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ON THE DISCRETIZATION OF DEGENERATE SWEEPING PROCESSES

M. Kunze and M.D.P. Monteiro Marques

Abstract: We prove existence theorems for evolution equations of the form $-u'(t) \in \partial\delta_{C(t)}(Au(t))$ with some maximal monotone and strongly monotone operator $A\colon D(A) \to 2^H$.

1 – Introduction and main results

We study the evolution problem

$$-u'(t) \in \partial\delta_{C(t)}(Au(t)) \quad \text{a.e. in } [0,T], \quad u(0) = u_0 \in D(A) \;, \tag{1}$$

where A is a maximal monotone and strongly monotone operator in the real Hilbert space H, and $t \mapsto C(t)$ is a Hausdorff-continuous multifunction with closed convex values. Equations of this form arise from problems of the type $-x'(t) \in \partial\delta_C(g(t,x(t)))$, which play an important rôle in elasticity theory, cf. [8, 3, 4] for more information. To solve (1) means that we have to find $u \in W^{1,1}([0,T];H)$ and $v \in L^2([0,T];H)$ such that $u(0) = u_0$,

$$u(t) \in D(A) \text{ a.e.}, \quad v(t) \in Au(t) \cap C(t) \text{ a.e. and } -u'(t) \in \partial\delta_{C(t)}(v(t)) \text{ a.e. in } [0,T] \,.$$

Our general assumptions are

(H1) $A\colon D(A) \to 2^H \backslash \{\emptyset\}$ is a maximal monotone operator (abbreviated mmop) such that $A = \partial\psi$ for some lsc, convex and proper $\psi\colon H \to \mathbb{R} \cup \{\infty\}$, and there exists a $\beta > 0$ such that

$$\langle Ax - Ay,\, x - y\rangle \geq \beta\,|x-y|^2 \quad \text{for } x,y \in D(A) \;. \tag{2}$$

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(**H2**) For every $t \in [0, T]$, $\emptyset \neq C(t) \subset H$ is a closed convex set, and $t \mapsto C(t)$ is Lipschitz continuous, in that for some $L \geq 0$

$$\mathrm{d_H}\Big(C(t), C(s)\Big) \leq L|t-s| \quad \text{for} \;\; t, s \in [0, T] \;, \tag{3}$$

and we obtain the following result.

Theorem 1. *Let (H1) and (H2) be satisfied. If in addition*

(**H3a**) $C(0)$ *is bounded, or*

(**H3b**) *there exists a function* $M\colon\ [0,\infty[\ \to [0,\infty[$ *which maps bounded sets into bounded sets such that*

$$\|Ax\| = \sup\Big\{|y|\colon\ y \in Ax\Big\} \leq M(|x|) \quad \textit{for} \;\; x \in D(A) \;, \tag{4}$$

and

(**H4a**) $D(A) \cap \overline{B}_R(0)$ *is relatively compact for every* $R > 0$*, or*

(**H4b**) $C(t) \cap \overline{B}_R(0)$ *is compact for every* $t \in [0, T]$ *and* $R > 0$*,*

then (1) has a Lipschitz continuous solution, for every $u_0 \in D(A)$ *with* $Au_0 \cap C(0) \neq \emptyset$*.*

In the bounded linear case we can do much better. Here the result is

Theorem 2. *Let* $A\colon\ H \to H$ *be linear, bounded and selfadjoint such that* $\langle Ax, x\rangle \geq \beta\,|x|^2$ *for* $x \in H$*. If (H2) holds for* $C(\cdot)$ *and if* $Au_0 \in C(0)$*, then (1) has a unique solution, and this solution is Lipschitz continuous.*

To discuss these theorems, we first remark that our proof relies on a concrete and constructive discretization method, contrary to [4], where related results were obtained in a more complicated way by Yosida–Moreau approximation of A and $\partial\delta_{C(t)}$. Theorem 1 in particular covers all results from [4] with A being a subdifferential. Moreover, the conditions (H3a) and (H4a) are easier to satisfy in applications. We also note that taking $A = \mathrm{id}$ in Theorem 2 gives the known existence theorem (cf. [6, 7] or [5, p. 141] and the references given therein) for the classical sweeping process in infinite dimensions (including uniqueness), and therefore seems to be a more natural extension than [4]; the latter covered the classical sweeping process only in case that $\dim H < \infty$. We also refer to [4] for additional references concerning non-standard variants of the classical sweeping process.

Already simple examples show that this Theorem 2 (and hence also Theorem 1) might be wrong if $\beta = 0$ in (2), cf. Example 3 in Section 2 below. Although (H3) and (H4) are needed only for proof-technical reasons, we guess that (H1) and (H2) are not enough to ensure the existence of a solution to (1). These conditions (H3) and (H4) play a rôle as follows: from (H1) and (H2) alone it is possible to construct two approximating sequences satisfying $u_n(t) \in D(A)$, $v_n(t) \in Au_n(t)$ and $v_n(t)$ "almost in" $C(t)$. Moreover, $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in norm and variation. Then (H3) is used to ensure that also $(v_n)_{n\in\mathbb{N}}$ is uniformly bounded, and thus w.l.o.g. $u_n \to u$ and $v_n \to v$ weakly in $L^2([0,T];H)$ for some functions u and v. But to conclude $v(t) \in Au(t)$ a.e., one of the weak convergences has to be improved to a strong convergence, and for this (H4) is needed. It is clear that in concrete special cases, e.g. if $\mathcal{A}$ (the realization of A in $L^2([0,T];H)$) has weakly-weakly-closed graph (as is the case for linear A), then no additional compactness condition is needed. It should be noted that (H3b) is a restrictive condition, since it enforces $D(A) = H$. [Indeed, $A^{-1}\colon H \to D(A)$ is a mmop, and $A = (A^{-1})^{-1}$ is locally bounded, so $H = R(A^{-1}) = D(A)$.]

We also remark that our results remain true, if $t \mapsto C(t)$ is only assumed to be absolutely continuous, i.e. $\mathrm{d_H}(C(t), C(s)) \le |a(t) - a(s)|$ for some ac. function $a\colon [0,T] \to \mathbb{R}$, the difference being only that the solution obtained is also only ac. instead of Lipschitz continuous. Condition (3) was only imposed to simplify the proof.

The paper is organized as follows. In Section 2 we introduce some notation and state some preliminary results which will be used to establish the existence of the discretization resp. to prove convergence of the approximants. Moreover, we included some easy counterexamples concerning uniqueness of solutions and the case $\beta = 0$ in (2). Section 3 contains the construction of the approximations and the derivation of uniform bounds under assumptions (H1) and (H2). The proofs of Theorem 1 and Theorem 2 are carried out in Sections 4 and Section 5, respectively.

2 – Notation and preliminaries

Our notations are quite standard, cf. [1, 2, 5]. So $\langle\cdot,\cdot\rangle$ denotes the inner product in H, and for a mmop A, $D(A)$ resp. $R(A) = \bigcup_{x\in D(A)} Ax$ are the domain of definition resp. the range of A. For a closed convex $C \subset H$ the set $\partial\delta_C(x) = N_C(x) = \{y \in H\colon \langle y, c - x\rangle \le 0 \;\forall\, c \in C\}$, $x \in C$, denotes the normal cone to C at x. Also,

$$\mathrm{d_H}(C_1, C_2) = \max\Big\{\sup_{x\in C_2} \operatorname{dist}(x, C_1),\ \sup_{x\in C_1} \operatorname{dist}(x, C_2)\Big\}$$

with $\operatorname{dist}(x, C_1) = \inf\{|x-y|\colon\, y \in C_1\}$ for $C_1, C_2 \subset H$ is the Hausdorff distance between the sets C_1 and C_2.

To establish the existence of approximate solutions to (1), we introduce the following notation.

Definition 1. Let A be a mmop in H such that (2) holds, and let $C \subset H$ be closed convex. A map $D(A) \ni u \mapsto P_{A,C}(u) \in D(A)$ is called the *approximation operator*, if for every $u \in D(A)$ there exists a $P_{A,C}(u) = w \in D(A)$ such that $u - w \in N_C(Aw)$, i.e. $u - w \in N_C(v)$ for some $v \in Aw \cap C$.

Remark 1. Due to (2) the element $P_{A,C}(u)$ is unique. Indeed, assume that also for some $\overline{w} \in D(A)$ we have $u - \overline{w} \in N_C(\overline{v})$ for some $\overline{v} \in A\overline{w} \cap C$. Because $N_C(\cdot)$ is monotone it follows that $0 \le \langle [u-w] - [u-\overline{w}],\, v - \overline{v}\rangle = \langle v - \overline{v},\, \overline{w} - w\rangle$. Since $v \in Aw$ and $\overline{v} \in A\overline{w}$, this implies $w = \overline{w}$ by means of (2).

Remark 2. $P_{A,C}(\cdot)$ exists iff $R(A^{-1}+N_C) \supset D(A)$. To see this, we note first that by (2), A^{-1} is locally bounded, hence A is onto by [2, Théorème 2.3]. Thus $A^{-1}\colon\, H \to D(A)$ is monotone, single-valued, and $1/\beta$-Lipschitz, because of (2). Thus $A^{-1} + N_C$ is a mmop with domain of definition C, cf. [2, Lemme 2.4]. Now $R(A^{-1}+N_C) \supset D(A)$ iff for $u \in D(A)$ we find $v \in C$ such that $u \in A^{-1}v + N_C(v)$. Letting $w = A^{-1}v$ this yields $v \in Aw$ and $u - w \in N_C(v)$.

This gives a simple criterion on when the approximation operator can be defined.

Lemma 1. *Let A be a mmop such that (2) holds. If $C \subset H$ is nonempty, closed, convex and bounded, then $P_{A,C}(\cdot)$ exists.*

Proof: In this case, $A^{-1} + N_C$ is a mmop with bounded domain C, hence $A^{-1} + N_C$ is onto by [2, Corollaire 2.2]. ∎

The dependence of $P_{A,C}(\cdot)$ on C is studied in the next lemma.

Lemma 2. *Let A be a mmop satisfying (2) and let $C_1, C_2 \subset H$ be such that $P_{A,C_2}(\cdot)$ exists. If $u_1 \in D(A)$ with $Au_1 \cap C_1 \neq \emptyset$, then*

$$|u_1 - P_{A,C_2}(u_1)| \le \frac{1}{\beta}\, \mathrm{d_H}(C_1, C_2)\ .$$

Proof: Fix $v_1 \in Au_1 \cap C_1$ and $v_2 \in Au_2 \cap C_2$ such that $u_1 - u_2 \in N_{C_2}(v_2)$, with $u_2 = P_{A,C_2}(u_1)$. Hence $\langle u_1 - u_2,\, z - v_2\rangle \le 0$ for all $z \in C_2$, and thus by (2) for these z

$$\beta\,|u_1-u_2|^2 \le \langle u_1-u_2,\, v_1-v_2\rangle \le \langle u_1-u_2,\, v_1-z\rangle \le |u_1-u_2|\,|v_1-z|\ .$$

Consequently, since $v_1\in C_1$,

$$|u_1-u_2| \le \frac{1}{\beta}\,\mathrm{dist}\,(v_1,C_2) \le \frac{1}{\beta}\,\mathrm{d_H}(C_1,C_2)\ ,$$

as was claimed. ∎

Next we collect some further preliminary results on mmops and convex functions which will be needed later on.

Lemma 3. *Let $\psi:\ H\to\mathbb{R}\cup\{\infty\}$ be lsc, convex and proper. If $u\in W^{1,2}([0,T];H)$ and $u(t)\in D(\partial\psi)$ a.e. in $]0,T[$, and if there exists $v\in L^2([0,T];H)$ such that $v(t)\in\partial\psi(u(t))$, then the function $t\mapsto\psi(u(t))$ is a.c. on $[0,T]$. Moreover,*

$$\frac{d}{dt}[\psi\circ u](t) = \langle u'(t),\, v(t)\rangle \qquad \text{a.e. in}\ \ ]0,T[\ .$$

Proof: Cf. [2, Lemme 3.3, p. 73]. ∎

Lemma 4. *Let $\psi:\ H\to\mathbb{R}\cup\{\infty\}$ be lsc, convex and proper. If $x_n\to x$ weakly in H, then $\liminf_{n\to\infty}\psi(x_n)\ge\psi(x)$.*

Proof: By using Mazur's theorem, or [1, Chapter 1, Proposition 1.5]. ∎

The next lemma will be used to approximate possibly unbounded $C(t)$.

Lemma 5. *Let $t\mapsto C(t)$ satisfy (H2). Then there exists an $n_0\in\mathbb{N}$ such that for all $n\ge n_0$ we have $C_n(t):=C(t)\cap\overline{B}_n(0)\neq\emptyset$ for $t\in[0,T]$, and*

$$\mathrm{d_H}(C_n(t),C_n(s)) \le 8\,\mathrm{d_H}(C(t),C(s)) \le 8\,L\,|t-s|\,, \qquad t,s\in[0,T]\ .$$

Proof: We choose a continuous selection $z\colon\ [0,T]\to H$ of $C(\cdot)$, e.g. by solving the usual sweeping process with the convex and Lipschitz-continuous moving set $t\mapsto C(t)$. Fixing $n_1\ge|z|_\infty+2$, in particular the first claim holds for $n\ge n_1$. To show the estimate on $\mathrm{d_H}$, we will use [9, Section 8, p. 169/170], cf. the proof of this result. First, $C(t)\cap\mathrm{int}\,\overline{B}_n(0)\neq\emptyset$ for $n\ge n_1$ and $t\in[0,T]$, since $z(t)$ is contained in this set. Moreover, the function $t\mapsto\mathrm{diam}(C(t)\cap\overline{B}_n(0))$ is bounded by $D = 2n$. Finally,

$$\begin{aligned} e(C(t), H\backslash\overline{B}_n(0)) &= \sup_{x\in C(t)} \operatorname{dist}(x, H\backslash\overline{B}_n(0)) \geq \operatorname{dist}(z(t), H\backslash\overline{B}_n(0)) \\ &\geq n - |z|_\infty > n - (n_1 - 1) = n - n_1 + 1 =: \varrho \; . \end{aligned}$$

Thus by choosing $\alpha = \varrho/2$ in the formula derived in the proof of [9, Proposition on p. 169], for $t \in [0, T]$ by (H2)

$$\begin{aligned} \mathrm{d_H}(C_n(t), C_n(s)) &\leq \left(\frac{\varrho + D}{\varrho/2}\right) \mathrm{d_H}(C(t), C(s)) = 2\left(\frac{3n - n_1 + 1}{n - n_1 + 1}\right) \mathrm{d_H}(C(t), C(s)) \\ &\leq 8\,\mathrm{d_H}(C(t), C(s)) \leq 8\,L\,|t - s| \; , \end{aligned}$$

at least, if $n \geq 3(n_1 - 1)$. Thus we can define $n_0 = 3(n_1 - 1) \geq n_1$ to obtain the claim. ∎

Finally we will make some remarks concerning uniqueness of solution to (1) and the assumption on the strong monotonicity of A. Since A is in general multivalued, one can not expect solutions (u, v) to be unique. This is shown by the following simple

Example 1: Let $H = \mathbb{R}$, $T = 1$, $D(A) = \{0\}$, $A(0) = \mathbb{R}$ and $u_0 = 0$. Then $A = \partial\delta_0$ is a subdifferential and (2) holds with every $\beta > 0$. Hence necessarily $u(t) = 0$ in $[0, 1]$ for a solution (u, v), but v only has to be a selection of $C(\cdot)$.

The next example shows that we cannot allow $\beta = 0$ in (2), i.e. it is not enough that A be only maximal monotone.

Example 2: Let $H = \mathbb{R}$, $T = 1$, $Au = u + 1$ for $u \leq -1$, $Au = 0$ for $-1 \leq u \leq 1$ and $Au = u - 1$ for $u \geq 1$. Also let $C(t) = [t, 1]$ for $0 \leq t \leq 1$. Then A is a maximal monotone graph and hence a subdifferential. Moreover, (2) holds with $\beta = 0$ and $C(\cdot)$ is $\mathrm{d_H}$-Lipschitz with constant $L = 1$. Suppose that (1) has a solution (u, Au) with initial values $u_0 = 0 \in D(A)$ and $v_0 = Au_0 = 0 \in C(0)$. Then, by continuity of u, we have $|u(t)| \leq 1$ for $0 \leq t \leq \delta$ for some $\delta > 0$. Therefore $v(t) = Au(t) = 0$ a.e. in $[0, \delta]$, contradicting $v(t) \in C(t) = [t, 1]$ a.e.

In dimensions $d \geq 2$ it is possible to modify the last example also to have a counterexample for a bounded linear selfadjoint A.

Example 3: Let $H = \mathbb{R}^2$, $T = 1$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $C(t) = [0, 1] \times [t, 1]$ for $0 \leq t \leq 1$. Here A is linear, bounded, selfadjoint, and (2) holds with $\beta = 0$. Also

$C(\cdot)$ is $\mathrm{d_H}$-Lipschitz. Again there can be no solution (u, Au) of (1) with initial value $u_0 = (0,0) \in D(A)$, since this would imply $Au(t) = (u_1(t), 0) \in C(t)$ a.e. in $[0,1]$, a contradiction.

3 – Discretization and bounds

In this section we will assume that (H1) and (H2) are satisfied. Under these hypotheses we will establish the existence of approximative solutions, and we will derive several auxiliary results on the approximations.

For every $n \in \mathbb{N}$ we define an approximative solution $u_n : [0,T] \to H$ as follows. Fix the partition $0 = t_0^n < t_1^n < ... < t_n^n = T$ of $[0,T]$ with $t_i^n = i\,T/n$ for $i = 0, ..., n$. Then

$$|t_{i+1}^n - t_i^n| = \frac{T}{n} \quad \text{for} \ \ i = 0, ..., n-1 \ . \tag{5}$$

By Lemma 5, $C_n(t) := C(t) \cap \overline{B}_n(0) \neq \emptyset$, $t \in [0,T]$, for all sufficiently large $n \in \mathbb{N}$. Since by hypothesis $Au_0 \cap C(0) \neq \emptyset$, there exists $v_0 \in H$ such that we have $v_0 \in Au_0 \cap C_n(0)$ for large n. Because all sets $C_n(t)$ are bounded, it follows from Lemma 1 that the approximation operators $P_{A,C_n(t)}(\cdot)$ exist for large n, say for $n \geq n_0$.

Let $u_0^n = u_0 \in D(A)$. Thus by Lemma 1 there is $u_1^n = P_{A,C_n(t_1^n)}(u_0^n) \in D(A)$, and hence we find $v_1^n \in Au_1^n \cap C_n(t_1^n)$ such that $u_0^n - u_1^n \in N_{C_n(t_1^n)}(v_1^n)$. Moreover, since $Au_0^n \cap C_n(0) \neq \emptyset$, Lemma 2 and Lemma 5 imply

$$|u_0^n - u_1^n| \leq \beta^{-1}\, \mathrm{d_H}(C_n(0), C_n(t_1^n)) \leq \frac{8}{\beta}\, \mathrm{d_H}(C(0), C(t_1^n)) \ .$$

Because again $u_1^n \in D(A)$ and $Au_1^n \cap C_n(t_1^n) \neq \emptyset$, we can proceed in this way to get $u_i^n \in D(A)$ and $v_i^n \in Au_i^n \cap C_n(t_i^n)$ for $i = 1, ..., n$ such that

$$\begin{gathered} u_i^n - u_{i+1}^n = u_i^n - P_{A,C_n(t_{i+1}^n)}(u_i^n) \in N_{C_n(t_{i+1}^n)}(v_{i+1}^n) \quad \text{and} \\ |u_i^n - u_{i+1}^n| \leq \frac{8}{\beta}\, \mathrm{d_H}(C(t_i^n), C(t_{i+1}^n)) \end{gathered} \tag{6}$$

for $i = 0, ..., n-1$. We then define

$$u_n(t) = u_i^n \ \text{ and } \ v_n(t) = v_i^n \quad \text{for} \ \ t \in [t_i^n, t_{i+1}^n[, \quad i = 0, ..., n-1 \ , \tag{7}$$

with $v_0^n := v_0$. Moreover, we also let $u_n(T) = u_n^n$ and $v_n(T) = v_n^n$. Set

$$\theta_n(t) = t_i^n \ \text{ for } t \in [t_i^n, t_{i+1}^n[, \quad i = 0, ..., n-1, \qquad \text{and} \qquad \theta_n(T) = T \ .$$

The above definitions yield $u_n(0) = u_0$, $v_n(0) = v_0$,

$$u_n(t) \in D(A) \quad \text{and} \quad v_n(t) \in Au_n(t) \cap C_n(\theta_n(t)) \subset Au_n(t) \cap C(\theta_n(t)) \quad \text{for} \quad t \in [0,T] \;. \tag{8}$$

This implies, by (H2) and (5),

$$v_n(t) \in C(t) + \overline{B}_{LT/n}(0) \quad \text{for} \quad n \in \mathbb{N}, \quad t \in [0,T] \;. \tag{9}$$

Moreover, as a consequence of (2),

$$|u_n(t) - u_m(t)| \le \beta^{-1}\, |v_n(t) - v_m(t)| \quad \text{for} \quad n,m \in \mathbb{N}, \quad t \in [0,T] \;. \tag{10}$$

We also have

$$|u_n(t) - u_n(s)| \le \frac{8L}{\beta}\, [t - s + T/n] \quad \text{for} \quad 0 \le s \le t \le T \;. \tag{11}$$

Indeed, if w.l.o.g. $s \in [t_i^n, t_{i+1}^n[$ and $t \in [t_j^n, t_{j+1}^n[$ with $j \ge i+1$, then we obtain from (6) and (3)

$$\begin{aligned} |u_n(t) - u_n(s)| = |u_j^n - u_i^n| \le \sum_{k=i}^{j-1} |u_k^n - u_{k+1}^n| \le \frac{8}{\beta} \sum_{k=i}^{j-1} \mathrm{d_H}(C(t_k^n), C(t_{k+1}^n)) \le \\ \le \frac{8L}{\beta} \sum_{k=i}^{j-1} [t_{k+1}^n - t_k^n] = \frac{8L}{\beta}\, [t_j^n - t_i^n] \le \frac{8L}{\beta}\, [t - t_i^n] \le \frac{8L}{\beta}\, [t - s + T/n] \;, \end{aligned}$$

and hence (11). To obtain the convergence of the sequence $(u_n)_{n \ge n_0}$ constructed above, we first note that (6) and (3) imply uniformly in n, as in the proof of (11),

$$\mathrm{var}\,(u_n; 0, T) = \sum_{i=0}^{n-1} |u_{i+1}^n - u_i^n| \le \frac{8L}{\beta}\, T \;.$$

Since $u_n(0) = u_0$, we also get

$$|u_n(t)| \le |u_0| + \frac{8L}{\beta}\, T \quad \text{for} \quad n \ge n_0, \quad t \in [0,T] \;, \tag{12}$$

so that the sequence $(u_n)_{n \ge n_0}$ is uniformly bounded in norm and variation. Hence, cf. [5, Theorem 0.2.1], we find a function $u\colon [0,T] \to H$ of bounded variation and a subsequence, for simplicity again indexed with $n \in \mathbb{N}$ ($n \ge n_0$), such that

$$u_n(t) \to u(t) \quad \text{weakly for all} \quad t \in [0,T] \;, \tag{13}$$

and

$$u_n \to u \quad \text{weakly in} \;\; L^2([0,T];H) \; . \tag{14}$$

In particular,

$$u(0) = u_0 \;, \tag{15}$$

and by (11)

$$|u(t)-u(s)| \le \liminf_{n\to\infty} |u_n(t)-u_n(s)| \le \frac{8L}{\beta}\,[t-s] \quad \text{for} \;\; 0\le s\le t\le T \; .$$

Thus u is Lipschitz continuous, and hence differentiable a.e. with bounded derivative u'.

To take a first step towards proving the validity of $-u'(t)\in N_{C(t)}(Au(t))$ a.e., we will show

Lemma 6. *In the situation considered above, for all continuous* $z\colon [0,T]\to H$ *being a selection of* $C(\cdot)$

$$\int_{[0,T]} z(t)\cdot du(t) \ge \psi(u(T)) - \psi(u_0) \; . \tag{16}$$

In addition, if $u(t)\in D(A)$ *a.e. in* $[0,T]$*, and if for some* $v\in L^2([0,T];H)$ *one has* $v(t)\in Au(t)\cap C(t) = \partial\psi(u(t))\cap C(t)$ *a.e. in* $[0,T]$*, then*

$$-u'(t) \in N_{C(t)}(v(t)) \quad \text{a.e. in} \;\; [0,T] \; . \tag{17}$$

Proof: Since z is bounded, we have $z(t)\in C_n(t)$, $t\in[0,T]$, for all $n\in\mathbb{N}$ sufficiently large. We first remark that by construction $v^n_{i+1}\in Au^n_{i+1} = \partial\psi(u^n_{i+1})$. Thus $\langle v^n_{i+1},\, u^n_{i+1}-x\rangle \ge \psi(u^n_{i+1})-\psi(x)$ for all $x\in H$, and hence (6) implies

$$\langle \overline{z},\, u^n_{i+1}-u^n_i\rangle \ge \langle v^n_{i+1},\, u^n_{i+1}-u^n_i\rangle \ge \psi(u^n_{i+1})-\psi(u^n_i) \;\; \text{for} \; \overline{z}\in C_n(t^n_{i+1}) \; . \tag{18}$$

Define the approximation $z_n(t) := z(t^n_{i+1}) \in C_n(t^n_{i+1})$ for $t\in\,]t^n_i, t^n_{i+1}]$ and $z_n(0) := z(0)$. Then $z_n(t)\to z(t)$ uniformly on $[0,T]$ by (5). Since $t=0$ is not an atom of du_n and since u_n is right-continuous, we obtain from (18)

$$\begin{aligned}
\int_{[0,T]} z_n(t)\cdot du_n(t) &= \int_{]0,T]} z_n(t)\cdot du_n(t) = \sum_{i=0}^{n-1} \int_{]t^n_i,t^n_{i+1}]} z_n(t)\cdot du_n(t) = \\
&= \sum_{i=0}^{n-1} z(t^n_{i+1})\cdot \int_{]t^n_i,t^n_{i+1}]} du_n(t) = \sum_{i=0}^{n-1} \Big\langle z(t^n_{i+1}),\, u_n(t^n_{i+1})-u_n(t^n_i)\Big\rangle \\
&\ge \sum_{i=0}^{n-1} [\psi(u^n_{i+1})-\psi(u^n_i)] = \psi(u^n_n)-\psi(u^n_0) = \psi(u_n(T))-\psi(u_0) \; .
\end{aligned} \tag{19}$$

By the uniform convergence $z_n \to z$ we find $|\int_{[0,T]} z_n \cdot du_n - \int_{[0,T]} z \cdot du_n| \to 0$, and the continuity of u and [5, Theorem 0.2.1] yield $\lim_{n\to\infty} \int_{[0,T]} z \cdot du_n = \int_{[0,T]} z \cdot du$. Thus, by (13) and Lemma 4 it results from (19) that $\int_{[0,T]} z(t) \cdot du(t) \geq \psi(u(T)) - \psi(u_0)$ for every continuous selection $z\colon [0,T] \to H$ of $C(\cdot)$. Thus we have shown (16).

In case that there exists a $v \in L^2([0,T];H)$ such that $v(t) \in \partial\psi(u(t))$ a.e., this can be used as follows. Since also $u(t) \in D(A)$ a.e. in $[0,T]$, by Lemma 3 $\psi \circ u$ is a.c. with

$$\frac{d}{dt}[\psi \circ u](t) = \langle u'(t),\, v(t)\rangle \quad \text{ a.e. in } \; [0,T] \;.$$

Thus for all continuous selections $z\colon [0,T] \to H$ of $C(\cdot)$

$$\int_{[0,T]} z(t) \cdot du(t) \geq \psi(u(T)) - \psi(u_0) = \int_{[0,T]} \langle u'(t),\, v(t)\rangle \, dt \;.$$

This yields (17) analogously to [10, Proposition 6], cf. [5, p. 144]. ∎

Next we will state two results about properties of limit functions.

Lemma 7. *In the situation considered above, if for some $v \in L^2([0,T];H)$*

$$v_n \to v \quad \textit{weakly in } \; L^2([0,T];H) \;,$$

then

$$v(t) \in C(t) \quad \textit{a.e. in } \; [0,T] \;. \tag{20}$$

Proof: Fix $\varepsilon > 0$ and let $\mathcal{C}_\varepsilon = \{\phi \in L^2([0,T];H)\colon\ \phi(t) \in C(t) + \overline{B}_\varepsilon(0) \text{ a.e.}\}$. Then $\mathcal{C}_\varepsilon$ is closed and convex, hence weakly closed, and $v_n \in \mathcal{C}_\varepsilon$ for large n, by (9). Thus $v \in \mathcal{C}_\varepsilon$ for all $\varepsilon > 0$. Since every $C(t)$ is closed, the claim follows. ∎

Lemma 8. *In the situation considered above, if for some $v \in L^2([0,T];H)$ one has $v_n \to v$ weakly in $L^2([0,T];H)$ and $u_n \to u$ strongly in $L^2([0,T];H)$, or $v_n \to v$ strongly in $L^2([0,T];H)$, then*

$$v(t) \in Au(t) \quad \textit{a.e. in } \; [0,T] \;. \tag{21}$$

Proof: Consider the realization of A in $L^2([0,T];H)$, i.e. $\mathcal{A}\xi = \{\phi \in L^2([0,T];H)\colon\ \phi(t) \in A\xi(t) \text{ a.e.}\}$ for $\xi \in D(\mathcal{A}) = \{\widetilde{\xi} \in L^2([0,T];H)\colon\ \widetilde{\xi}(t) \in D(A)$ a.e.$\}$. Then $\mathcal{A}$ is maximal monotone in $L^2([0,T];H)$, cf. [2, Exemple 2.3.3], and

$(u_n, v_n) \in \mathcal{A}$ by (8). Thus $(u, v) \in \mathcal{A}$ in the first case, because graph($\mathcal{A}$) is strongly-weakly-closed, cf. [2, Proposition 2.5]. In the second case the argument is the same, by (14), since graph($\mathcal{A}$) is also weakly-strongly-closed. ∎

Later on we will also need continuous approximations of u. For that, we define

$$\overline{u}_n(t) = \frac{t - t_i^n}{t_{i+1}^n - t_i^n} \, (u_{i+1}^n - u_i^n) + u_i^n \quad \text{ for } \; t \in [t_i^n, t_{i+1}^n], \quad i = 0, ..., n-1 \;.$$

By (7), (6), (3) and (5) we obtain

$$|\overline{u}_n(t) - u_n(t)| \le \frac{8LT}{\beta n} \quad \text{ for all } \; n \in \mathbb{N}, \quad t \in [0, T] \;. \tag{22}$$

Hence (13) yields

$$\overline{u}_n(t) \to u(t) \quad \text{ weakly for all } \; t \in [0, T] \;. \tag{23}$$

Moreover, by (22) and (11),

$$|\overline{u}_n(t) - \overline{u}_n(s)| \le \frac{8L}{\beta} \, [t - s + 3T/n] \quad \text{ for } \; n \in \mathbb{N}, \quad 0 \le s \le t \le T \;.$$

Therefore the sequence $(\overline{u}_n)_{n \in \mathbb{N}} \subset C([0, T]; H)$ is equicontinuous.

4 – Proof of Theorem 1

In this section, we will derive consequences of (H3) and (H4) which directly yield the claim of Theorem 1 in the considered cases.

4.1. Consequences of (H3)

Assume first that $C(0)$ is bounded. Then (H2) shows that for some $R_1 > 0$

$$\bigcup_{t \in [0,T]} C(t) \subset \overline{B}_{R_1}(0) \;.$$

This implies by (8)

$$|v_n(t)| \le R_1 \quad \text{ for } \; n \in \mathbb{N}, \quad t \in [0, T] \;. \tag{24}$$

The $v_n(t)$ are also uniformly bounded, if (4) holds. Indeed, let $R_2 = |u_0| + \frac{8L}{\beta} T$. Then by assumption $M([0, R_2]) \subset [0, R_3] \subset \mathbb{R}$ for some sufficiently large $R_3 > 0$.

Hence $v_n(t) \in Au_n(t)$ and (12) imply that (24) is satisfied with R_1 replaced by R_3. Consequently, under either condition we obtain w.l.o.g. that for some $v \in L^2([0,T];H)$ we have $v_n \to v$ weakly in $L^2([0,T];H)$. Thus, by Lemma 7, (20) holds, i.e. $v(t) \in C(t)$ a.e.

4.2. Consequences of (H4)

We will show that under one of the conditions (H4a) or (H4b),

$$u(t) \in D(A) \quad \text{and} \quad v(t) \in Au(t) \qquad \text{a.e. in} \quad [0,T] \;, \tag{25}$$

with v from Section 4.1. This in turn gives, by Section 4.1 and Lemma 6, the differential inclusion (17). Summing up, then (15), (20), (25) and (17) prove that (u,v) is a solution of (1). In particular, the argument to be given will show that in order that (25) holds, it is enough to prove that $\{u_n(t):\ n \geq n_0\} \subset H$ is relatively compact for every $t \in [0,T]$.

So assume first that (H4a) holds, i.e. $D(A) \cap \overline{B}_R(0) \subset H$ is relatively compact for every $R > 0$. To prove uniform convergence of the equicontinuous sequence $(\overline{u}_n)_{n\geq n_0} \subset C([0,T];H)$ of continuous approximations, we take $R = |u_0| + \frac{8L}{\beta}\,T$. For all $N \geq n_0$, by (22), (8), and (12), we have

$$\begin{aligned}\Big\{\overline{u}_n(t)\colon\ n \geq N\Big\} &\subset \Big\{u_n(t)\colon\ n \geq N\Big\} + \overline{B}_{8LT/\beta N}\,(0)\\ &\subset (D(A) \cap \overline{B}_R(0)) + \overline{B}_{8LT/\beta N}\,(0)\;.\end{aligned}$$

Therefore $\{\overline{u}_n(t):\ n \geq n_0\} \subset H$ is relatively compact for every $t \in [0,T]$, and consequently by (23) and by Arzelà–Ascoli's theorem, w.l.o.g. $|\overline{u}_n - u|_\infty \to 0$ as $n \to \infty$. This, together with (22), implies $u_n \to u$ in $L^2([0,T];H)$, so that by Lemma 8 we obtain (25).

Next assume that (H4b) holds, i.e. $C(t) \cap \overline{B}_R(0) \subset H$ is relatively compact for every $t \in [0,T]$ and $R > 0$. By Section 4.1 we know that (24) is satisfied with some suitable $R_1 > 0$. We take $R = 2R_1$ and consider $N \in \mathbb{N}$ so large that $LT/N \leq R_1$. Then by (9) for all sufficiently large $N \in \mathbb{N}$

$$\begin{aligned}\Big\{v_n(t)\colon\ n \geq N\Big\} &\subset \Big(C(t) + \overline{B}_{LT/N}(0)\Big) \cap \overline{B}_{R_1}(0)\\ &\subset \Big(C(t) \cap \overline{B}_R(0)\Big) + \overline{B}_{LT/N}(0)\;.\end{aligned}$$

This implies by assumption that $\{v_n(t)\colon\ n \geq n_0\} \subset H$ is relatively compact for all $t \in [0,T]$. Consequently, (10) yields that the same is true for $\{u_n(t)\colon\ n \geq n_0\}$, and this was all we had to show according to the above remark.

Hence the proof of Theorem 1 is finished in all considered cases. ∎

5 – Proof of Theorem 2

We first note that in particular $A = \partial\psi$ for the lsc, convex and proper $\psi(x) = \frac{1}{2}\langle Ax, x\rangle$. Thus (H1) and (H2) hold, and therefore the results derived in Section 3 are valid. Also $\|Ax\| = |Ax| \le |A|\,|x|$, so that the second condition in (H3) is satisfied with $M(r) = |A|\,r$. Hence also the uniform bound (24) for the $|v_n(t)|$ from Section 4.1 remains true, and we may assume $v_n \to v$ weakly in $L^2([0,T];H)$ for some $v \in L^2([0,T];H)$. As for (25) it is enough to show that $v(t) = Au(t)$ a.e. in $[0,T]$, because $D(A) = H$. This can be achieved as follows. Since $v_n(t) = Au_n(t)$ by (8) and $u_n(t) \to u(t)$ weakly for every $t \in [0,T]$ by (13), the symmetry of A yields $v_n(t) \to Au(t)$ weakly for every $t \in [0,T]$, and this implies $v(t) = Au(t)$ a.e. in $[0,T]$, as it was to be shown in order to prove that (u,v) is a solution of (1).

To prove uniqueness, suppose that (u,v) and $(\overline{u},\overline{v})$ are solutions. Then a.e. in $[0,T]$ we have $u'(t)\cdot(z - v(t)) \ge 0$ for all $z \in C(t)$. Since $\overline{v}(t) \in C(t)$ a.e. we find $u'(t)\cdot(\overline{v}(t) - v(t)) \ge 0$ a.e. Exchanging the rôles and adding both inequalities we get $\langle v(t) - \overline{v}(t),\, u'(t) - \overline{u}'(t)\rangle \le 0$ a.e. with $v(t) = Au(t)$ and $\overline{v}(t) = A\overline{u}(t)$. Moreover, a.e. in $[0,T]$

$$\frac{d}{dt}\Big\langle Au(t) - A\overline{u}(t),\, u(t) - \overline{u}(t)\Big\rangle = 2\Big\langle Au(t) - A\overline{u}(t),\, u'(t) - \overline{u}'(t)\Big\rangle \le 0 \;,$$

and therefore integration and $u(0) = \overline{u}(0) = u_0$ gives for $t \in [0,T]$

$$\beta|u(t) - \overline{u}(t)|^2 \le \Big\langle Au(t) - A\overline{u}(t),\, u(t) - \overline{u}(t)\Big\rangle \le 0 \;.$$

This completes the proof of Theorem 2. ∎

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M. Kunze,
Mathematisches Institut der Universität Köln,
Weyertal 86, D - 50931 Köln – GERMANY

and

M.D.P. Monteiro Marques,
C.M.A.F. and Faculdade de Ciências da Universidade de Lisboa,
Av. Prof. Gama Pinto 2, P - 1699 Lisboa – PORTUGAL