PORTUGALIAE MATHEMATICA
Vol. 54 Fasc. 2 – 1997

SOME INEQUALITIES FOR IMMERSED SURFACES

V. Marenich* and I.V. Guadalupe**

Abstract: For immersions of surfaces in Riemannian manifolds with pinched curvature we establish some generalizations of relations between the integral of the square of the norm of the mean curvature vector and the area of the surface with topological invariants, known for immersions into space forms. For immersions with codimension two into space forms of nonpositive curvature these leads to a nonexistence of minimal immersions with everywhere non-zero normal curvature tensor.

1 – Introduction

Let Σ be an oriented surface which is isometrically immersed into an orientable n-dimensional Riemannian manifold E with δ-pinched curvature $K_\sigma(E)$ (see §2 for definition). We establish some generalizations of relations between the integral of the square of the norm of the mean curvature vector H and the area of the surface with topological invariants, known for immersions into space forms. Let K_N be the normal curvature of Σ (see §2 for definition), $\chi(\Sigma)$ denote the Euler characteristic of the tangent bundle and $\chi(\nu)$ denote the Euler characteristic of the plane bundle when the codimension of Σ in E is 2. We prove the following generalization of theorem 1 of Rodriguez and Guadalupe [3].

Theorem 1. *Let $f : \Sigma \to E$ be an isometric immersion of a compact oriented surface Σ into an orientable n-dimensional Riemannian manifold E with δ-pinched curvature $K_\sigma(E)$, i.e.,*

$$-1-\delta \le K_\sigma(E) \le -1$$

Received: December 1, 1995; *Revised*: August 9, 1996.
* Research partially supported by CNPq, Brasil.
** Research partially supported by IMPA-CNPq, Brasil.

or

$$-\delta/2 \le K_\sigma(E) \le \delta/2$$

or

$$1 \le K_\sigma(E) \le 1+\delta$$

(δ could be arbitrarily large).

Then the following is true:

$$\int_\Sigma \|H\|^2\, d\Sigma + \left(\frac{8}{3}\delta - 1\right) \operatorname{Area}(\Sigma) \ge 2\pi\,\chi(\Sigma) + \left|\int_\Sigma K_N\, d\Sigma\right| \tag{1.1}$$

or

$$\int_\Sigma \|H\|^2\, d\Sigma + \left(\frac{19}{6}\delta\right) \operatorname{Area}(\Sigma) \ge 2\pi\,\chi(\Sigma) + \left|\int_\Sigma K_N\, d\Sigma\right| \tag{1.2}$$

or

$$\int_\Sigma \|H\|^2\, d\Sigma + \left(\frac{11}{3}\delta + 1\right) \operatorname{Area}(\Sigma) \ge 2\pi\,\chi(\Sigma) + \left|\int_\Sigma K_N\, d\Sigma\right| . \tag{1.3}$$

Remark. If H is parallel and $n = 4$, we have

$$\left(\|H\|^2 + \frac{8}{3}\delta - 1\right) \operatorname{Area}(\Sigma) \ge 2\pi\Big(\chi(\Sigma) + \chi(\nu)\Big) \tag{1.4}$$

or

$$\left(\|H\|^2 + \frac{19}{6}\delta\right) \operatorname{Area}(\Sigma) \ge 2\pi\Big(\chi(\Sigma) + \chi(\nu)\Big) \tag{1.5}$$

or

$$\left(\|H\|^2 + \frac{11}{3}\delta + 1\right) \operatorname{Area}(\Sigma) \ge 2\pi\Big(\chi(\Sigma) + \chi(\nu)\Big) \tag{1.6}$$

correspondingly.

Corollary 1 ([3])**.** *Let E has constant sectional curvature $K_\sigma(E) = c$. If $f\colon \Sigma \to E$ is an isometric immersion, then*

$$\int_\Sigma \|H\|^2\, d\Sigma + c \operatorname{Area}(\Sigma) \ge 2\pi\,\chi(\Sigma) + \left|\int_\Sigma K_N\, d\Sigma\right| \tag{1.7}$$

with equality if and only if K_N does not change sign and the ellipse of the curvature is a circle at every point.

Corollary 2. *Let $f:\Sigma\to E$ be an isometric immersion of a compact oriented surface Σ into E^4 with δ-pinched curvature $K_\sigma(E)$, i.e.,*

$$-1-\delta\le K_\sigma(E)\le -1$$

or

$$-\delta/2\le K_\sigma(E)\le \delta/2$$

or

$$1\le K_\sigma(E)\le 1+\delta\ .$$

Then, if $K_N>0$ at every point, we have

$$\int_\Sigma \|H\|^2\,d\Sigma\ge 12\pi+\Big(1-\frac{8}{3}\delta\,\mathrm{Area}(\Sigma)\Big) \tag{1.8}$$

or

$$\int_\Sigma \|H\|^2\,d\Sigma\ge 12\pi-\frac{19}{6}\delta\,\mathrm{Area}(\Sigma) \tag{1.9}$$

$$\int_\Sigma \|H\|^2\,d\Sigma\ge 12\pi-\Big(\frac{11}{3}+1\Big)\delta\,\mathrm{Area}(\Sigma) \tag{1.10}$$

correspondingly.

Corollary 3 ([2])**.** *Let $f:\Sigma\to S^n(1)$ be a minimal immersion of a compact oriented surface $\Sigma\sim S^2$ into the unit sphere $S^n(1)$ with $K_N\ne 0$. Then we have*

$$\mathrm{Area}(\Sigma)\ge 12\pi \tag{1.11}$$

and the equality holds if and only if Σ is contained in some totally geodesic $S^4(1)\subset S^n(1)$.

Theorem 2. *Let $f:\Sigma\to E^4$ be a minimal immersion of a compact oriented surface Σ into E^4 with constant sectional curvature $K_\sigma(E)=-1$. Then we have*

$$|\chi(\nu)|<|\chi(\Sigma)|\ . \tag{1.12}$$

Corollary 4. *Every minimal immersion of a compact oriented surface Σ into E^4 with constant sectional curvature $K_\sigma(E)=-1$ must have a point where the normal curvature K_N equals zero.*

2 – Preliminaries

Let $f: \Sigma \to E$ be an isometric immersion of a surfaces Σ into an oriented n-dimensional Riemannian manifold E with non-constant sectional curvature $K_\sigma(E)$. Denote by $\nabla^\perp$ the connection in the normal bundle ν of this immersion, and by $R^\perp$ the curvature tensor of $\nabla^\perp$. Let also $B\colon T\Sigma \times T\Sigma \to \nu$ be the second fundamental form of the immersion and A_ξ — the symmetric endomorphism of $T\Sigma$ defined by $\langle B(X,Y),\xi\rangle = \langle A_\xi X, Y\rangle$, where $\langle\ ,\ \rangle$ is the inner product in TE. If $\{e_1, e_2\}$ is some orthonormal frame of Σ and $B_{ij} = B(e_i, e_j),\ i,j = 1,2$, then easy to see that for the mean curvature vector $H = \frac{1}{2}\,\mathrm{trace} B$ we have

$$4\,\|H\|^2 = |B_{11} + B_{22}|^2 \tag{2.1}$$

and from the Gauss equation it follows that

$$K = \langle B_{11}, B_{22}\rangle - |B_{12}|^2 + K_\sigma(E) \tag{2.2}$$

for the Gaussian curvature K of Σ.

Recall that if $R_p^\perp \neq 0$ then at this point $p \in \Sigma$ we can define a 2-plane ν_p^* — an orthogonal complement in ν_p to the annihilator of the tensor $R_p^\perp$. If $R^\perp \neq 0$ everywhere then the subbundle ν^* is globally defined and we have an orthogonal bundle splitting $\nu = \nu^* + \nu^0$ of the normal bundle ν of Σ, where ν^0 is the annihilator of $R^\perp$, see [1] for details. By definition, the normal curvature at the point $p \in \Sigma$ is

$$K_N(p) = \langle R^\perp(e_1, e_2)e_3, e_4\rangle(p)\ ,$$

where $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are tangent and normal frames of Σ at the point p correspondingly. If $T\Sigma$ and ν^* are oriented, then the normal curvature K_N is globally defined and has sign. In codimension 2, $\nu = \nu^*$ and to define K_N we do not need $R^\perp \neq 0$ condition. In higher codimensions, if $R^\perp \neq 0$ and Σ is orientable, then we will always choose orientations in $T\Sigma$ an in ν^* such that $K_N \geq 0$.

From the Ricci equation we have

$$K_N = K_N(E) + |B_{11} - B_{22}|\,|B_{12}| \tag{2.3}$$

with

$$K_N(E) = \langle R(e_1, e_2)e_3, e_4\rangle\ , \tag{2.4}$$

where $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are orthonormal frames of $T_p\Sigma$ and ν_p, respectively, and R is the curvature tensor of E.

Definition. We say that n-dimensional Riemannian manifold E has δ-pinched sectional curvature $K_\sigma(E)$ if

$$-1-\delta \le K_\sigma(E) \le -1$$

or

$$-\delta/2 \le K_\sigma(E) \le \delta/2$$

or

$$1 \le K_\sigma(E) \le 1+\delta \ .$$

Lemma 3 (Berger's inequality)**.** *If E is an n-dimensional Riemannian manifold with δ-pinched curvature, then*

$$|K_N(E)| \le \frac{8}{3}\,\delta \ . \tag{2.5}$$

Proof: Indeed, for vectors X, Y, Z and W of unit length and normal to each other the following is true

$$\begin{aligned}\langle R(X,Y)Z,\, W\rangle = \langle R(X{+}Z,Y)X+Z,\, W\rangle - \langle R(X,Y)X,\, W\rangle \\ - \langle R(Z,Y)Z,\, W\rangle - \langle R(Z,Y)X,\, W\rangle \ ,\end{aligned}$$

and

$$\begin{aligned}\langle R(X,Z)W,\, Y\rangle = \langle R(X{+}W,Z)X+W,\, Y\rangle - \langle R(X,Z)X,\, Y\rangle \\ - \langle R(W,Z)W,\, Y\rangle - \langle R(W,Z)X,\, Y\rangle \ .\end{aligned}$$

After summation and adding $\langle R(X,Y)Z, W\rangle$ to both sides this gives

$$3\langle R(X,Y)Z,\, W\rangle = R_1 + R_2 + R_3 \ ,$$

where R_1 and R_2 contains only curvature terms with three different arguments and

$$R_3 = \langle R(X,W)Y,\, Z\rangle + \langle R(X,Z)W,\, Y\rangle + \langle R(X,Y)Z,\, W\rangle = 0$$

is the Bianchi identity. Now, for every curvature term with only three different arguments we have

$$2\langle R(X,Y)Y,\, Z\rangle = \langle R(X{+}Z,Y)Y,\, X{+}Z\rangle - \langle R(X,Y)Y,\, X\rangle - \langle R(Z,Y)Y,\, Z\rangle \ .$$

But easy to see that when the curvature is δ-pinched, then

$$\Big|\langle R(X,Y)Y,\, Z\rangle\Big| \le \delta \ ,$$

that gives $R_1 + R_2 \le 8\,\delta$ and (2.5) follows. ∎

3 – Proof of Theorems

Proof of Theorem 1: Let $\{e_1, e_2\}$ be an orthonormal frame at p and $u = (B_{11} - B_{22})/2$ and $v = B_{12}$. Hence, from (2.1), (2.2), (2.3) and (2.5) we have

$$\begin{aligned} 0 &\le \Big(|B_{11} - B_{22}| - 2|B_{12}|\Big)^2 \\ &= |B_{11} - B_{22}|^2 + 4|B_{12}|^2 - 4|B_{11} - B_{22}|\,|B_{12}| \\ &\le |B_{11}|^2 + |B_{22}|^2 + 2|B_{12}|^2 - 2K - 4|K_N| + 4\frac{8\delta}{3} + 2K_\sigma(E) \ . \end{aligned} \tag{3.1}$$

On the other hand

$$\begin{aligned} 4|H|^2 &= |B_{11} + B_{22}|^2 = |B_{11}|^2 + |B_{22}|^2 + 2\langle B_{11}, B_{22}\rangle \\ &= |B_{11}|^2 + |B_{22}|^2 + 2|B_{12}|^2 + 2K - 2K_\sigma(E) \ . \end{aligned} \tag{3.2}$$

Hence, by (3.1) and (3.2), it follows that

$$|H|^2 + \frac{8\delta}{3} + K_\sigma(E) \ge K + |K_N| \ . \tag{3.3}$$

Integrating (3.3) over Σ, we get (1.1), (1.2) and (1.3) from δ-pinched conditions. ∎

Proof of Corollary 1: In this case $K_\sigma(E) = c$ and $K_N(E) = 0$, and (1.7) follows from (3.3). ∎

Proof of Corollary 2: If E has constant sectional curvature, then according to [1] we have $2\chi(\Sigma) = \chi(\nu) = 4$ and (1.8), (1.9) and (1.10) follows from (1.1), (1.2) and (1.3) respectively. ∎

Proof of Corollary 3: From [2] we have

$$\int_\Sigma K_N \, d\Sigma \ge 4\pi\, \chi(\Sigma) \tag{3.4}$$

and (1.11) follows from (1.7) and (3.4). ∎

Proof of Theorem 2: In this case $H = 0$ and $K_\sigma(E) = -1$. The inequality (3.3) is replacing by

$$|K_N| + K \le -1 \ . \tag{3.5}$$

Then from (3.5) and $K < 0$ it follows

$$|K_N| < -K \ . \tag{3.6}$$

Integrating (3.6) over Σ, we get

$$\left|\frac{1}{2\pi}\int_\Sigma K_N\, d\Sigma\right| \le \frac{1}{2\pi}\int_\Sigma |K_N|\, d\Sigma < -\frac{1}{2\pi}\int_\Sigma K\, d\Sigma \tag{3.7}$$

and (1.12) follows from (3.7). ∎

Proof of Corollary 4: Suppose that $R_p^\perp \neq 0$ for all $p \in \Sigma$, then according to [1] in E with constant sectional curvature we have $\chi(\nu) = 2\chi(\Sigma)$. This implies that $|\chi(\nu)| = 2|\chi(\Sigma)| > |\chi(\Sigma)|$, which contradicts to (1.12). ∎

REFERENCES

[1] Asperti, A.C., Ferus, D. and Rodriguez, L. – Surfaces with non-zero normal curvature tensor, *Atti Accad-Naz. Dei Lincei,* 73 (1982), 109–115.

[2] Guadalupe, I.V. – Minimal immersions of surfaces into n-dimensional space forms, *Rend. Sem. Mat. Uni. Padova,* 94 (1995), 155–163.

[3] Rodriguez, L. and Guadalupe, I.V. – Normal curvature of surfaces in space forms, *Pacific J. Math.,* 106 (1983), 95–103.

Valery Marenich,
IMECC – UNICAMP, Universidade Estadual de Campinas,
CP. 6065, 13081-970 Campinas, SP – BRASIL
E-mail: marenich@ime.unicamp.br

and

Irwen Valle Guadalupe,
IMECC – UNICAMP, Universidade Estadual de Campinas,
CP. 6065, 13081-970 Campinas, SP – BRASIL
E-mail: irwen@ime.unicamp.br