}^{\mathrm{t}}({}^{\mathrm{t}}f_{\beta\alpha})\Big) \tag{2.10}$$

yields an *inductive system of topological algebras* with

$$F = \varinjlim_\alpha E_\alpha^\wedge = \bigcup_{\alpha \in I} \widehat{f}_\alpha(E_\alpha^\wedge) \tag{2.11}$$

the corresponding *inductive limit topological algebra*, where $\widehat{f}_\alpha$ stands for the canonical map. Then, $h_\alpha \equiv {}^{\mathrm{t}}({}^{\mathrm{t}}f_\alpha)$ maps each $E_\alpha^\wedge$, $\alpha \in I$, into $E^\wedge \equiv \mathcal{G}(E)$, so that one obtains a *continuous algebra morphism onto*

$$h = \varinjlim_\alpha h_\alpha \colon \ F = \varinjlim_\alpha E_\alpha^\wedge \to \mathcal{G}(E) \equiv E^\wedge \ , \tag{2.12}$$

becoming $1-1$, when E is *semi-simple* [3: p. 481, Theorem 7.1].

Now, the *Šilov boundary* $\partial(E)$ *of* E is the least *boundary set of* E, that is the smallest closed subset of $\mathfrak{M}(E)$ on which every $\widehat{x}$, $x \in E$, attains its maximum absolute value [8: p. 189, Definitions 2.1 and 2.2]. Its elements, Šilov points, are characterized by the fact that, for every open neighbourhood V of f in $\mathfrak{M}(E)$, there exists $x \in E$, such that $M_{\widehat{x}} \subseteq V$, with

$$M_{\widehat{x}} = \left\{ f \in \mathfrak{M}(E) \colon |\widehat{x}(f)| = \sup_{h \in \mathfrak{M}(E)} |\widehat{x}(h)| \equiv p_{\mathfrak{M}(E)}(\widehat{x}) \right\} , \tag{2.13}$$

being a closed subset of $\mathfrak{M}(E)$, by the continuity of $\widehat{x}$, $x \in E$ [8: p. 190, (2.4), Lemma 2.1].

On the other hand, the smallest *weak boundary set of* E, that is, a non-closed subset of $\mathfrak{M}(E)$ with the property of a boundary set, is defined as the *Bishop boundary of* E, denoted by $\mathcal{B}(E)$. A criterion for its existence is that (cf. also (2.13))

$$M_{\widehat{x}} \cap G_\delta(\mathfrak{M}(E)) \neq \emptyset, \qquad \text{for every} \ \ x \in E \ , \tag{2.14}$$

in the context of a unital Urysohn algebra E with the respective Gel'fand transform algebra $E^\wedge$ σ-complete [4: Theorem 3.1]. In this respect, $G_\delta(\mathfrak{M}(E))$ is the set of G_δ points in $\mathfrak{M}(E)$, while by an *Urysohn algebra* E we mean a topological algebra E such that for every $f \in \mathfrak{M}(E)$ and open neighbourhood V of f, there exists $x \in E$ with

$$0 \leq \widehat{x} \leq 1 \,, \qquad \widehat{x}(f) = 1 \qquad \text{and} \qquad \widehat{x} = 0|_{V^c} \ . \tag{2.15}$$

By the preceding, *every Bishop point of a unital Urysohn algebra* E *is a* G_δ*-point*, while *the converse holds true if* $E^\wedge$ *is* σ*-complete* [4: Corollary 3.2]. Furthermore, in the context of Urysohn algebras, if the Bishop boundary exists, then it is dense in the spectrum of the given algebras (ibid.). The relation between Bishop and Šilov boundaries of topological algebras is that whenever the Bishop boundary exists it is dense in the Šilov boundary, since the boundary sets are the closures of the weak boundary sets [4: Corollary 3.3].

On the other hand, the *Choquet boundary* of a topological algebra E, denoted by $\mathrm{Ch}(E)$, consists of the continuous characters of E which are represented only by the respective *Dirac measures* [5: (4.1)]. Moreover, *in a unital topological algebra* E *with compact spectrum* $\mathfrak{M}(E)$ *and* $E^\wedge$ σ*-complete an* $f \in \mathfrak{M}(E)$ *is in the Choquet boundary if, and only if, for every open neighbourhood* U *of* f *there exists* $x \in E$ *with* $f \in M_{\widehat{x}} \subseteq U$ (see [2: p. 208, Theorem 7.3]).

Now, a *spectral algebra* is a topological algebra E whose spectrum $\mathfrak{M}(E)$ is a *spectral set*, in the sense that

$$Sp_E(x) = \widehat{x}(\mathfrak{M}(E)), \quad x \in E \ , \tag{2.16}$$

where $Sp_E(x)$ stands for the spectrum of $x \in E$ [2: p. 13, Definition 2.1], [8: p. 47, Definitions 1.1, 1.2]. Every unital commutative advertibly complete locally m-convex algebra is a spectral algebra [8: p. 104, Corollary 6.4].

3 – Šilov boundary of an inductive limit topological algebra

Given an inductive system of unital topological algebras $(E_\alpha, f_{\beta\alpha})$ relative to a directed index set I, with $E = \varinjlim_\alpha E_\alpha$ the corresponding inductive limit unital topological algebra, we seek conditions guaranteeing that the restrictions of the homeomorphism (2.7) to the respective Šilov boundaries $\partial(E_\alpha)$, $\alpha \in I$, whenever they exist, are also homeomorphisms.

In this respect, we first examine when the family $(\partial(E_\alpha), {}^{\mathrm{t}}f_{\beta\alpha}|_{\partial(E_\beta)})$ constitutes a projective system of subsets of $\mathfrak{M}(E_\alpha)$; that is when

$${}^{\mathrm{t}}f_{\beta\alpha}(\partial(E_\beta)) \subseteq \partial(E_\alpha) \ , \tag{3.1}$$

for any $\alpha \leq \beta$ in I. A sufficient condition is given in the following.

Lemma 3.1. *Let $(E_\alpha, f_{\beta\alpha})$ be an inductive system of unital topological algebras with Šilov boundaries $\partial(E_\alpha)$, $\alpha \in I$, such that the connecting maps $f_{\beta\alpha}$ and ${}^{\mathrm{t}}f_{\beta\alpha}$, with $\alpha \leq \beta$ in I, are surjective. Then, the family $(\partial(E_\alpha), {}^{\mathrm{t}}f_{\beta\alpha}|_{\partial(E_\beta)})$ yields a projective system of topological spaces.*

Proof: By taking $u_\beta \in \partial(E_\beta)$, then ${}^{\mathrm{t}}f_{\beta\alpha}(u_\beta) = u_\beta \circ f_{\beta\alpha} \in \mathfrak{M}(E_\alpha)$. If U_α is an open neighbourhood of $u_\beta \circ f_{\beta\alpha}$ in $\mathfrak{M}(E_\alpha)$, we show that there exists $x_\alpha \in E_\alpha$ such that (cf. (2.13) and the comments before)

$$M_{\widehat{x_\alpha}} \subseteq U_\alpha \ . \tag{3.2}$$

Indeed, by the continuity of ${}^{\mathrm{t}}f_{\beta\alpha}$ there exists an open neighbourhood U_β of u_β in $\mathfrak{M}(E_\beta)$ such that ${}^{\mathrm{t}}f_{\beta\alpha}(U_\beta) \subseteq U_\alpha$, and since $u_\beta \in \partial(E_\beta)$, there is $x_\beta \in E_\beta$ with $M_{\widehat{x_\beta}} \subseteq U_\beta$. Hence,

$${}^{\mathrm{t}}f_{\beta\alpha}(M_{\widehat{x_\beta}}) \subseteq {}^{\mathrm{t}}f_{\beta\alpha}(U_\beta) \subseteq U_\alpha \ , \tag{3.3}$$

where

$$ {}^{\mathrm{t}}f_{\beta\alpha}(M_{\widehat{x}_\beta}) = M_{\widehat{x}_\beta} \circ f_{\beta\alpha} = \left\{ u_\beta \circ f_{\beta\alpha} \colon\, u_\beta \in M_{\widehat{x}_\beta} \right\} . \tag{3.4} $$

Now by the hypothesis for $f_{\beta\alpha}$, there exists $x_\alpha \in E_\alpha$ such that $x_\beta = f_{\beta\alpha}(x_\alpha)$. Hence $|\widehat{x}_\beta(u_\beta)| = |u_\beta(x_\beta)| = |(u_\beta \circ f_{\beta\alpha})(x_\alpha)| = |\widehat{x}_\alpha(u_\beta \circ f_{\beta\alpha})| \le p_{\mathfrak{M}(E_\alpha)}(\widehat{x}_\alpha)$, so that $p_{\mathfrak{M}(E_\beta)}(\widehat{x}_\beta) \le p_{\mathfrak{M}(E_\alpha)}(\widehat{x}_\alpha)$, while $p_{\mathfrak{M}(E_\alpha)}(\widehat{x}_\alpha) \le p_{\mathfrak{M}(E_\beta)}(\widehat{x}_\beta)$ follows from the surjectivity of ${}^{\mathrm{t}}f_{\beta\alpha}$. Thus (cf. also (3.3) and (3.4)),

$$ M_{\widehat{x}_\alpha} = M_{\widehat{x}_\beta} \circ f_{\beta\alpha} = {}^{\mathrm{t}}f_{\beta\alpha}(M_{\widehat{x}_\beta}) \subseteq U_\alpha \ , \tag{3.5} $$

proving (3.2), therefore ${}^{\mathrm{t}}f_{\beta\alpha}(u_\beta) \in \partial(E_\alpha)$, which implies the assertion. ∎

The surjectivity of ${}^{\mathrm{t}}f_{\beta\alpha}$, $\alpha \le \beta$ in I, implies the surjectivity of the canonical map ρ_α, $\alpha \in I$ [7: p. 210, Lemma 9], so that in view of (2.8) and the homeomorphism j, one obtains the surjectivity of

$$ {}^{\mathrm{t}}f_\alpha = \rho_\alpha \circ j \approx \rho_\alpha \,, \qquad \alpha \in I \ . \tag{3.6} $$

Thus, we have the following result, useful in the sequel.

Lemma 3.2. *Let $(E_\alpha, f_{\beta\alpha})$ be an inductive system of unital topological algebras and $E = \varinjlim_\alpha E_\alpha$ the respective inductive limit unital topological algebra. Moreover, suppose that the connecting maps ${}^{\mathrm{t}}f_{\beta\alpha}$, with $\alpha \le \beta$ in I, are onto. Then, for every $x \in E = \varinjlim_\alpha E_\alpha$, with $x = f_\alpha(x_\alpha)$ for some $\alpha \in I$, one has*

$$ \rho_\alpha(M_{\widehat{x}}) \approx {}^{\mathrm{t}}f_\alpha(M_{\widehat{x}}) = M_{\widehat{x}} \circ f_\alpha = M_{\widehat{x}_\alpha} \ . \tag{3.7} $$

Proof: Let $x \in E$ with $x = f_\alpha(x_\alpha)$, for some $\alpha \in I$ and $\upsilon \in \mathfrak{M}(E)$. Then, $|\widehat{x}(\upsilon)| = |\upsilon(x)| = |(\upsilon \circ f_\alpha)(x_\alpha)| = |\widehat{x}_\alpha({}^{\mathrm{t}}f_\alpha(\upsilon))| \le p_{\mathfrak{M}(E_\alpha)}(\widehat{x}_\alpha)$, hence $p_{\mathfrak{M}(E)}(\widehat{x}) \le p_{\mathfrak{M}(E_\alpha)}(\widehat{x}_\alpha)$ and since, by hypothesis and (3.6), ${}^{\mathrm{t}}f_\alpha$ is onto, $p_{\mathfrak{M}(E)}(\widehat{x}) = p_{\mathfrak{M}(E_\alpha)}(\widehat{x}_\alpha)$, $\alpha \in I$, implying (3.7) in view of (2.13). ∎

Based on the previous two lemmas one obtains the next result.

Theorem 3.1. *Consider the context of Lemma 3.1 and let $E = \varinjlim_\alpha E_\alpha$ be the corresponding inductive limit unital topological algebra, with Šilov boundary $\partial(E)$. Then, one obtains*

$$ \partial(E) = \varprojlim_\alpha \partial(E_\alpha) \ , \tag{3.8} $$

within a homeomorphism of the topological spaces involved, provided by (2.7).

Proof: Considering the restriction of the homeomorphism (2.7) to $\partial(E)$, one gets a *bicontinuous injection*

$$j' \equiv \varprojlim_{\alpha} {}^{\mathrm{t}}f_\alpha|_{\partial(E)} \colon \partial(E) \to \varprojlim_{\alpha} \partial(E_\alpha) \ , \tag{3.9}$$

since each one of ${}^{\mathrm{t}}f_\alpha$, $\alpha \in I$, *takes* $\partial(E)$ *into* $\partial(E_\alpha)$. Indeed, if $u \in \partial(E)$, then ${}^{\mathrm{t}}f_\alpha(u) = u \circ f_\alpha \equiv u_\alpha \in \mathfrak{M}(E_\alpha)$ and by the continuity of ${}^{\mathrm{t}}f_\alpha$, ${}^{\mathrm{t}}f_\alpha^{-1}(U_\alpha)$ is an open neighbourhood of u in $\mathfrak{M}(E)$, U_α being an open neighbourhood of u_α in $\mathfrak{M}(E_\alpha)$. Thus, there exists $x \in E = \bigcup_{\alpha \in I} f_\alpha(E_\alpha)$ such that $M_{\widehat{x}} \subseteq {}^{\mathrm{t}}f_\alpha^{-1}(U_\alpha)$, hence $M_{\widehat{x}} \circ f_\alpha \equiv {}^{\mathrm{t}}f_\alpha(M_{\widehat{x}}) \subseteq U_\alpha$. Since $x = f_\gamma(x_\gamma)$, for some $\gamma \in I$, there exists $\beta \in I$, with $\beta \geq \alpha, \gamma$, such that $x = f_\beta(f_{\beta\gamma}(x_\gamma)) = f_\beta(x_\beta)$ and by Lemma 3.2 $M_{\widehat{x_\beta}} = {}^{\mathrm{t}}f_\beta(M_{\widehat{x}})$, hence ${}^{\mathrm{t}}f_{\beta\alpha}(M_{\widehat{x_\beta}}) = ({}^{\mathrm{t}}f_{\beta\alpha} \circ {}^{\mathrm{t}}f_\beta)(M_{\widehat{x}}) = {}^{\mathrm{t}}f_\alpha(M_{\widehat{x}}) \subseteq U_\alpha$. By hypothesis and (3.5) one gets $M_{\widehat{x_\alpha}} \subseteq U_\alpha$, thus $u_\alpha \in \partial(E_\alpha)$, proving the assertion.

Now, we show that

$$\varprojlim_{\alpha} \partial(E_\alpha) \subseteq \partial(E) \ . \tag{3.10}$$

If $u \in \varprojlim_{\alpha} \partial(E_\alpha)$ and U is an open neighbourhood of u in $\varprojlim_{\alpha} \partial(E_\alpha)$, then it contains a basic open neighbourhood $\rho_\alpha^{-1}(U_\alpha)$, for some $\alpha \in I$, with U_α a basic open neighbourhood of $\rho_\alpha(u) = u_\alpha = u \circ f_\alpha$ in $\mathfrak{M}(E_\alpha)$ (cf. [8: p. 87, Lemma 3.1]). Since $u_\alpha \in \partial(E_\alpha)$, there exists $x_\alpha \in E_\alpha$ such that $M_{\widehat{x_\alpha}} \subseteq U_\alpha$, hence $\rho_\alpha^{-1}(M_{\widehat{x_\alpha}}) \subseteq \rho_\alpha^{-1}(U_\alpha) \subseteq U$. Setting $x = f_\alpha(x_\alpha) \in E = \varinjlim_{\alpha} E_\alpha$, one obtains (Lemma 3.2) $M_{\widehat{x}} \subseteq \rho_\alpha^{-1}(M_{\widehat{x_\alpha}}) \subseteq U$, hence $u \in \partial(E)$, proving (3.10) and thus (3.8). ∎

4 – Bishop boundary of an inductive limit topological algebra

We investigate the situation exhibited in Theorem 3.1 for the Bishop boundaries of the topological algebras considered. By the standard property of the Bishop boundary to be dense in the spectrum of an Urysohn algebra, in the context of the latter algebras one certainly has due to (2.7) the relation

$$\overline{\mathcal{B}(E)} = \mathfrak{M}(E) \cong \varprojlim_{\alpha} \mathfrak{M}(E_\alpha) = \varprojlim_{\alpha} \overline{\mathcal{B}(E_\alpha)} \ , \tag{4.1}$$

taking into account that the inductive limit of Urysohn algebras is also an Urysohn algebra, as the following result shows.

Lemma 4.1. *The inductive limit topological algebra* $E = \varinjlim_{\alpha} E_\alpha$ *of an inductive system of Urysohn algebras* $(E_\alpha, f_{\beta\alpha})$ *is an Urysohn algebra.*

Proof: Let $u \in \mathfrak{M}(E) \cong \varprojlim_{\alpha} \mathfrak{M}(E_\alpha)$ and U an open neighbourhood of u in $\mathfrak{M}(E)$. Then (cf. [8: p. 87, Lemma 3.1]), U contains a basic open neighbourhood $\rho_\alpha^{-1}(U_\alpha)$, for some $\alpha \in I$, with U_α a basic open neighbourhood of $\rho_\alpha(u) = u_\alpha = u \circ f_\alpha$ in $\mathfrak{M}(E_\alpha)$. By the hypothesis for E_α (cf. also (2.15)), there exists $x_\alpha \in E_\alpha$ such that $0 \le \widehat{x}_\alpha \le 1$, $\widehat{x}_\alpha(u_\alpha) = 1$ and $\widehat{x}_\alpha = 0|_{U_\alpha^c}$. By setting $x = f_\alpha(x_\alpha)$, one gets (cf. (2.9)) $0 \le \widehat{x} \le 1$, $\widehat{x}(u) = 1$ and $\widehat{x} = 0|_{U^c}$, since $U^c \subseteq \rho_\alpha^{-1}(U_\alpha^c)$, proving the assertion. ∎

In our attempt to eliminate the closure in (4.1), we first note that the family $(\mathcal{B}(E_\alpha), {}^{\mathrm{t}}f_{\beta\alpha})$ constitutes a projective system of topological spaces under suitable conditions exhibited in [4: Theorem 4.2]. Namely, we have.

Lemma 4.2. *Let* $(E_\alpha, f_{\beta\alpha})$ *be an inductive system of unital spectral Urysohn algebras such that for every* $\alpha \le \beta$ *in* I, E_β *has non-empty Bishop boundary and the Gel'fand transform algebras of* E_α, $E_\alpha^\wedge$, *are* σ*-complete. Moreover, let the connecting maps* $f_{\beta\alpha}$ *be "spectral radii preserving" in the sense that for every* $\alpha \le \beta$ *in* I,

$$r_{E_\alpha}(x_\alpha) = r_{E_\beta}(f_{\beta\alpha}(x_\alpha))\,, \qquad x_\alpha \in E_\alpha\ , \tag{4.2}$$

while the respective transpose maps ${}^{\mathrm{t}}f_{\beta\alpha}$ *are open injections. Then, the Bishop boundaries of* E_α, $\alpha \le \beta$, *exist and the family* $(\mathcal{B}(E_\alpha), {}^{\mathrm{t}}f_{\beta\alpha})$ *constitutes a projective system of topological spaces.*

Proof: According to (2.14) the existence of $\mathcal{B}(E_\alpha)$, $\alpha \le \beta$ in I, is accomplished by proving that

$$M_{\widehat{x}_\alpha} \cap G_\delta(\mathfrak{M}(E_\alpha)) \ne \emptyset\,, \qquad x_\alpha \in E_\alpha\ . \tag{4.3}$$

Since for every $\alpha \le \beta$ in I, $\mathcal{B}(E_\beta)$ exists, one obtains an analogous relation to (4.3) for $\mathfrak{M}(E_\beta)$ and every $x_\beta \in E_\beta$, hence for $f_{\beta\alpha}(x_\alpha)$, $x_\alpha \in E$; namely one has (cf. also (2.13))

$$\Big(M_{f_{\beta\alpha}(x_\alpha)} = M_{\widehat{x}_\alpha \circ {}^{\mathrm{t}}f_{\beta\alpha}}\Big) \cap G_\delta(\mathfrak{M}(E_\beta)) \ne \emptyset\,, \qquad x_\alpha \in E_\alpha\ . \tag{4.4}$$

By the hypothesis for ${}^{\mathrm{t}}f_{\beta\alpha}$ one gets

$${}^{\mathrm{t}}f_{\beta\alpha}\Big(G_\delta(\mathfrak{M}(E_\beta))\Big) \subseteq G_\delta(\mathfrak{M}(E_\alpha))\ , \tag{4.5}$$

while, for every $\upsilon \in M_{f_{\beta\alpha}(x_\alpha)}$, (4.2) implies that $|\widehat{x}_\alpha({}^{\mathrm{t}}f_{\beta\alpha}(\upsilon))| = |f_{\beta\alpha}(x_\alpha)(\upsilon)| = \sup_{u\in\mathfrak{M}(E_\beta)} |f_{\beta\alpha}(x_\alpha)(u)| = r_{E_\beta}(f_{\beta\alpha}(x_\alpha)) = r_{E_\alpha}(x_\alpha) = \sup_{u\in\mathfrak{M}(E_\alpha)} |\widehat{x}_\alpha(u)|$, hence

$$(4.6) \qquad {}^{\mathrm{t}}f_{\beta\alpha}(M_{f_{\alpha\beta}(x_\alpha)}) \equiv M_{f_{\beta\alpha}(x_\alpha)} \circ f_{\beta\alpha} \subseteq M_{\widehat{x}_\alpha} \ ,$$

for every $x_\alpha \in E_\alpha$. Now, by applying ${}^{\mathrm{t}}f_{\beta\alpha}$ to (4.4), one obtains (4.3) in view also of (4.5) and (4.6), therefore $\mathcal{B}(E_\alpha)$, $\alpha \leq \beta$, exists.

Concerning the last assertion, we have to prove that

$$(4.7) \qquad {}^{\mathrm{t}}f_{\beta\alpha}(\mathcal{B}(E_\beta)) \subseteq \mathcal{B}(E_\alpha)\,, \qquad \alpha \leq \beta\ .$$

Indeed, if $\upsilon \in \mathcal{B}(E_\beta)$, then υ is a G_δ point in $\mathfrak{M}(E_\beta)$, and by (4.5) ${}^{\mathrm{t}}f_{\beta\alpha}(\upsilon) \in G_\delta(\mathfrak{M}(E_\alpha))$. The σ-completeness of E_α, for every $\alpha \leq \beta$ in I, implies that ${}^{\mathrm{t}}f_{\beta\alpha}(\upsilon) \in \mathcal{B}(E_\alpha)$ (cf. comments following (2.15)), so that (4.7) holds true. ∎

Scholium 4.1. In the previous lemma, by assuming that all the Bishop boundaries of the topological algebras considered exist, we can avoid our assumption that E_α, $\alpha \in I$, are spectral and also our assumption (4.2).

On the basis of the Lemma 4.2, we get the following.

Theorem 4.1. *Suppose we have the context of Lemma 4.2. Moreover, let $E = \varinjlim_\alpha E_\alpha$ be the corresponding inductive limit unital topological algebra with Gel'fand transform algebra $E^\wedge$ σ-complete and Bishop boundary $\mathcal{B}(E)$. Then, one gets*

$$(4.8) \qquad \mathcal{B}(E) = \varprojlim_\alpha \mathcal{B}(E_\alpha)$$

within a homeomorphism of the topological spaces involved, provided by (2.7).

Proof: The restriction of the homeomorphism (2.7) to $\mathcal{B}(E)$ yields a *bicontinuous injection*

$$(4.9) \qquad j_1 \equiv \varprojlim_\alpha {}^{\mathrm{t}}f_\alpha|_{\mathcal{B}(E)} \colon \mathcal{B}(E) \to \varprojlim_\alpha \mathcal{B}(E_\alpha)\ ,$$

since each one of ${}^{\mathrm{t}}f_\alpha$, $\alpha \in I$, takes $\mathcal{B}(E)$ into $\mathcal{B}(E_\alpha)$: Indeed, if $u \in \mathcal{B}(E) \subseteq \mathfrak{M}(E) \cong \varprojlim_\alpha \mathfrak{M}(E_\alpha)$, then $u = (u_\alpha)_{\alpha\in I}$ is a G_δ point in $\mathfrak{M}(E)$, hence its projections u_α, $\alpha \in I$, are G_δ points in the corresponding $\mathfrak{M}(E_\alpha)$ (cf. e.g. [3: p. 6, Proposition 5.1]). Thus, by the σ-completeness of $E_\alpha^\wedge$, $\alpha \in I$, ${}^{\mathrm{t}}f_\alpha(u) = u \circ f_\alpha \equiv u_\alpha \in \mathcal{B}(E_\alpha)$ (cf. also (3.6)), proving the assertion.

On the other hand, by taking $u = (u_\alpha) \in \varprojlim_\alpha \mathcal{B}(E_\alpha)$, $\rho_\alpha(u) = u_\alpha \in \mathcal{B}(E_\alpha)$ is a G_δ point in $\mathfrak{M}(E_\alpha)$, $\alpha \in I$. Then, $\rho_\alpha^{-1}(u_\alpha) = u$ is G_δ in $\mathfrak{M}(E)$, hence, by the σ-completeness of $E^\wedge$, $u \in \mathcal{B}(E)$, implying that

$$\varprojlim_\alpha \mathcal{B}(E_\alpha) \subseteq \mathcal{B}(E) \ , \tag{4.10}$$

and this completes the proof. ∎

5 – Choquet boundary of an inductive limit topological algebra

In this section we examine whether the situation considered in Theorem 3.1 holds true for the Choquet boundaries of the topological algebras involved. Namely, if the map (2.7) maps homeomorphically the Choquet boundary of an inductive limit unital topological algebra onto the projective system of the Choquet boundaries of the factor algebras.

We first have the following, as a consequence of a relevant result in [6: Theorem 3.4, (3.12)] exhibiting that a continuous algebra morphism between topological algebras preserves the Choquet boundaries.

Lemma 5.1. *Let $(E_\alpha, f_{\beta\alpha})$ be an inductive system of unital topological algebras and $\mathrm{Ch}(E_\alpha)$, $\alpha \in I$, the corresponding Choquet boundaries. Then, the family $(\mathrm{Ch}(E_\alpha), {}^{\mathrm{t}}f_{\beta\alpha}|_{\mathrm{Ch}(E_\beta)})$ constitutes a projective system of topological spaces.*

Proof: It suffices to prove that

$${}^{\mathrm{t}}f_{\beta\alpha}(\mathrm{Ch}(E_\beta)) \subseteq \mathrm{Ch}(E_\alpha) \ . \tag{5.1}$$

If $u_\beta \in \mathrm{Ch}(E_\beta)$, then by definition, there exists $\mu_\beta \in \mathcal{M}_c^+(\mathfrak{M}(E_\beta))$ such that $\mu_\beta = \delta_{u_\beta}$. Considering ${}^{\mathrm{t}}f_{\beta\alpha}(u_\beta) = u_\beta \circ f_{\beta\alpha} \in \mathfrak{M}(E_\alpha)$, one obtains $\mu_\alpha = \mu_\beta \circ {}^{\mathrm{t}}({}^{\mathrm{t}}f_{\beta\alpha}) \in \mathcal{M}_c^+(\mathfrak{M}(E_\alpha))$, with $\mathrm{supp}(\mu_\alpha) \subseteq {}^{\mathrm{t}}f_{\beta\alpha}(\mathrm{supp}(\mu_\beta))$, such that $\mu_\alpha(h_\alpha) = \mu_\beta(h_\alpha \circ {}^{\mathrm{t}}f_{\beta\alpha}) = \delta_{u_\beta}(h_\alpha \circ {}^{\mathrm{t}}f_{\beta\alpha}) = h_\alpha({}^{\mathrm{t}}f_{\beta\alpha}(u_\beta)) = \delta_{{}^{\mathrm{t}}f_{\beta\alpha}(u_\beta)}(h_\alpha)$, for every $h_\alpha \in \mathcal{C}(\mathfrak{M}(E_\alpha))$. Hence, ${}^{\mathrm{t}}f_{\beta\alpha}(u_\beta) \in \mathrm{Ch}(E_\alpha)$, proving the desired relation (5.1). ∎

Theorem 5.1. *Consider the context of Lemma 5.1 and let $E = \varinjlim_\alpha E_\alpha$ be the corresponding inductive limit unital topological algebra with Choquet boundary $\mathrm{Ch}(E)$. Then, one gets the relation*

$$\mathrm{Ch}(E) \underset{\text{homeo}}{\underset{\rightarrow}{\subseteq}} \varprojlim_\alpha \mathrm{Ch}(E_\alpha) \ , \tag{5.2}$$

within a homeomorphism of the respective topological spaces provided by (2.7). In particular, if the connecting maps ${}^{\mathrm{t}}f_{\beta\alpha}$, $\alpha \leq \beta$ in I, are onto, $E^\wedge$ and $E_\alpha^\wedge$,

$\alpha \in I$, are σ-complete and $\mathfrak{M}(E_\alpha)$ compact, then

$$\mathrm{Ch}(E) = \varprojlim_{\alpha} \mathrm{Ch}(E_\alpha) \ , \tag{5.3}$$

within a homeomorphism of the topological spaces concerned.

Proof: By applying a similar argument to that in the proof of Lemma 5.1, one has that the maps ${}^t f_\alpha : \mathfrak{M}(E) \to \mathfrak{M}(E_\alpha)$, $\alpha \in I$, preserve the respective Choquet boundaries. Thus, ${}^t f_\alpha(u) \in \mathrm{Ch}(E_\alpha)$, for every $u \in \mathrm{Ch}(E)$, and by the next commutative diagram

$$\begin{tikzcd}
& Ch(E) \arrow[r, "i"] \arrow[d, "{}^t f_\alpha|_{Ch(E)}"] \arrow[ddl, "j|_{Ch(E)}"'] & \mathfrak{M}(E) \arrow[d, "{}^t f_\alpha"'] \arrow[ddr, "\varprojlim_\alpha {}^t f_\alpha \equiv j"] & \\
& Ch(E_\alpha) \arrow[r, "i_\alpha"'] & \mathfrak{M}(E_\alpha) & \\
\varprojlim_\alpha Ch(E_\alpha) \arrow[ur, "\rho_\alpha"'] \arrow[rrr, "\varprojlim_\alpha i_\alpha"'] & & & \varprojlim_\alpha \mathfrak{M}(E_\alpha) \arrow[ul, "\rho_\alpha"]
\end{tikzcd} \tag{5.4}$$

one obtains a *bicontinuous injection* $j|_{\mathrm{Ch}(E)} \colon \mathrm{Ch}(E) \to \varprojlim_{\alpha} \mathrm{Ch}(E_\alpha)$, that is

$$\mathrm{Ch}(E) \underset{\text{homeo}}{\overset{\subseteq}{\to}} \varprojlim_{\alpha} \mathrm{Ch}(E_\alpha) \ . \tag{5.5}$$

Conversely, let $u = (u_\alpha)_{\alpha \in I} \in \varprojlim_{\alpha} \mathrm{Ch}(E_\alpha)$ and U an open neighbourhood of u in $\varprojlim_{\alpha} \mathrm{Ch}(E_\alpha)$. Then, U contains a basic open neighbourhood of $\rho_\alpha(u) = u_\alpha = u \circ f_\alpha$ in $\mathfrak{M}(E_\alpha)$ (cf. [8: p. 87, Lemma 3.1]). Since $u_\alpha \in \mathrm{Ch}(E_\alpha)$, there exists $x_\alpha \in E_\alpha$, such that $u_\alpha \in M_{\widehat{x}_\alpha} \subseteq U_\alpha$ (see Preliminaries). Setting $x = f_\alpha(x_\alpha) \in E = \varinjlim_{\alpha} E_\alpha$, one obtains (cf. Lemma 3.2) $u \in M_{\widehat{x}} \subseteq \rho_\alpha^{-1}(M_{\widehat{x}_\alpha}) \subseteq \rho_\alpha^{-1}(U_\alpha) \subseteq U$. Thus, $u \in \mathrm{Ch}(E)$, according to the hypothesis for $E^\wedge$ and the compactness of $\mathfrak{M}(E) = \varprojlim_{\alpha} \mathfrak{M}(E_\alpha)$, $\mathfrak{M}(E_\alpha)$ being compact. Hence

$$\varprojlim_{\alpha} \mathrm{Ch}(E_\alpha) \subseteq \mathrm{Ch}(E) \ , \tag{5.6}$$

which in connection with (5.5) proves (5.3). ∎

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Note (added in proof). The characterization of a Choquet point, that has been applied in Theorem 5.1, holds true, more generally, when the spectra of the topological algebras concerned ae just *Q-spaces*. (See R.I. Hadjigeorgiou, *Boundaries and peak points of topological algebras* (to appear), in conjunction with L. Gillman, M. Jerison, *Rings of Continuous Functions*, Springer-Verlag, 1976, and Jun-Iti Nagata, *Modern General Topology*, North-Holland, 1968).

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