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ON THE DIOPHANTINE EQUATION $D_1x^2 + D_2^m = 4y^n$

M. Mignotte

Abstract: Consider positive integers D_1, D_2 and let h be the class-number of the imaginary quadratic field $Q(\sqrt{-D_1\,D_2})$. We study the equation $D_1x^2 + D_2^m = 4y^n$ when m is an odd integer, n an odd prime number, $\gcd(D_1x, D_2y) = 1$, $n \nmid h$, and also $y > 1$.

Le Maohua [L] obtained the following result.

Theorem A. *Let D_1 and D_2 be positive integers. Consider the diophantine equation*

$$(*) \qquad D_1\,x^2 + D_2^m = 4\,y^n \;,$$

where m is an odd integer, n an odd prime number, $\gcd(D_1x, D_2y) = 1$, $n \nmid h$, where h is the class-number of the imaginary quadratic field $Q(\sqrt{-D_1\,D_2})$, and $y > 1$. Then the equation $(*)$ has a finite number of solutions (D_1, D_2, x, y, m, n). Moreover, the solutions satisfy $4y^n < \exp\exp 470$ and $n < 8 \cdot 10^6$.

The aim of the present note is to sharpen the upper bound on n and, which is more important to prove that (under some extra hypothesis) for each pair (D_1, D_2) there is at most one solution n. Namely, our result is the following.

Theorem 1. *Consider the diophantine equation $D_1x^2 + D_2^m = 4y^n$ with the same hypotheses as in Theorem A. Then, in the range $2 \le y \le 5000$ the only solutions with $n > 5$ are $n = 7$ for $y \in \{2, 3, 4, 5\}$ and $n = 13$ for $y = 2$, and the solutions for $n = 5$ are obtained for*

$$y \in \Big\{2, 3, 4, 5, 7, 8, 11, 13, 18, 21, 29, 34, 47, 55, 76, 89, 123, 144, 199, 233, \\ 322, 377, 521, 610, 843, 987, 1364, 1597, 2207, 2584, 3571, 4181\Big\} \;.$$

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For $5000 < y < 43000$ *there is at most one solution* $n \geq 5$. *For* $y \geq 43000$ *there is at most one solution* $n \geq 3$. *Moreover, in all cases* $n \leq 52000$.

Our proof follows rather closely Le Maohua's. It is proved in [L] that the previous hypotheses lead to a relation of the type

$$\frac{\varepsilon^n - \bar{\varepsilon}^n}{\varepsilon - \bar{\varepsilon}} = \begin{cases} \pm n, & \text{if } m > 1 \text{ and } n \mid D_2, \\ \pm 1, & \text{otherwise ,} \end{cases} \tag{1}$$

where

$$\varepsilon = \frac{X_1\sqrt{D_1} + Y_1\sqrt{-D_2}}{2}, \qquad \bar{\varepsilon} = \frac{X_1\sqrt{D_1} - Y_1\sqrt{-D_2}}{2}, \tag{2}$$

with X_1 and Y_1 positive and

$$D_1 X_1^2 + D_2 Y_1^2 = 4y = 4|\varepsilon|^2, \qquad \gcd(D_1X_1, D_2Y_1) = 1 .$$

Put $\varepsilon = |\varepsilon|\, e^{i\theta}$ (with $|\theta| \leq \pi$). Notice that $\sin\theta = \frac{Y_1\sqrt{D_2}}{2|\varepsilon|}$, $\cos\theta = \frac{X_1\sqrt{D_1}}{2|\varepsilon|}$ so that $0 < \theta < \pi/2$, and also that (1) implies $|2\sin(n\theta)| \leq n\,|\varepsilon|^{-n}\,|\varepsilon - \bar{\varepsilon}|$. Thus, when $2\,n\,|\varepsilon^{1-n}| \leq 1$ the relations

$$|e^{2in\theta} - 1| = |2\sin(n\theta)| \leq n|\varepsilon|^{-n}\,|\varepsilon - \bar{\varepsilon}| \leq 2n|\varepsilon|^{1-n}$$

imply $|\sin(n\theta)| \leq 1/2$, so that there exists an integer k^*, with $0 \leq k^* < n/2$ (recall that n is odd) such that

$$2|n\theta - k^*\pi| \leq 2\arcsin(1/2) \times n|\varepsilon|^{1-n}\,|\varepsilon - \bar{\varepsilon}| < 1.048\,n|\varepsilon|^{-n}\,|\varepsilon - \bar{\varepsilon}| \leq 2.096\,n|\varepsilon|^{1-n} .$$

This prove that the "linear form in logs"

$$\Lambda = n\log\left(\frac{\varepsilon}{\bar{\varepsilon}}\right) - k\,i\,\pi = (2\,n\,\theta - k\,\pi)\,i\ , \tag{3}$$

where $k = 2k^*$ (thus $0 \leq k < n$), satisfies

$$\log|\Lambda| \leq -n\log|\varepsilon| + \log\Big(\pi\, n|\varepsilon - \bar{\varepsilon}|/3\Big) \leq -n\log|\varepsilon| + \log\Big(2.096\,n|\varepsilon|\Big)\ , \tag{4}$$

if $n \geq 11$ when $y = 2$, if $n \geq 7$ when $y = 3$, if $n \geq 5$ when $4 \leq y \leq 6$ and for any $n \geq 3$ for $y \geq 7$.

We prove that $k \neq 0$. Indeed, if $k = 0$ the relation $\Lambda = 2n\theta i$ and (4) imply $|\varepsilon|^{n-1} \leq 1.1$ which does not hold.

To get a lower bound for $|\Lambda|$ we apply [L-M-N], Th. 3, which gives (we denote by $h(\alpha)$ the logarithmic absolute height of an algebraic number)

$$\log|\Lambda| \geq -9\, a\, H^2 \ ,$$

where

$$a = \max\Big\{20,\ 12.85|\log(\varepsilon/\overline{\varepsilon})| + Dh(\varepsilon/\overline{\varepsilon})\Big\}, \qquad D = \Big[Q(\overline{\varepsilon}/\varepsilon) : Q\Big]/2 \ ,$$

$$H = \max\Big\{17,\ D\log\Big(\frac{|k|}{2a} + \frac{n}{25.7\,\pi}\Big) + 4.6\,D + 3.25\Big\} \ ,$$

here

$$h(\varepsilon/\overline{\varepsilon}) = \log|\varepsilon|\,, \qquad 0 < \log(\overline{\varepsilon}/\varepsilon) < \pi \ ,$$

and $D = 1$ since $\overline{\varepsilon}/\varepsilon$ satisfies $y(\overline{\varepsilon}/\varepsilon)^2 - \frac{1}{2}(D_1X_1^2 - D_2Y_1^2)(\varepsilon/\overline{\varepsilon}) + y = 0$. This gives

$$\log|\Lambda| \geq -9\max\{20,\ 25.70 + \log|\varepsilon|\} \times \max\{17,\ \log n + 4.57\}^2 \ . \tag{5}$$

Comparing (4) and (5) we get the fundamental inequality

$$\begin{aligned} n\log|\varepsilon| \leq 9\Big(&\max\{20,\ 25.7\,\theta + \log|\varepsilon|\} \times \max\{17,\ \log n + 4.57\}^2\Big) \\ &+ \log\Big(2.096\,n|\varepsilon|\Big) \ . \end{aligned} \tag{6}$$

Using the inequalities $y \geq 2$ and $\theta < \pi/2$, we get $a \leq 118.5\log|\varepsilon|$, and (6) implies

$$n < 683000 \ . \tag{7}$$

But, for larger y we get a better upper bound for n, for example

$$y \geq 50 \ \Rightarrow\ n \leq 110000\,, \qquad y \geq 5000 \ \Rightarrow\ n \leq 52000 \ . \tag{8}$$

Also, $y > 10^4$ implies $n < 48300$ and $y > 10^{500}$ implies $n < 3000$.

Using the upper bound (7), a numerical verification shows that equation (1) has no solution with $n > 5$ and $y \leq 50$ except $n = 7$ for $y \in \{2,3,4,5\}$ and $n = 13$ for $y = 2$, and that the solutions for $n = 5$ are obtained for $y \in \{2,3,4,5,7,8,11,13,18,21,29,34,47\}$.

Thus, we may suppose that $y > 50$, in this case it is easy to verify that if n is a solution of (4) then n is a denominator of a principal convergent of the real number $2\theta/\pi$, which permits to speed up the verification.

Using the first of inequalities (8), we have verified that equation (1) has no solution with $n > 5$ in the range $50 < y \leq 5000$, and that, in this range, the solutions with $n = 5$ occur for

$$y \in \Big\{55, 76, 89, 123, 144, 199, 233, 322, 377, 521, 610,$$
$$843, 987, 1364, 1597, 2207, 2584, 3571, 4181\Big\}\,.$$

Suppose now that $y > 5000$ and that n' is a second solution to (1) with $n' > n \geq 3$, and n' prime. Then, by (8), we have $n' < 52000$, and there exists an even integer k' such that $0 < k' < n'$ and

$$(3') \qquad \Lambda' = n' \log\Big(\frac{\varepsilon}{\overline{\varepsilon}}\Big) - k'\, i\, \pi$$

satisfies

$$(4') \qquad \log|\Lambda'| \leq -n' \log|\varepsilon| + \log\Big(2.096\, n'|\varepsilon|\Big)\,.$$

Then (because $0 < k < n$ and $k' \neq 0$)

$$2\pi \leq |n\,k' - n'k|\,\pi = |n'\Lambda - n\,\Lambda'| \leq 2.096\, n\, n'|\varepsilon|^{-n+1}(1 + |\varepsilon|^{-2}) \leq 2.1\, n\, n'|\varepsilon|^{-n+1}\,.$$

Thus

$$n' \geq \pi\,\frac{|\varepsilon|^{n-1}}{1.05\, n}\,.$$

This leads to a contradiction for $n \geq 5$. Thus there is at most one solution $n \geq 5$ for $y > 5000$. In the same way, there is at most one solution $n \geq 3$ for $y \geq 43000$.

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Maurice Mignotte,
Université Louis Pasteur, Mathématique,
67084 Strasbourg – FRANCE
E-mail: `mignotte@math.u-strasbg.fr`