PORTUGALIAE MATHEMATICA
Vol. 54 Fasc. 3 – 1997

ABOUT STEADY TRANSPORT EQUATION II — — SCHAUDER ESTIMATES IN DOMAINS WITH SMOOTH BOUNDARIES

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Abstract: This paper is a continuation of our work [7]. We investigate steady transport equation

$$\lambda z + w \cdot \nabla z + a z = f , \quad \lambda > 0 ,$$

in various domains (bounded or unbounded) with sufficiently smooth compact or noncompact boundaries. The coefficients w and a are "sufficiently smooth" functions, "small" in appropriate norms. There is no transport of z through the boundary. Under these assumptions, we study existence, regularity, uniqueness and asymptotic behaviour (when the domain is unbounded) of solutions in spaces of Holder continuous functions. The corresponding estimates are derived. The results presented here have found a series of applications in the compressible fluid dynamics.

1 – Introduction

This work is a continuation of our previous paper [7], where we have studied existence, uniqueness, asymptotic behaviour and regularity of solutions to the steady transport equation in Sobolev and homogenous Sobolev spaces, and their duals. Here we investigate the steady transport equation

$$\lambda z + w \cdot \nabla z + a z = f , \tag{1.1}$$

($\lambda > 0$) in Holder spaces of continuous functions, in various types of domains $\Omega \subset \mathbb{R}^n$. The same results, we derive for this equation, are valid also for systems,

Received: May 18, 1996.
AMS Classification: 35Q35, 35L, 76N.

cf. [7] equation (1.3). In such a case, the unknown quantity is the vector field $z = (z_1, ..., z_m)$, $m = 2, 3, ...$ and w, a are the matrix fields $w = (w_{ij})$, $a = (a_{ik})$, $i, k = 1, ..., m$, $j = 1, ..., n$. The generalization, which is easy, is left to the reader.

The study of the equation (1.1) is based on the apriory estimates for z in Sobolev spaces formulated in [7] (see also [2] when Ω is bounded), on the interpolation arguments and on several fundamental properties of Slobodeckij spaces (which are the Besov spaces of a particular type) — see Triebel [9], or Bergh and Lofstrom [3]. All these tools are recalled in Section 2.

For a large class of domains Ω (bounded or unbounded) — see Section 2 — it is proved, that for the right hand side f in a suitable Holder space and for given coefficients w, a "small" in suitable Holder spaces (moreover, w has to be such that its normal component vanishes at the boundary), the equation (1.1) admits just one solution z in the same Holder space as that one of the right hand side f. Moreover, the corresponding estimates hold. For this theorem, see Section 3, Theorem 3.1 and for its proof, Section 4. In Section 5, we investigate the decay of solutions at infinity when the domain is unbounded. We put a particular stress to the exterior domains and to the whole space. The results are formulated in Theorem 5.1.

These are the main achievements of the present paper. The results are directly applicable to the investigation of the steady compressible flows, see [4] or [8]. As far as the author knows, they have been missing in the mathematical literature.

2 – Notations and preliminary results

2.1. Functional spaces

• By $B_R(x)$, we denote the ball in $\mathbb{R}^n$ with the center x and the radius R, $B^R(x) = \mathbb{R}^n \backslash B_R(x)$; $B_R(0)$ is denoted shortly by B_R and $B^R(0)$ by B^R.

• Let Ω be a domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$ and with ν the outer normal to it, or the whole $\mathbb{R}^n$ ($n \geq 2$). We denote by $\mathcal{C}^\infty(\Omega)$ a space of the infinitely differentiable functions on Ω and by $\mathcal{C}_0^\infty(\overline{\Omega})$ a space of infinitely differentiable functions (up to the boundary) with compact support in $\overline{\Omega}$. $\mathcal{C}_0^\infty(\Omega)$ is the space of smooth functions with compact support in Ω. When equipped with the usual weak topology, it is denoted by $\mathcal{D}(\Omega)$; by $\mathcal{D}'(\Omega)$ we denote its dual space, the usual space of distributions. By $\mathcal{S}(\mathbb{R}^n)$ we denote the space of rapidly decreasing functions equipped with the system of seminorms $\sup_{x\in\mathbb{R}^n} |p(x)\, \nabla^j z(x)|$ where

$j = 0, 1, \ldots$ and $p(x)$ are the polynoms on $\mathbb{R}^n$; $\mathcal{S}(\Omega)$ is the space of restrictions on Ω of the functions belonging to $\mathcal{S}(\mathbb{R}^n)$. The space of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$ is the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^n)$.

- By $\mathcal{C}^k(\overline{\Omega})$ or $\mathcal{C}^{k,0}(\overline{\Omega})$, $k = 0, 1, \ldots$, we denote the Banach space of differentiable functions (up to the boundary) up to order k, with the finite norm

$$|z|_{\mathcal{C}^k,\Omega} = \sum_{j=0}^{k} \sup_{x\in\Omega} |\nabla^j z(x)| \tag{2.1}$$

(in the definition we suppose that the boundary $\partial\Omega$ of Ω possesses at least $\mathcal{C}^k$ regularity).

By the Holder space $\mathcal{C}^{k,\alpha}(\overline{\Omega})$, $k = 0, 1, \ldots$, $\alpha \in (0, 1)$, we denote a Banach space of differentiable functions up to order k up to the boundary, with the finite norm

$$|z|_{\mathcal{C}^\alpha,\Omega} = |z|_{\mathcal{C}^k,\Omega} + \mathcal{H}_{\alpha,\Omega}(\nabla^k z)\ , \tag{2.2}$$

where

$$\mathcal{H}_{\alpha,\Omega}(z) \sup_{x,y\in\Omega} \frac{|z(x) - z(y)|}{|x - y|^\alpha}\ . \tag{2.3}$$

In this definition we have supposed that the boundary $\partial\Omega$ has at least $\mathcal{C}^{k,\alpha}$ regularity. We often use the Banach space $\mathcal{C}_0^{k,\alpha}(\overline{\Omega})$ which is defined as the completion of $\mathcal{C}_0^\infty(\overline{\Omega})$ in the norm $\mathcal{C}^{k,\alpha}$, i.e.

$$\mathcal{C}_0^{k,\alpha}(\Omega) = \overline{\mathcal{C}_0^\infty(\overline{\Omega})}^{|\cdot|_{\mathcal{C}^{k,\alpha}}}\ ; \tag{2.4}$$

it is a subspace of $\mathcal{C}^{k,\alpha}(\overline{\Omega})$ equipped with the norm (2.2). Similarly $\mathcal{C}_0^{k,\alpha}(\overline{\mathbb{R}^n}) = \mathcal{C}_0^{k,\alpha}(\mathbb{R}^n)$ is a completion of $\mathcal{C}_0^\infty(\mathbb{R}^n)$ in the norm $|\cdot|_{\mathcal{C}^{k,\alpha}}$. Notice that $\mathcal{C}^{k,\alpha}(\overline{\Omega}) = \mathcal{C}_0^{k,\alpha}(\overline{\Omega})$ for Ω a bounded domain.

- By $L^p(\Omega) = W^{0,p}(\Omega)$, $1 \le p \le \infty$, we denote the usual Lebesgue space with the norm $\|\cdot\|_{0,p}$ and by $W^{k,p}(\Omega)$ (resp. $W_0^{k,p}(\Omega)$), $k = 1, 2, \ldots$, the Sobolev spaces equipped with the norms

$$\|\cdot\|_{k,p} = \sum_{j=0}^{k} \|\nabla^j z\|_{0,p}\ . \tag{2.5}$$

Index zero denotes zero traces.

• By $W^{s,p}(\mathbb{R}^n)$, $0 < s < \infty$, s noninteger, $1 < p < \infty$, we denote so called Slobodeckij spaces. They are Banach spaces of distributions with the finite norm

$$\|z\|_{W^{s,p},\mathbb{R}^n} = \|z\|_{[s],p} + \mathcal{W}_{s,p}(z) \ , \tag{2.6}$$

where

$$\mathcal{W}_{s,p}(z) = \left(\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left|\nabla^{[s]}z(x) - \nabla^{[s]}z(y)\right|}{|x-y|^{n+\{s\}p}}\,dx\,dy\right)^{1/p} \ . \tag{2.7}$$

Here, we have decomposed s in such a way that

$$s = [s] + \{s\} \ , \tag{2.8}$$

with $[s]$ an integer or 0 and $0 < \{s\} < 1$. See e.g. [9], p.36 for more details.

• By $H^{s,p}(\mathbb{R}^n)$, $0 \le s < \infty$, $1 < p < \infty$, we denote a space of all tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ with the finite norm

$$\|z\|_{s,p,\mathbb{R}^n} = \left\|F^{-1}\left[(1+|\xi|^2)^{s/2}\,Fz\right]\right\|_{0,p,\mathbb{R}^n} \ . \tag{2.9}$$

Here, we have denoted by Fz the Fourier transform of z

$$Fz = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\xi}\,z(x)\,dx \tag{2.10}$$

and by F^{-1} its inverse. These spaces are called the spaces of Bessel potentials (see [9], p.37).

• We usually simplify the notation of norms with respect to the domain. If the domain is Ω, then e.g. $\|\cdot\|_{k,p,\Omega}$ is denoted simply by $\|\cdot\|_{k,p}$, $|\cdot|_{\mathcal{C}^{k,\alpha},\Omega}$ is abbreviated by $|\cdot|_{\mathcal{C}^{k,\alpha}}$. If the norm refers to another domain than Ω, then it appears as a further index of the norm, e.g. $\|\cdot\|_{k,p,\mathbb{R}^n}$ means the norm in $W^{k,p}(\mathbb{R}^n)$, etc.

Remark 2.1. We recall several important properties of the above spaces:

(i) $H^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$ for $1 < p < \infty$ and $s = 0, 1, ...$ algebraically and topologically, see [9], p.87–88.

(ii) $\mathcal{S}(\mathbb{R}^n)$ is dense both in $H^{s,p}(\mathbb{R}^n)$ and $W^{s,p}(\mathbb{R}^n)$ ($1 < p < \infty$, $0 < s < \infty$) see [9], p.48.

(iii) If G is a bounded domain and $z \in L^\infty(\Omega)$. Then $z \in L^p(\Omega)$ for any $1 \le p < \infty$ and

$$\lim_{p\to\infty} \|z\|_{0,p,G} = \|z\|_{0,\infty,G} \ , \tag{2.11}$$

see e.g. [5], p.84.

(**iv**) If $p > n$ then we have continuous imbedding for the Sobolev spaces

$$W^{k,p}(\mathbb{R}^n) \subset \mathcal{C}^{k-1,\alpha'}(\overline{\mathbb{R}^n}) \ , \tag{2.12}$$

with $\alpha' = \frac{p-n}{p}$ (if $\frac{p-n}{p} < 1$) or with $\alpha' \in (0,1)$ (if $\frac{p-n}{p} \geq 1$), see e.g. [5], p.293.

(**v**) For any open subset $G \subset \mathbb{R}^n$ (with sufficiently smooth boundary ∂G, say of the regularity $\mathcal{C}^{k,\alpha'}$), the imbedding

$$\mathcal{C}^{k,\alpha'}(\overline{G}) \subset \mathcal{C}^{k,\alpha}(\overline{G}) \ , \tag{2.13}$$

with $0 < \alpha < \alpha' < 1$, is compact, see e.g. [5], p.39.

2.2. Interpolation

We recall a particular case of what is called the K-method or the real method of interpolation. We refer to Bergh, Lofstrom [3], p.38–42 or Triebel [9], p.62–63, for more details and proofs. For two Banach spaces A_0, A_1 (for simplicity suppose $A_1 \subset A_0$ with the continuous imbedding) with norms $\|\cdot\|_{A_0}$ and $\|\cdot\|_{A_1}$, respectively, we define a functor

$$K(t,a) = \inf_{a=a_0+a_1} \Big(\|a_0\|_{A_0} + \|a_1\|_{A_1} \Big) \ .$$

Here $t \in \mathbb{R}^1$ and the infimum is taken over all decompositions of $a \in A_0$ into the sum of $a_0 \in A_0$ and $a_1 \in A_1$.

Let $0 < \theta < 1$ and $1 \leq p < \infty$. Then it is possible to define an interpolation space $[A_0, A_1]_{\theta,p}$ as the space of all elements of A_0 with the finite norm

$$\|a\|_{[A_0,A_1]_{\theta,p}} = \left(\int_0^\infty \Big(t^{-\theta} K(t,a) \Big)^p \frac{dt}{t} \right)^{1/p} \ . \tag{2.14}$$

It can be shown that $[A_0, A_1]_{\theta,p}$ with norm (2.14) is a Banach space.

The main theorem of the theory of interpolation reads (see e.g. [9], p.63):

Lemma 2.1. *Let $0 < \theta < 1$, $1 \leq p < \infty$. Let A_0, A_1, $\mathcal{A}_0$, $\mathcal{A}_1$ be Banach spaces with the norms $\|\cdot\|_{A_0}$, $\|\cdot\|_{A_1}$, $\|\cdot\|_{\mathcal{A}_0}$, $\|\cdot\|_{\mathcal{A}_1}$, respectively, such that $A_1 \subset A_0$, $\mathcal{A}_1 \subset \mathcal{A}_0$. Let $\mathcal{L}$ be a bounded linear operator which maps A_0 into $\mathcal{A}_0$ and A_1 into $\mathcal{A}_1$, i.e.*

$$\|\mathcal{L}a\|_{\mathcal{A}_0} \leq M_0 \|a\|_{A_0} \,, \qquad M_0 > 0 \ , \tag{2.15}$$

for any $a \in A_0$ *and*

$$\|\mathcal{L}a\|_{\mathcal{A}_1} \le M_1 \|a\|_{A_1}\,, \qquad M_1 > 0\,, \tag{2.16}$$

for any $a \in A_1$. *Then* $\mathcal{L}$ *is a bounded linear operator from* $[A_0, A_1]_{\theta,p}$ *into* $[\mathcal{A}_0, \mathcal{A}_1]_{\theta,p}$ *such that*

$$\|\mathcal{L}a\|_{[\mathcal{A}_0,\mathcal{A}_1]_{\theta,p}} \le M_0^{1-\theta}\, M_1^{\theta} \|a\|_{[A_0,A_1]_{\theta,p}}\ . \tag{2.17}$$

It is well known (cf. [3], Th. 6.2.4 and [9], p.90) that the Slobodeckij spaces can be obtained by the interpolation of Sobolev spaces $W^{k,p}(\mathbb{R}^n)$. We have the following lemma:

Lemma 2.2. *Let* $k = 0, 1, ...,$ $0 < \alpha < 1$. *Then*

$$W^{k+\alpha,p}(\mathbb{R}^n) = \Big[W^{k,p}(\mathbb{R}^n), W^{k+1,p}(\mathbb{R}^n)\Big]_{\alpha,p} \tag{2.18}$$

algebraically and topologically.

The next auxiliary result is needed in the proof of the existence theorem.

Lemma 2.3. *Let* $z \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. *Then,* $z \in \mathcal{C}^{0,\alpha}(\overline{\mathbb{R}}^n) \cap W^{\alpha,p}(\mathbb{R}^n)$ *for any* $\alpha \in (0,1)$, $p \in [1,\infty)$, *and*

$$\lim_{p\to\infty} \mathcal{W}_{(\alpha,p)}(z) = \mathcal{H}_\alpha(z)\ . \tag{2.19}$$

The statement of Lemma 2.3 remains true for any $z \in \mathcal{C}^{0,\alpha}(\overline{\mathbb{R}}^n)$ *with compact support in* $\mathbb{R}^n$.

Proof: If $z \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, then obviously $z \in \mathcal{C}_0^{0,\alpha}(\mathbb{R}^n) \cap W^{\alpha,p}(\mathbb{R}^n)$. Let $\epsilon \in (0,1)$ and $R > 1$ such that $\operatorname{supp} z \subset B_R$. We have, for $x \ne y$:

$$\begin{aligned} \frac{1}{(2R)^n} \frac{|z(x)-z(y)|^p}{|x-y|^{\alpha p}} &\le \frac{|z(x)-z(y)|^p}{|x-y|^{n+\alpha p}} \le \\ &\le \frac{1}{\epsilon^n} \frac{|z(x)-z(y)|^p}{|x-y|^{\alpha p}} + \left(\sum_{i=1}^n \left|\frac{\partial z}{\partial x_i}\Big(x+\xi_i(x-y)\Big)\right|\right)^p |x-y|^{p-n-\alpha p}\ , \end{aligned} \tag{2.20}$$

where $0 < \xi_i < 1$ are suitable real numbers. Hence

$$\begin{aligned} \left(\frac{1}{2R}\right)^{n/p} \left\|\frac{z(x)-z(y)}{|x-y|^\alpha}\right\|_{0,p,\mathbb{R}^n\times\mathbb{R}^n} &\le \mathcal{W}_{(\alpha,p)}(z) \le \\ \le \left(\frac{1}{\epsilon}\right)^{n/p} \left\|\frac{z(x)-z(y)}{|x-y|^\alpha}\right\|_{0,p,\mathbb{R}^n\times\mathbb{R}^n} &+ \epsilon^{1-\alpha-n/p} (\operatorname{meas}(\operatorname{supp} z))^2\, |\nabla z|_{\mathcal{C}^0} \end{aligned} \tag{2.21}$$

for p "sufficiently large". Passing to the limit $p \to \infty$, we get by (2.19)

$$\mathcal{H}_\alpha(z) \le \lim_{p\to\infty} \mathcal{W}_{(\alpha,p)}(z) \le \mathcal{H}_\alpha(z) + \epsilon^{(1-\alpha)}\, C(z) \tag{2.22}$$

for any $\epsilon \in (0,1)$. Here C is a positive constant dependent of z. Hence by $\epsilon \to 0$, we obtain

$$\mathcal{W}_{(\alpha,p)}(z) \to \mathcal{H}_\alpha(z) \quad \text{as} \;\; p \to \infty \; .$$

The lemma is proved. ∎

2.3. Domains

We say that $\Omega \subset \mathbb{R}^n$ $(n \ge 2)$ is of class $\mathcal{D}^{k,\alpha'}$, $k = 1, 2, ...,$ $0 \le \alpha' < 1$, if and only if

(**i**) $\partial\Omega \in \mathcal{C}^{k,\alpha'}$;

(**ii**) for any $l = 0, 1, ..., k$, $1 \le p < \infty$, there exists a continuous extension

$$\mathcal{E}\colon\, \mathcal{C}^{l,\alpha'}(\overline{\Omega}) \cap W^{l,p}(\Omega) \to \mathcal{C}^{l,\alpha'}(\overline{\mathbb{R}}^n) \cap W^{l,p}(\mathbb{R}^n) \; . \tag{2.23}$$

Notice that the following domains $\Omega \subset \mathbb{R}^n$ are of class $\mathcal{D}^{k,\alpha'}$;

(**a**) the whole space $\mathbb{R}^n$ and the halfspace $\mathbb{R}^n_+$;

(**b**) the bounded domains Ω with the boundary $\partial\Omega \in \mathcal{C}^{k,\alpha'}$;

(**c**) an exterior domain to a compact reagion Ω_c with the boundary $\partial\Omega_c \in \mathcal{C}^{k,\alpha'}$;

(**d**) the pipes with the finite cross sections:

$$\Omega = \Omega' = \Big\{x = (x', x)\colon\, x' = (x_1, ..., x_{n-1}),\; x_n \in \mathbb{R}^1, \\ 0 < \delta < |x'| \le \phi(x_n),\; \phi \in \mathcal{C}^{k,\alpha'}(\overline{\mathbb{R}}^1)\Big\} \; ;$$

(**e**) an exterior domain to a pipe described in (e), i.e. $\Omega = \mathbb{R}^n \backslash \Omega'$.

All statements (a)–(e) can be proved in the standard way. We recall some elements of these proofs for the sake of completeness. If $\Omega = \mathbb{R}^n_+ = \{(x', x_n),$ $x' \in \mathbb{R}^{n-1}$, $x^n \ge 0\}$, we take the following extension:

$$\mathcal{E}u(x) = \begin{cases} u(x) & \text{if } x_n \ge 0, \\ \displaystyle\sum_{s=1}^{k+1} \lambda_s\, u(x', -s x_n) & \text{if } x_n < 0 \; , \end{cases} \tag{2.24}$$

where $\lambda_s \in \mathbb{R}^1$ are such that

$$\sum_{s=1}^{k+1} \lambda_s(-s)^\beta = 1 \quad \text{for any} \quad \beta = 0, ..., k \ .$$

One easy verifies that $\nabla_{x'}^{r_1} \nabla_{x_n}^{r_2}(\mathcal{E}u) \in \mathcal{C}^{0,\alpha'}(\overline{\mathbb{R}}^n)$ or $(\in L^p(\mathbb{R}^n))$ if and only if $\nabla_{x'}^{r_1} \nabla_{x_n}^{r_2} u \in \mathcal{C}^{0,\alpha'}(\overline{\mathbb{R}}_+^n)$ or $(\in L^p(\mathbb{R}_+^n))$, $r_1, r_2 = 0, 1, ...$, and corresponding estimates hold, i.e.

$$\begin{aligned} &|\nabla_{x'}^{r_1} \nabla_{x_n}^{r_2}(\mathcal{E}u)|_{\mathcal{C}^{0,\alpha'},\mathbb{R}^n} \le c|\nabla_{x'}^{r_1} \nabla_{x_n}^{r_2} u|_{\mathcal{C}^{0,\alpha'},\mathbb{R}_+^n} \ , \\ &\|\nabla_{x'}^{r_1} \nabla_{x_n}^{r_2}(\mathcal{E}u)\|_{0,p,\mathbb{R}^n} \le c\|\nabla_{x'}^{r_1} \nabla_{x_n}^{r_2} u\|_{0,p,\mathbb{R}_+^n} \ . \end{aligned} \tag{2.25}$$

This yields the statement for $\Omega = \mathbb{R}_+^n$. If Ω is of one of the types (b), (c), (d), (e), we proceed by the partition of unity to the local description of the boundary, transforming thus the problems to the family of the similar extension problems on the half-space and on the whole space. We thus get the existence of an extension (2.23) which satisfies

$$\begin{aligned} &|\mathcal{E}u|_{\mathcal{C}^{l,\alpha'},\mathbb{R}^n} \le c|u|_{\mathcal{C}^{l,\alpha'},\Omega} \ , \\ &\|\mathcal{E}u\|_{l,p,\mathbb{R}^n} \le c\|u\|_{l,p,\Omega} \ . \end{aligned} \tag{2.26}$$

For more details see [7], Ex. 2.1.

3 – Main Theorem

The main goal of the present paper is to prove the following theorem:

Theorem 3.1. *Let $k = 0, 1, ...$, $\alpha \in (0, 1)$, $\Omega \in \mathcal{D}^{k+1,0}$ and*

$$w \in \mathcal{C}^{k+1}(\overline{\Omega}), \quad w \cdot \nu|_{\partial\Omega} = 0 \ (\text{if } \Omega \neq \mathbb{R}^n), \quad a \in \mathcal{C}_0^{k,\alpha}(\overline{\Omega}) \ , \tag{3.1}$$

$$f \in \mathcal{C}_0^{k,\alpha}(\overline{\Omega}) \ . \tag{3.2}$$

Then there exists $\gamma_1 > 0$ (dependent of k, α) such that we have: If

$$\gamma_1 \, \theta < \lambda \, , \qquad \theta = |\nabla w|_{\mathcal{C}^k} + |a|_{\mathcal{C}^{k,\alpha}} \tag{3.3}$$

then there exists just one solution $z \in \mathcal{C}^{k,\alpha}(\overline{\Omega})$ ([3]) *of problem (1.1) satisfying the estimate*

$$|z|_{\mathcal{C}^{k,\alpha}} \le \frac{1}{\lambda - \gamma_1 \theta} \, |f|_{\mathcal{C}^{k,\alpha}} \; . \tag{3.4}$$

Remark 3.1. If Ω is bounded, then the condition $a, f \in \mathcal{C}_0^{k,\alpha}(\overline{\Omega})$ is equivalent to $a, f \in \mathcal{C}^{k,\alpha}(\overline{\Omega})$. Let Ω be an exterior domain or the whole space. The reader easily verifies that a sufficient condition for a function $a \in \mathcal{C}^{k,\alpha}(\overline{\Omega})$ to belong to $\mathcal{C}_0^{k,\alpha}(\overline{\Omega})$ is e.g. $|a|_{\mathcal{C}^{k,\alpha},B_1(\xi)} \to 0$ as $|\xi| \to \infty$. Similar critera hold for other types of unbounded domains.

4 – Proof of Theorem 3.1

We start with two remarks

- It is enough to carry out the proof only for $\Omega = \mathbb{R}^n$. The general case $\Omega \in \mathcal{D}^{k+1,0}$ can be treated by means of continuous extensions, see (2.23), (2.26) and the reasoning used in the proof of Theorem 4.2 in [7]. The details are left to the reader.
- It suffices to perform the proof only with $f \in \mathcal{C}_0^{\infty}(\overline{\Omega})$. The general case $f \in \mathcal{C}_0^{k,\alpha}(\overline{\Omega})$ can be then established by the density argument, (cf. (2.4)).

The proof with $\Omega = \mathbb{R}^n$ and $f \in \mathcal{C}_0^{\infty}(\overline{\Omega})$ is devided into three steps. In the first step, we recall the existence theorem in Sobolev spaces proved in [7] (see also [2] for Ω bounded). In the second step, we consider the equation $\lambda z + w \cdot \nabla z = f$. We establish for it, the existence theorem in Slobodeckij spaces, by using the result from the first step and the interpolation of Sobolev spaces, cf. Lemma 2.2. Then, the corresponding existence statement in Holder spaces follows by using the limit process described in Lemma 2.3. The existence theorem for the complete system (1.1) follows from the previous result and the Banach contraction principle.

([3]) If $k = 0$, the solution is apriory weak, i.e. it satisfies the integral identity

$$\int_\Omega z \Big[\lambda\, \phi - w \cdot \nabla\phi + (a - \operatorname{div} w)\, \phi\Big]\, dx = \int_\Omega f\, \phi\, dx$$

for any $\phi \in \mathcal{C}_0^{\infty}(\overline{\Omega})$; in particular, it fullfils the equation (1.1) in the sense of distributions. On the other hand, once $z \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$, the identity (1.1) yields $w \cdot \nabla z \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$. Therefore, also in this case, equation (1.1) is satisfied everywhere in Ω.

First step

Since $f \in \mathcal{C}_0^\infty(\overline{\Omega})$, we have in particular $f \in W^{k,p}(\Omega)$ for any p, $1 < p < \infty$. The following statement is proved in [7], Th. 5.1, Th. 5.8 and Remarks 3.1, 8.1:

Lemma 4.1. *Let $l = 0, 1, ..., r = 0, 1, ..., l$, $1 < p < \infty$, $\Omega = \mathbb{R}^n$ and*

$$w \in \mathcal{C}^l(\overline{\mathbb{R}^n}), \quad a \in \mathcal{C}^l(\overline{\mathbb{R}^n}) \qquad (l \geq 1) \ ,$$

or

$$w \in \mathcal{C}^1(\overline{\mathbb{R}^n}), \quad a \in \mathcal{C}^0(\overline{\mathbb{R}^n}) \qquad (l = 0) \ ,$$

and

$$f \in W^{l,p}(\mathbb{R}^n) \ .$$

(**a**) *Then there exists $\gamma_1 > 0$ (dependent of l and independent of p) such that we have: If*

$$\gamma_1 \, \overline{\theta}_l < \lambda \ ,$$

where

$$\overline{\theta}_l = |\nabla w|_{\mathcal{C}^{l-1}} + |a|_{\mathcal{C}^l} \quad (l \geq 1) \, , \qquad \overline{\theta}_0 = |\nabla w|_{\mathcal{C}^0} + |a|_{\mathcal{C}^0} \quad (l = 0) \ ,$$

then the problem (1.1) possesses just one solution

$$z \in W^{l,p}(\mathbb{R}^n)$$

which satisfies the estimate

$$|z|_{r,p} \leq \frac{1}{\lambda - \gamma_1 \, \overline{\theta}_l} \, |f|_{r,p} \ . \tag{4.1}$$

(**b**) *If $\operatorname{supp} f \in B_R$, $R > 0$ and $\operatorname{supp} w \in B_R$, then $\operatorname{supp} z \in B_R$.*

Second step

Let $R > 0$ and ψ_R be a cut-off function $\psi_R(x) = \psi(x/R)$ where $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $0 \leq \psi \leq 1$ and

$$\psi(x) = \begin{cases} 1 & \text{in } B_1, \\ 0 & \text{in } B^2 \ . \end{cases} \tag{4.2}$$

We easy see that

$$\operatorname{supp} \psi_R \in B_{2R} \, , \qquad \psi_R = 1 \ \text{ in } B_R \ ,$$

$$\operatorname{supp} \nabla^r \psi_R \subset B_{2R} \backslash B_R, \quad |\nabla^r \psi_R(x)| \leq c R^{-r} \qquad (r = 1, 2, ...) \ . \tag{4.3}$$

Consider first the equation

$$\lambda\, z + w\cdot\nabla z = \mathcal{F} \quad \text{in } \ \mathbb{R}^n \ , \tag{4.4}$$

with $\mathcal{F}\in\mathcal{C}_0^\infty(\mathbb{R}^n)$, for unknown function z. Its approximation

$$\lambda\, z_R + (w\,\psi_R)\cdot\nabla z_R = \mathcal{F} \quad \text{in } \ \mathbb{R}^n \tag{4.5}$$

is a transport equation for unknown function z_R. Put

$$\overline{\theta}'_{l,R} = |\nabla(w\,\psi_R)|_{\mathcal{C}^{l-1}} \ \ (l\geq 1)\,, \qquad \overline{\theta}'_{l,R} = |\nabla(w\,\psi_R)|_{\mathcal{C}^0} \ \ (l=0)$$

and

$$\overline{\theta}'_{l} = |\nabla w|_{\mathcal{C}^{l-1}} \ \ (l\geq 1)\,, \qquad \overline{\theta}'_{l} = |\nabla w|_{\mathcal{C}^0} \ \ (l=0)\ .$$

An easy calculation shows

$$\overline{\theta}'_{l,R}\to\overline{\theta}'_l \quad \text{as } \ R\to\infty\ . \tag{4.6}$$

Applying Lemma 4.1 to the equation (4.5), we get for R "sufficiently large", existence of a universal constant $\gamma_1>0$ (independent of p, R, w, $\mathcal{F}$) such that it holds: If $\gamma_1\,\overline{\theta}'_{k+1}<\lambda/2$, then there exists a (unique) solution $z_R\in W^{k+1,p}(\mathbb{R}^n)$ of problem (4.5), with arbitrary $p\in(1,\infty)$.

Moreover $\operatorname{supp} z_R \in B_{2R}$ provided $\operatorname{supp}\mathcal{F}\in B_{2R}$. This solution satisfies estimates

$$\begin{aligned}
\|z_R\|_{k,p} &\leq \frac{1}{\lambda-\gamma_1\,\overline{\theta}'_{k+1}}\,\|\mathcal{F}\|_{k,p}\ ,\\
\|z_R\|_{k+1,p} &\leq \frac{1}{\lambda-\gamma_1\,\overline{\theta}'_{k+1}}\,\|\mathcal{F}\|_{k+1,p}\ .
\end{aligned} \tag{4.7}$$

Writing (4.5) for the differences $z_R-z_{R'}$ ($R'>R>0$), we get

$$\lambda(z_R-z_{R'}) = -\psi_R\,w\cdot\nabla(z_R-z_{R'}) + (\psi_R-\psi_{R'})\,w\cdot\nabla z_{R'}\ . \tag{4.8}$$

We calculate the following auxiliary estimates

$$\begin{aligned}
-\int_{\mathbb{R}^n}\psi_R\,w\cdot[\nabla(z_R-z_{R'})]\,|z_R-z_{R'}|^{p-2}\,(z_R-z_{R'})\,dx &=\\
&= \frac{1}{p}\int_{\mathbb{R}^n}\operatorname{div}(\psi_R\,w)\,|z_R-z_{R'}|^p\,dx\\
&\leq \frac{1}{p}\Big[|\operatorname{div} w|_{\mathcal{C}^0} + |w\cdot\nabla\psi_R|_{\mathcal{C}^0}\Big]\,\|z_R-z_{R'}\|_{0,p}^p
\end{aligned}$$

and

$$-\int_{\mathbb{R}^n} (\psi_R - \psi_{R'})\, w \cdot \nabla_{z_{R'}} |z_R - z_{R'}|^{p-2} (z_R - z_{R'})\, dx \le$$
$$\le c|w|_{\mathcal{C}^0} \, \|\nabla z_{R'}\|_{0,p,B^R} \Big(\|z_R\|^{p-1}_{0,p,B^R} + \|z_{R'}\|^{p-1}_{0,p,B^R} \Big) \ .$$

The right hand side of the last inequality tends to zero as $R, R' \to \infty$, cf. (4.3) and (4.7). Therefore, multiplying the equation (4.8) by $|z_R - z_{R'}|^{p-2}(z_R - z_{R'})$ and integrating over $\mathbb{R}^n$, we get, that z_R is a Cauchy sequence in $L^p(\mathbb{R}^n)$. Hence, there exists $z \in L^p(\mathbb{R}^n)$ such that

$$z_R \to z \quad \text{strongly in } \ L^p(\mathbb{R}^n) \ , \tag{4.9}$$

as $R \to \infty$. Obviously, z satisfies equation (4.4) in the weak sense. Moreover, (4.7) suggests that the operator $\mathcal{P}$ which maps $\mathcal{F}$ onto z_R (where z_R is the solution of the equation (4.5)), is a continuous linear operator of $W^{k,p}(\mathbb{R}^n)$ into $W^{k,p}(\mathbb{R}^n)$ and of $W^{k+1,p}(\mathbb{R}^n)$ into $W^{k+1,p}(\mathbb{R}^n)$. We therefore have by (4.7) and (2.17), (2.18):

$$\|z_R\|_{k+\alpha,p,\mathbb{R}^n} \le \frac{1}{\lambda - \gamma_1 \, \overline{\theta}'_{k+1}} \, \|\mathcal{F}\|_{k+\alpha,p,\mathbb{R}^n} \ .$$

Recalling that $\|b\|_{k+\alpha,p} = \|b\|_{k,p} + \mathcal{W}_{\alpha,p}(\nabla^k b)$ and that z_R has a compact support, we get by (2.11) and (2.19), as $p \to \infty$:

$$|z_R|_{\mathcal{C}^{k,\alpha},\mathbb{R}^n} \le \frac{1}{\lambda - \gamma_1 \, \overline{\theta}'_{k+1}} \, |\mathcal{F}|_{\mathcal{C}^{k,\alpha},\mathbb{R}^n} \ . \tag{4.10}$$

From the imbeddings (2.12), (2.13), with p "sufficiently large", and taking into account the estimate $(4.7)_2$, we deduce that for any B_R, there exist a chosen subsequence $\{R_i\}_{i=1}^{\infty}$ ($R_i \to \infty$ as $i \to \infty$), and a $\overline{z} \in \mathcal{C}^{k,\alpha}(\overline{B}_R)$ such that

$$z_{R_i} \to \overline{z} \quad \text{strongly in } \ \mathcal{C}^{k,\alpha}(\overline{B}_R) \ . \tag{4.11}$$

In virtue of (4.9),

$$z = \overline{z} \quad \text{a.e. in } \ B_R \ .$$

In the other words, $z \in \mathcal{C}^{k,\alpha}(\overline{B}_R)$ for any $R > 0$. Moreover, in virtue of (4.10), z satisfies the estimate

$$|z|_{\mathcal{C}^{k,\alpha},B_R} \le \frac{1}{\lambda - \gamma_1 \, \overline{\theta}'_{k+1}} \, |\mathcal{F}|_{\mathcal{C}^{k,\alpha},\mathbb{R}^n} \ , \tag{4.12}$$

uniformly with respect to R. This means, in particular, that

$$z \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{R}}^n)$$

and the validity of the estimate

$$(4.13) \qquad |z|_{\mathcal{C}^{k,\alpha},\mathbb{R}^n} \le \frac{1}{\lambda - \gamma_1\, \overline{\theta}'_{k+1}}\, |\mathcal{F}|_{\mathcal{C}^{k,\alpha},\mathbb{R}^n} \ .$$

Third step

Consider a linear map

$$(4.14) \qquad \mathcal{L}\colon\ q \to z \ ,$$

formally defined as follows: for a given q, the function z is a solution of the transport equation

$$(4.15) \qquad \lambda\, z + w \cdot \nabla z = f - a\, q \ ,$$

with $f \in \mathcal{C}_0^{k,\alpha}(\mathbb{R}^n)$. If $a \in \mathcal{C}_0^{k,\alpha}(\mathbb{R}^n)$ and $q \in \mathcal{C}_0^{k,\alpha}(\mathbb{R}^n)$, then $aq \in \mathcal{C}_0^{k,\alpha}(\mathbb{R}^n)$. Hence (4.14) defines a linear map of $\mathcal{C}_0^{k,\alpha}(\mathbb{R}^n)$ into itself. Moreover, it holds

$$|a\, q|_{\mathcal{C}^{k,\alpha},\mathbb{R}^n} \le c|a|_{\mathcal{C}^{k,\alpha},\mathbb{R}^n}\, |q|_{\mathcal{C}^{k,\alpha},\mathbb{R}^n} \ , \qquad c > 0 \ ,$$

with c dependent only of k, α. The estimate (4.13) applies to the equation (4.15). It furnishes

$$(4.16) \qquad (\lambda - \gamma_1\, \overline{\theta}'_{k+1})\, |z|_{\mathcal{C}^{k,\alpha},\mathbb{R}^n} \le |f|_{\mathcal{C}^{k,\alpha},\mathbb{R}^n} + c|a|_{\mathcal{C}^{k,\alpha},\mathbb{R}^n}\, |q|_{\mathcal{C}^{k,\alpha},\mathbb{R}^n} \ .$$

This yields the contraction of $\mathcal{L}$ provided

$$c|a|_{\mathcal{C}^{k,\alpha},\mathbb{R}^n} < \lambda - \gamma_1\, \overline{\theta}'_{k+1} \ .$$

By the Banach contraction priniciple (see e.g. Zeidler [10]), $\mathcal{L}$ possesses a unique fixed point $q = z$ which obviously satisfies equation (1.1). Now, the estimate (3.4) follows directly from the inequality (4.16) written at the fixed point $q = z$. Theorem 2.1 is thus proved. ∎

5 – The decay of solutions

In this section we investigate the decay of solutions guaranteed by Theorem 3.1. We limit ourselves to the case $\Omega = \mathbb{R}^n$. The generalisations to the arbitrary unbounded domains $\Omega \in \mathcal{D}^{1,0}$ are possible by the same reasoning. We let them to the interested reader.

Let $m \in (0,\infty)$ and $\Omega = \mathbb{R}^n$. We denote by $\mathcal{C}^0_m(\mathbb{R}^n) = \mathcal{C}^0_m(\overline{\mathbb{R}}^n)$ the Banach space of all continuous functions on $\mathbb{R}^n$ with the finite norm

$$|z|_{\mathcal{C}^0_m} = \sup_{x\in\mathbb{R}^n} \Big|(1+|x|)^m\, z\Big|_{\mathcal{C}^0} \,. \tag{5.1}$$

By the Holder space $\mathcal{C}^{0,\alpha}_m(\mathbb{R}^n) = \mathcal{C}^{0,\alpha}_m(\overline{\mathbb{R}}^n)$, $0<\alpha<1$, we denote a Banach space of continuous functions on $\mathbb{R}^n$ with finite norm

$$|z|_{\mathcal{C}^{0,\alpha}_m} = |z|_{\mathcal{C}^0_m} + \sup_{\xi\in\mathbb{R}^n} \Big\{(1+|\xi|)^m\, \mathcal{H}_{\alpha,B_1(\xi)}(z)\Big\} \,, \tag{5.2}$$

where

$$\mathcal{H}_{\alpha,B_1(\xi)}(z) = \sup_{x,y\in B_1(\xi)} \frac{|z(x)-z(y)|}{|x-y|^\alpha} \,. \tag{5.3}$$

Let $\gamma>0$, but fixed. We realize that

$$|\cdot|_{\mathcal{C}^0_m} \quad \text{and} \quad |\cdot|'_{\mathcal{C}^0_m} = \sup_{\xi\in\mathbb{R}^n}\Big\{\Big|(1+|\xi|)^m\,\cdot\Big|_{\mathcal{C}^0,B_\gamma(\xi)}\Big\} \tag{5.5}$$

are equivalent norms in $\mathcal{C}^0_m(\overline{\mathbb{R}}^n)$ and

$$\begin{gathered} |\cdot|_{\mathcal{C}^{0,\alpha}_m} \quad \text{and} \\ |\cdot|'_{\mathcal{C}^{0,\alpha}_m} = |\cdot|'_{\mathcal{C}^0_m} + \sup_{\xi\in\mathbb{R}^n}\Big\{(1+|\xi|)^m\,\mathcal{H}_{\alpha,B_\gamma(\xi)}(\cdot)\Big\} \end{gathered} \tag{5.6}$$

are equivalent norms in $\mathcal{C}^{0,\alpha}_m(\overline{\mathbb{R}}^n)$; i.e.

$$\begin{aligned} c_1|\cdot|'_{\mathcal{C}^0_m} &\le |\cdot|_{\mathcal{C}^0_m} \le c_2|\cdot|'_{\mathcal{C}^0_m} \,, \\ c_1|\cdot|'_{\mathcal{C}^{0,\alpha}_m} &\le |\cdot|_{\mathcal{C}^{0,\alpha}_m} \le c_2|\cdot|'_{\mathcal{C}^{0,\alpha}_m} \,. \end{aligned} \tag{5.7}$$

The coefficients c_1, c_2 in (5.7) are positive and depend only of m, γ and α.

Theorem 5.1. *Let $m\in(0,\infty)$, $\alpha\in(0,1)$, $\Omega=\mathbb{R}^n$ and*

$$\begin{gathered} w \in \mathcal{C}^1(\overline{\Omega})\,, \qquad a \in \mathcal{C}^{0,\alpha}_0(\overline{\Omega}) \,, \\ f \in \mathcal{C}^{0,\alpha}_m(\overline{\Omega}) \,. \end{gathered} \tag{5.8}$$

Then there exists $\overline{\gamma}_1 > 0$ (dependent of α, m) such that we have: If

$$\overline{\gamma}_1\,\theta' < \lambda\,, \qquad \theta' = |\nabla w|_{\mathcal{C}^0} + |a|_{\mathcal{C}^{0,\alpha}} \,, \tag{5.9}$$

then there exists just one solution $z \in \mathcal{C}^{0,\alpha}_m(\overline{\Omega})$ of the problem (1.1) satisfying the estimate

$$|z|_{\mathcal{C}^{0,\alpha}_m} \le \frac{c}{\lambda - \overline{\gamma}_1\,\theta'}\,|f|_{\mathcal{C}^{0,\alpha}_m}\,. \tag{5.10}$$

Here $c > 0$ depends only of α, m.

Proof: Let ψ be a cut-off function (4.2). For fixed $\xi \in \mathbb{R}^n$, we define

$$\psi_{\xi,R}(x) = \psi\Big(\frac{x-\xi}{R}\Big)\,, \qquad \widetilde{\psi}_{\xi,R}(x) = \psi\Big(\frac{2(x-\xi)}{R}\Big)\,. \tag{5.11}$$

One easy sees that

$$\begin{aligned} \psi_{\xi,R}(x) &= \begin{cases} 1 & \text{in } B_R(\xi), \\ 0 & \text{in } B^{2R}(\xi)\,, \end{cases} \\ \widetilde{\psi}_{\xi,R}(x) &= \begin{cases} 1 & \text{in } B_{R/2}(\xi), \\ 0 & \text{in } B^{R}(\xi)\,. \end{cases} \end{aligned} \tag{5.12}$$

Moreover

$$\operatorname{supp} \nabla\psi_{\xi,R} \subset B_{2R}(\xi)\backslash B_R(\xi)\,, \qquad \operatorname{supp} \nabla\widetilde{\psi}_{\xi,R} \subset B_R(\xi)\backslash B_{R/2}(\xi) \tag{5.13}$$

and

$$|\nabla^r\psi_{\xi,R}(x)|, \ \ |\nabla^r\widetilde{\psi}_{\xi,R}(x)| \le c\,R^{-r} \qquad (r = 1, 2, ...)\,, \tag{5.14}$$

where $c > 0$ is independent of R.

Let $f \in \mathcal{C}^{0,\alpha}_m(\overline{\mathbb{R}}^n)$ i.e. in particular, $f \in \mathcal{C}^{0,\alpha}_0(\mathbb{R}^n)$. Theorem 3.1 thus guarantees the existence of a (unique) solution $z \in \mathcal{C}^{0,\alpha}(\overline{\mathbb{R}}^n)$. Multiplying equation (1.1) by $\widetilde{\psi}_{\xi,R}$ we find out that $\widetilde{z} = z\,\widetilde{\psi}_{\xi,R}$ satisfies the following equation

$$\lambda\,\widetilde{z} + \psi_{\xi,R}\,w\cdot\nabla\widetilde{z} + a\,\widetilde{z} = f\,\widetilde{\psi}_{\xi,R} + z\,w\cdot\nabla\widetilde{\psi}_{\xi,R} \quad \text{ in } \ B_{2R}(\xi)\,. \tag{5.15}$$

The right hand side of equation (5.15) belongs to $\mathcal{C}^{0,\alpha}(\overline{B}_{2R}(\xi))$ and its $\mathcal{C}^{0,\alpha}(\overline{B}_{2R}(\xi))$-norm is estimated by

$$|f|_{\mathcal{C}^{0,\alpha},\mathbb{R}^n} + \frac{c}{R}\,|z|_{\mathcal{C}^{0,\alpha},B_{2R}(\xi)}\,|w|_{\mathcal{C}^{1},B_{2R}(\xi)}\,, \tag{5.16}$$

with c independent of R. The coellicient θ (see (3.3)–(3.4)) for the equation (5.15), reads $\theta_R = |\nabla(w\,\psi_{\xi,R})|_{\mathcal{C}^0,B_{2R}(\xi)} + |a|_{\mathcal{C}^{0,\alpha},B_{2R}(\xi)}$; it is less or equal than

$$\widetilde{\theta}' = c\Big(|\nabla w|_{\mathcal{C}^0,\mathbb{R}^n} + \frac{1}{R}\,|w|_{\mathcal{C}^0,\mathbb{R}^n} + |a|_{\mathcal{C}^{0,\infty},\mathbb{R}^n}\Big)\,. \tag{5.17}$$

Theorem 3.1 applied to the equation (5.15) yields (after some calculation) the following statement: there exists a positive number $\overline{\gamma}_1$ (independent of R, α, ξ, w, a, f), such that if $\overline{\gamma}_1\,\widetilde{\theta}' < \lambda/2$, then it holds

$$|\widetilde{z}|_{\mathcal{C}^{0,\alpha},B_{2R}(\xi)} \le \frac{c'}{\lambda-\overline{\gamma}_1\,\widetilde{\theta}'}\,|\widetilde{f}\,\widetilde{\psi}_{\xi,R}|_{\mathcal{C}^{0,\alpha},B_{2R}(\xi)} \tag{5.18}$$

(with $c'>0$ independent of R, α, ξ, w, a, f). This implies immediately

$$|z|_{\mathcal{C}^{0,\alpha},B_{R/2}(\xi)} \le \frac{c}{\lambda-\overline{\gamma}_1\,\widetilde{\theta}'}\,|f|_{\mathcal{C}^{0,\alpha},B_{2R}(\xi)}\ . \tag{5.19}$$

Multiplying the last inequality by $(1+|\xi|)^m$, we get

$$\begin{aligned}(1+|\xi|)^m\,|z|_{\mathcal{C}^0_m,B_{R/2}(\xi)} + (1+|\xi|)^m\,\mathcal{H}_{\alpha,B_{R/2}(\xi)}(z) &\le \\ \le \frac{c}{\lambda-\overline{\gamma}_1\,\widetilde{\theta}'}\Big\{|f|_{\mathcal{C}^0_m,B_{2R}(\xi)} &+ (1+|\xi|)^m\,\mathcal{H}_{\alpha,B_{2R}(\xi)}(f)\Big\}\ .\end{aligned} \tag{5.20}$$

Further, we realize that $\widetilde{\theta}' \to \theta'$ as $R\to\infty$. The supremum in (5.20), over all $\xi\in\mathbb{R}^n$, furnishes, when using the equivalence of norms (see (5.5), (5.6)), the estimate (5.10). Theorem 5.1 is thus proved. ∎

Remark 5.1.

- As we already pointed out, Theorem 5.1 can be reformulated for more general weights (see (5.18) in [7]) and for any unbounded domain in the class $\mathcal{D}^{1,0}$. These generalisations, with evident modifications in the proofs, are left to the interested reader. In particular, the theorem holds, as it states for Ω an exterior domain of $\mathbb{R}^n$ with $\partial\Omega\in\mathcal{C}^1$. In this case, one has to assume, in addition to (5.8), the supplementary condition $w\cdot\nu|_{\partial\Omega}=0$.
- Theorem 5.1 implies in particular: Any solution $z\in\mathcal{C}^{0,\alpha}(\overline{\mathbb{R}}^n)$ of the problem (1.1) with the right hand side $f\in\mathcal{C}^{0,\alpha}_m(\mathbb{R}^n)$ and the coefficients w, a satisfying (3.1), belongs automatically to the class $\mathcal{C}^{0,\alpha}_m(\mathbb{R}^n)$.

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