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ON THE DECAY ESTIMATES FOR THE WAVE EQUATION WITH A LOCAL DEGENERATE OR NONDEGENERATE DISSIPATION

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Abstract: In a bounded domain, we consider the wave equation with a local dissipation. We prove the polynomial decay of the energy for a degenerate dissipation and the exponential decay of the energy for a nondegenerate dissipation. The method of proof is direct and is based on multipliers technique, on some integral inequalities due to Haraux and on a judicious idea of Conrad and Rao.

1 – Introduction and statement of the main results

The main purpose of this paper is to give precise decay estimates for the wave equation with a dissipation localized in a neighbourhood of a suitable subset of the domain under consideration. Throughout the paper, we use the following notations. Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) having a smooth boundary $\Gamma = \partial\Omega$. We denote by ν the unit normal pointing into the exterior of Ω. We fix $x^0 \in \mathbb{R}^N$ and we set $m(x) = x - x^0$,

$$R = \sup\Big\{|m(x)|,\ x \in \Omega\Big\}, \quad \Gamma_+ = \Big\{x \in \Gamma;\ m(x)\cdot\nu(x) > 0\Big\} \ \text{ and } \ \Gamma_- = \Gamma\backslash\Gamma_+$$

$(u \cdot v = \displaystyle\sum_1^N u_i\, v_i$ for all $u, v \in \mathbb{R}^N)$. Let $a = a(x) \in L^\infty(\Omega)$ be a nonnegative

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bounded function such that

$$\exists\, p > 0\colon \int_\omega \frac{dx}{a^p} < \infty \tag{1.1}$$

or

$$a(x) \geq a_0 > 0 \quad \text{a.e. in } \ \omega\ , \tag{1.2}$$

where ω is a neighbourhood of Γ_+ and a_0 is a positive constant. By neighbourhood of Γ_+, we actually mean the intersection of Ω and a neighbourhood of Γ_+. Throughout the paper, we denote by $|a^{-1}|_p$ the quantity $(\int_\omega \frac{dx}{a^p})^{\frac{1}{p}}$ and by $|u|_r$ the norm of a function $u \in L^r(\Omega)$, $1 \leq r \leq \infty$.

Now consider the following damped wave equation

$$\begin{cases} y'' - \Delta y + a\, y' = 0 & \text{in } \Omega \times (0,\infty), \\ y = 0 & \text{on } \Gamma \times (0,\infty), \\ y(0) = y^0 & \text{in } \Omega, \\ y'(0) = y^1 & \text{in } \Omega\ . \end{cases} \tag{1.3}$$

When the function a satisfies (1.1) (resp. (1.2)), we say that the dissipation $a\, y'$ is degenerate (resp. nondegenerate). Let $\{y^0, y^1\} \in H_0^1(\Omega) \times L^2(\Omega)$. System (1.3) is then well-posed in the space $H_0^1(\Omega) \times L^2(\Omega)$; in fact, there exists a unique weak solution of (1.2) with

$$y \in \mathcal{C}([0,\infty); H_0^1(\Omega)) \cap \mathcal{C}^1([0,\infty); L^2(\Omega))\ . \tag{1.4}$$

Introduce the energy

$$E(t) = \frac{1}{2} \int_\Omega \Big\{ |y'(x,t)|^2 + |\nabla y(x,t)|^2 \Big\}\, dx\,, \qquad \forall\, t \geq 0\ . \tag{1.5}$$

The energy E is a nonincreasing function of the time variable t and we have for almost every $t \geq 0$,

$$E'(t) = -\int_\Omega a\, |y'|^2\, dx\ . \tag{1.6}$$

Our purpose in this paper is to prove that the energy decays

– polynomially when the function a satisfies (1.1),

– exponentially when the function a satisfies (1.2)

and to give a precise decay rate in each case.

The semi-group approach or microlocal analysis or differential inequalities are the methods used by the authors to establish exponential or polynomial decay when the damping is effective only in an open nonvoid subset of the domain Ω (cf. Bardos *et al* [1], Cheng *et al* [2], Haraux [6], Nakao [13] Zuazua [16, 17]). Here we give an alternative approach based on some integral inequalities due to Haraux [4, 5]; the advantage here is that we provide a direct proof without using either the semi-group theory nor a unique continuation result. Our method essentially relies on the multipliers technique (cf. Lions [11], Komornik [8]). We emphasize on the fact that apart from the constructive approach of Haraux [6], the authors working in this framework have used microlocal analysis or a compactness-uniqueness argument to prove the decay estimate of the energy (cf. Bardos *et al* [1], Nakao [13], Zuazua [16, 17]). The unique continuation property and the compactness argument used by Nakao and Zuazua permit to the authors to get rid of some lower order terms. Here, we proceed in a different way by introducing an auxiliary elliptic problem whose solution is used as multiplier. This type of approach was used by Conrad and Rao in [3] to study the nonlinear boundary stabilization of the wave equation.

For the sequel we need the following definition of Nakao [13]

Definition. Let a be a smooth function. We say that $\{y^0, y^1\} \in H^{m+1}(\Omega) \times H^m(\Omega)$ satisfies the compatibility condition of m^{th} order associated to (1.3) if

$$y^k \in H^{m+1-k}(\Omega) \cap H_0^1(\Omega), \quad \text{for } k \in \{0, 1, ..., m\}, \qquad \text{and} \qquad y^{m+1} \in L^2(\Omega)$$

where the functions y^k are given by

$$y^k = \Delta y^{k-2} - a\, y^{k-1}, \qquad k \in \{2, 3, ..., m+1\}\ . \tag{1.6}$$

We have the following existence and regularity result

Theorem 1.0. *Let m be a nonnegative integer. Let $\{y^0, y^1\} \in H^{m+1}(\Omega) \times H^m(\Omega)$ $(H^0(\Omega) = L^2(\Omega))$ satisfy the compatibility condition of m^{th} order associated to (1.3). Suppose that $a \in \mathcal{C}^{m-1}(\bar{\Omega})$ ($a \in L^\infty$ if $m = 0$).*

Then the solution y of (1.2) satisfies

$$y \in \bigcap_{k=0}^{m} \mathcal{C}^k([0,\infty); H^{m+1-k}(\Omega) \cap H_0^1(\Omega)) \cap \mathcal{C}^{m+1}([0,\infty); L^2(\Omega))\ . \tag{1.7}$$

Moreover, if we set

$$F_m = \left(\|y^1\|^2_{H^m(\Omega)} + \|y^0\|^2_{H^{m+1}(\Omega)}\right)^{\frac{1}{2}} \tag{1.8}$$

then there exits a positive constant c such that

$$\|y'(t)\|_{H^m(\Omega)} \le c\, F_m\ , \quad \|y(t)\|_{H^{m+1}(\Omega)} \le c\, F_m, \qquad \text{for a.e. } t \ge 0\ . \tag{1.9}$$

For the proof of this result we refer the interested reader to Pazy [14].

The main results of this paper are the following

Theorem 1.1. *Let m be a positive integer. Let $\{y^0, y^1\} \in H^{m+1}(\Omega) \times H^m(\Omega)$ satisfy the compatibility condition of m^{th} order associated to (1.3). Let ω be a neighbourhood of Γ_+. Suppose that $a \in \mathcal{C}^{m-1}(\bar{\Omega})$ satisfies* (1.1) *with*

$$\begin{cases} 0 < p < \infty & \text{if } N \in \{1, 2, ..., 2m\}, \\ N - 2m \le m\,p & \text{if } N \ge 2m + 1\ . \end{cases} \tag{1.10}$$

Then, for $1 \le N < 2m$, we have the decay estimate

$$E(t) \le K_0 \Big(|a^{-1}|_p \, F_m^{\frac{N}{mp}} + E(0)^{\frac{N}{2mp}} \Big)^{\frac{2mp}{N}} \, t^{-\frac{2mp}{N}}\ , \qquad \forall\, t > 0\ , \tag{1.11}$$

where K_0 is a positive constant independent of the initial data.

For $N \ge 2m$, the energy E satisfies

$$E(t) \le K_1 \Big(|a^{-1}|_p^2 \, F_m^{\frac{2N}{mp}} + E(0)^{\frac{N}{mp}} \Big)^{\frac{mp}{N}} \, t^{-\frac{mp}{N}}\ , \qquad \forall\, t > 0\ , \tag{1.12}$$

where K_1 is a positive constant independent of the initial data.

Theorem 1.2. *Let $\{y^0, y^1\} \in H_0^1(\Omega) \times L^2(\Omega)$. Assume that $a \in L_+^\infty(\Omega)$ satisfies* (1.2) *for some $a_0 > 0$. Let ω be a neighbourhood of Γ_+.*

Then there exists a positive constant τ_0, independent of the initial data such that

$$E(t) \le \left[\exp\Big(1 - \frac{t}{\tau_0} \Big) \right] E(0)\ , \qquad \forall\, t \ge 0\ . \tag{1.13}$$

Remark 1.1. Theorem 1.1 extends Theorem 1 of Nakao [13]. In fact under the same hypotheses on the data, Nakao only proved estimate (1.11). Moreover, the proof of Theorem 1.1 presented below is direct in the sense that we do not use any compactness argument whereas Nakao did in the proof of Theorem 1 of [13]. Therefore the constants in the estimations obtained by Nakao are not explicit. As in Nakao [12], we observe that as the solutions become smoother, the decay becomes more rapid so that the degeneracy of the function a is compensated by the regularity of the solutions. We also point out that for very high dimensions, the function a may not be too degenerate.

Remark 1.2. Since (1.2) implies (1.1), Theorem 1.2 can be viewed as a limiting case of Theorem 1.1 as p tends to infinity.

The remainder of the paper is organized as follows. In section 2, we give some lemmas which are useful for the proofs of Theorems 1.1 and 1.2. Section 3 is devoted to the proofs of Theorems 1.1 and 1.2.

2 – Some preliminary Lemmas

The proofs of Theorems 1.1 and 1.2 rely on the following lemmas.

Lemma 2.1 (Gagliardo-Nirenberg)**.** *Let* $1 \le q \le s \le \infty$, $1 \le r \le s$, $0 \le k < m < \infty$, *(k and m are nonnegative integers) and* $\delta \in [0,1]$. *Let* $v \in W^{m,q}(\Omega) \cap L^r(\Omega)$. *Suppose that*

$$k - \frac{N}{s} \le \delta\Big(m - \frac{N}{q}\Big) - \frac{N(1-\delta)}{r} \;. \tag{2.1}$$

Then $v \in W^{k,s}(\Omega)$ *and there exists a positive constant* C *such that*

$$\|v\|_{W^{k,s}(\Omega)} \le C\, \|v\|^{\delta}_{W^{m,q}(\Omega)}\, |v|_r^{1-\delta} \;. \tag{2.2}$$

Lemma 2.2. *Let* $E\colon\ [0,\infty[\ \to [0,\infty[$ *be a nonincreasing locally absolutely continuous function such that there are nonnegative constants* β *and* A *with*

$$\int_S^\infty E(t)^{\beta+1}\, dt \le A\, E(S)\,, \qquad \forall\, S \ge 0 \;. \tag{2.3}$$

Then we have

$$E(t) \le \begin{cases} \Big[\exp\Big(1 - \dfrac{t}{A}\Big)\Big]\, E(0), & \forall\, t \ge 0 \quad \textit{if } \beta = 0, \\[2ex] \Big(A\Big(1 + \dfrac{1}{\beta}\Big)\Big)^{\frac{1}{\beta}}\, t^{-\frac{1}{\beta}}, & \forall\, t > 0 \quad \textit{if } \beta > 0 \;. \end{cases} \tag{2.4}$$

This lemma is due to Haraux and its proof can be found in [4, 5] or [8, 9], [10]. This lemma reduces the proofs of Theorems 1.1–1.2 to the proofs of estimates of type (2.3).

From now on, we denote by S and T two real numbers such that $0 \le S < T < \infty$. We write E instead of $E(t)$.

Lemma 2.3. *Let* $\mu \geq 0$, $q \in (W^{1,\infty}(\Omega))^N$, $\alpha \in \mathbb{R}$ *and* $\xi \in W^{1,\infty}(\Omega)$. *We have the identities*

$$
\begin{aligned}
(2.5)\qquad & \int_\Omega y'\{2q\cdot\nabla y + \alpha\, y\}\, dx\, E^\mu\Big]_S^T + \\
&+ \int_{\Omega\times]S,T[} \Big(\operatorname{div}(q) - \alpha\Big) \Big\{|y'|^2 - |\nabla y|^2\Big\}\, E^\mu\, dx\, dt \\
&- \mu \int_{\Omega\times]S,T[} E^{\mu-1}\, E'\, y'\, \{2q\cdot\nabla y + \alpha\, y\}\, dx\, dt \\
&+ 2\int_{\Omega\times]S,T[} E^\mu\, \nabla y\cdot\nabla q_k\, \frac{\partial y}{\partial x_k}\, dx\, dt \\
&+ \int_{\Omega\times]S,T[} a\, y'\, \{2q\cdot\nabla y + \alpha\, y\}\, E^\mu\, dx\, dt = \\
&= \int_{\Gamma\times]S,T[} E^\mu (q\cdot\nu) \left(\frac{\partial y}{\partial \nu}\right)^2 d\Gamma\, dt\ .
\end{aligned}
$$

$$
\begin{aligned}
(2.6)\qquad & \int_\Omega y'\,\xi\, y\, dx\, E^\mu\Big]_S^T - \int_{\Omega\times]S,T[} \xi\Big\{|y'|^2 - |\nabla y|^2\Big\}\, E^\mu\, dx\, dt - \\
&- \mu \int_{\Omega\times]S,T[} E^{\mu-1}\, E'\, y'\, y\,\xi\, dx\, dt + \int_{\Omega\times]S,T[} y\,\nabla y\cdot\nabla\xi\, E^\mu\, dx\, dt + \\
&+ \int_{\Omega\times]S,T[} a\, y'\,\xi\, y\, E^\mu\, dx\, dt = 0\ .
\end{aligned}
$$

The proof of Lemma 2.3 is based on standard multipliers technique, the interested reader should refer to Lions [11] or Komornik [8]. We observe that the multiplier $\{2q\cdot\nabla y + \alpha\, y\}\, E^\mu$ $(\mu > 0)$ is often used for nonlinear problems (cf. [8]). The fact that this multiplier could be used for linear problems was already observed by Rao in [15].

Throughout the remaining part of the paper, c denotes different positive constants independent of the initial data and we use the following additional notations

$$
\omega_1 = \Big\{x\in\omega;\ a(x)\leq 1\Big\}, \qquad \omega_2 = \Big\{x\in\omega;\ a(x) > 1\Big\}\ .
$$

Lemma 2.4. *Under the hypotheses of Theorem 1.1 we have for $N < 2m$,*

$$\int_\omega |y'|^2\,dx \le |E'| + c\,|a^{-1}|_p^{\frac{p}{p+1}}\, F_m^{\frac{N}{m(p+1)}}\, E^{\frac{2m-N}{2m(p+1)}}\, |E'|^{\frac{p}{p+1}} \tag{2.7}$$

and for $N \ge 2m$,

$$\int_\omega |y'|^2\,dx \le |E'| + c\,|a^{-1}|_p^{\frac{p}{p+1}}\, F_m^{\frac{N}{m(p+1)}}\, E^{\frac{mp-(N-2m)}{2m(p+1)}}\, |E'|^{\frac{p}{2p+2}} \;. \tag{2.8}$$

Proof of Lemma 2.4: It is clear that for every $N \ge 1$, one has

$$\int_{\omega_2} |y'|^2\,dx \le |E'| \;. \tag{2.9}$$

For $1 \le N < 2m$, we have by Hölder inequality,

$$\begin{aligned}\int_{\omega_1} |y'|^2\,dx &\le |a^{-1}|_p^{\frac{p}{p+1}} \Big(\int_{\omega_1} a|y'|^{2+\frac{2}{p}}\,dx\Big)^{\frac{p}{p+1}}\\ &\le |a^{-1}|_p^{\frac{p}{p+1}}\, |y'|_\infty^{\frac{2}{p+1}}\, |E'|^{\frac{p}{p+1}} \;.\end{aligned} \tag{2.10}$$

In (2.10), we also used the fact that $H^m(\Omega) \subset L^\infty(\Omega)$ for $1 \le N < 2m$. Now, using Theorem 1.0 and the interpolation inequality (given by Lemma 2.1)

$$|\varphi|_\infty \le c\,|\varphi|_2^{\frac{2m-N}{2m}}\, \|\varphi\|_{H^m(\Omega)}^{\frac{N}{2m}}\,, \qquad \forall\,\varphi \in H^m(\Omega) \tag{2.11}$$

in (2.10), we obtain

$$\int_{\omega_1} |y'|^2\,dx \le c\,|a^{-1}|_p^{\frac{p}{p+1}}\, F_m^{\frac{N}{m(p+1)}}\, E^{\frac{2m-N}{2m(p+1)}}\, |E'|^{\frac{p}{p+1}} \;. \tag{2.12}$$

Combining (2.9) and (2.12), we find (2.7). Let us prove (2.8) now. It remains to estimate the quantity $\int_{\omega_1} |y'|^2\,dx$. We have by a twofold application of Hölder inequality,

$$\begin{aligned}\int_{\omega_1} |y'|^2\,dx &\le |a^{-1}|_p^{\frac{p}{p+1}} \Big(\int_{\omega_1} a\,|y'|^{2+\frac{2}{p}}\,dx\Big)^{\frac{p}{p+1}}\\ &\le |a^{-1}|_p^{\frac{p}{p+1}}\, |y'|_{\frac{2p+4}{p}}^{\frac{p+2}{p+1}}\, |E'|^{\frac{p}{2p+2}} \;.\end{aligned} \tag{2.13}$$

Observe that the second line of (2.13) is correct by Theorem 1.0, the Sobolev imbedding theorem and the hypothesis on p. Now, using in (2.13), the relations (1.7)–(1.9) and the interpolation inequality

$$|\varphi|_{\frac{2p+4}{p}} \le c\,|\varphi|_2^{\frac{mp-(N-2m)}{m(p+2)}}\, \|\varphi\|_{H^m(\Omega)}^{\frac{N}{m(p+2)}}\,, \qquad \forall\,\varphi \in H^m(\Omega)\;, \tag{2.14}$$

we find

$$\int_{\omega_1} |y'|^2\,dx \le c\,|a^{-1}|_p^{\frac{p}{p+1}}\, F_m^{\frac{N}{m(p+1)}}\, E^{\frac{mp-(N-2m)}{2m(p+1)}}\, |E'|^{\frac{p}{2p+2}} \;. \tag{2.15}$$

The combination of (2.9) and (2.15) yields the claimed inequality. ∎

3 – Proofs of Theorems 1.1 and 1.2

We recall that the method used to prove these theorems essentially relies on multipliers technique and on some integral inequalities due to Haraux.

Proof of Theorem 1.1: We proceed in several steps.

Step 1. Applying (2.5) with $\alpha = N-1$, $q(x) = m(x)$, observing that $\mathrm{div}(m) = N$ and using (1.5), we find

$$\begin{aligned} 2\int_S^T E^{\mu+1}\,dt = &-\int_\Omega y'\Big\{2m\cdot\nabla y + (N-1)\,y\Big\}\,dx\; E^\mu\Big]_S^T \\ &+\mu\int_{\Omega\times]S,T[} E^{\mu-1}\,E'\,y'\Big\{2m\cdot\nabla y + (N-1)\,y\Big\}\,dx\,dt \\ &-\int_{\Omega\times]S,T[} a\,y'\Big\{2m\cdot\nabla y + (N-1)\,y\Big\}\,E^\mu\,dx\,dt \\ &+\int_{\Gamma\times]S,T[} E^\mu(m\cdot\nu)\left(\frac{\partial y}{\partial\nu}\right)^2 d\Gamma\,dt \;. \end{aligned} \tag{3.1}$$

Since the energy is nonincreasing, using the result of Komornik [7], we find

$$\left|-\int_\Omega y'\Big\{2m\cdot\nabla y + (N-1)\,y\Big\}\,dx\; E^\mu\Big]_S^T\right| \le 4\,R\,E(0)^\mu\,E(S) \tag{3.2}$$

and

$$\left|\mu\int_{\Omega\times]S,T[} E^{\mu-1}\,E'\,y'\Big\{2m\cdot\nabla y + (N-1)\,y\Big\}\,dx\,dt\right| \le 2\,\mu\,R\,E(0)^\mu\,E(S) \;. \tag{3.3}$$

By Hölder inequality we have

$$\left|\int_{\Omega\times]S,T[} a\,y'\Big\{2m\cdot\nabla y + (N-1)\,y\Big\}\,E^\mu\,dx\,dt\right| \le c\int_S^T E^{\mu+\frac12}\,|E'|^{\frac12}\,dt$$

and the use of Young inequality, shows that

$$\left|\int_{\Omega\times]S,T[} a\,y'\Big\{2m\cdot\nabla y + (N-1)\,y\Big\}\,E^\mu\,dx\,dt\right| \le c\,E(0)^\mu\,E(S) + \int_S^T E^{\mu+1}\,dt \;. \tag{3.4}$$

Combining (3.2)–(3.4) and reporting the result obtained in (3.1), we obtain

$$\int_S^T E^{\mu+1}\,dt \le c\,E(0)^\mu\,E(S) + R\int_{\Gamma_+\times]S,T[} E^\mu \left(\frac{\partial y}{\partial\nu}\right)^2 d\Gamma\,dt\ . \tag{3.5}$$

At this stage, we observe, thanks to Lemma 2.2, that it suffices to obtain judicious estimates of the last term of the right hand side of (3.5) in terms of $E(S)$ and $\int_S^T E^{\mu+1}\,dt$ to complete the proof of Theorem 1.1.

Step 2. Let $h \in (W^{1,\infty}(\Omega))^N$ such that

$$h = \nu \ \text{ on } \Gamma_+\,, \qquad h\cdot\nu \ge 0 \ \text{ on } \Gamma\,, \qquad h = 0 \ \text{ in } \Omega\backslash\hat\omega\ , \tag{3.6}$$

where $\hat\omega$ is another neighbourhood of Γ_+ strictly contained in ω. (For the construction of the vectorfield h, the reader should refer to Lions [8], Chap. 1, Remark 3.2.)

Choose $\alpha = 0$ and $q = h$ in (2.5). Following Zuazua [16], we know that there exists a positive constant c_0 depending only on ω such that

$$\begin{aligned} R\int_{\Gamma_+\times]S,T[} E^\mu\left(\frac{\partial y}{\partial\nu}\right)^2 d\Gamma\,dt &\le R\int_{\Gamma\times]S,T[} E^\mu (h\cdot\nu)\left(\frac{\partial y}{\partial\nu}\right)^2 d\Gamma\,dt \le \\ &\le c_0\int_{\hat\omega\times]S,T[}\Big\{|y'|^2+|\nabla y|^2\Big\}E^\mu\,dx\,dt + 2\,R\int_\Omega y'\,h\cdot\nabla y\,dx\ E^\mu\Big]_S^T \\ &\quad -2\mu\,R\int_{\Omega\times]S,T[} E^{\mu-1}\,E'\,y'\,h\cdot\nabla y\,dx\,dt + 2\,R\int_{\Omega\times]S,T[} a\,y'\,h\cdot\nabla y\,E^\mu\,dx\,dt\ . \end{aligned} \tag{3.7}$$

Simple calculations using Young inequality show that

$$\begin{aligned} \left|-2\,R\int_\Omega y'\,h\cdot\nabla y\,dx\ E^\mu\Big]_S^T\right| + \left|2\,\mu\,R\int_{\Omega\times]S,T[} E^{\mu-1}\,E'\,y'\,h\cdot\nabla y\,dx\,dt\right| &\le \\ \le c\,E(0)^\mu\,E(S)\ . \end{aligned} \tag{3.8}$$

Using the Hölder inequality, in the last term of the right hand side of (3.7), we find

$$\left|2\,R\int_{\Omega\times]S,T[} a\,y'\,h\cdot\nabla y\,E^\mu\,dx\,dt\right| \le c\int_S^T E^{\mu+\frac12}\,|E'|^{\frac12}\,dt\ . \tag{3.9}$$

It is then an easy task to deduce from (3.9) that

$$\left|2\,R\int_{\Omega\times]S,T[} a\,y'\,h\cdot\nabla y\,E^\mu\,dx\,dt\right| \le \frac12\int_S^T E^{\mu+1}\,dt + c\,E(0)^\mu\,E(S)\ . \tag{3.10}$$

Combining (3.8), (3.9), (3.10) and reporting the obtained result in (3.5) yield

$$\int_S^T E^{\mu+1}\,dt \le c\,E(0)^\mu\,E(S) + c\int_{\hat\omega\times]S,T[}\Big\{|y'|^2+|\nabla y|^2\Big\}\,E^\mu\,dx\,dt\ . \tag{3.11}$$

Step 3. Introduce the function η, (constructed by Zuazua in [11], Chap. 7), which satisfies

$$\eta\in W^{1,\infty}(\Omega)\,,\qquad 0\le\eta\le 1\,,\qquad \eta=1\ \text{ in }\hat\omega\,,\qquad \eta=0\ \text{ in }\Omega\backslash\omega\ . \tag{3.12}$$

Applying (2.6) with $\xi=\eta^2$, we find

$$\begin{aligned}\int_{\Omega\times]S,T[}\eta^2\,|\nabla y|^2\,E^\mu\,dx\,dt &= -\int_\Omega y'\,\eta^2\,y\,dx\ E^\mu\Big]_S^T + \\ &+\int_{\Omega\times]S,T[}\eta^2\,|y'|^2\,E^\mu\,dx\,dt+\mu\int_{\Omega\times]S,T[}E^{\mu-1}\,E'\,y'\,y\,\eta^2\,dx\,dt\\ &-2\int_{\Omega\times]S,T[}\eta\,y\,\nabla y\cdot\nabla\eta\,E^\mu\,dx\,dt-\int_{\Omega\times]S,T[}a\,y'\,\eta^2\,y\,E^\mu\,dx\,dt\ .\end{aligned} \tag{3.13}$$

Simple calculations using Young inequality show that

$$\left|-\int_\Omega y'\,\eta^2\,y\,dx\ E^\mu\Big]_S^T+\mu\int_{\Omega\times]S,T[}E^{\mu-1}\,E'\,y'\,y\,\eta^2\,dx\,dt\right|\le c\,E(0)^\mu\,E(S) \tag{3.14}$$

and

$$\begin{aligned}\left|2\int_{\Omega\times]S,T[}\eta\,y\,\nabla y\cdot\nabla\eta\,E^\mu\,dx\,dt\right| &\le \frac12\int_{\Omega\times]S,T[}\eta^2\,|\nabla y|^2\,E^\mu\,dx\,dt\\ &\quad+2\,c\,|\nabla\eta|_\infty^2\int_{\omega\times]S,T[}|y|^2\,E^\mu\,dx\,dt\ .\end{aligned} \tag{3.15}$$

On the other hand, $\hat c$ denoting the constant in (3.11), we have

$$\left|2\,\hat c\int_{\Omega\times]S,T[}a\,y'\,\eta^2\,y\,E^\mu\,dx\,dt\right|\le c\,E(0)^\mu\,E(S)+\frac12\int_S^T E^{\mu+1}\,dt \tag{3.16}$$

Reporting (3.14)–(3.16) in (3.13), we find

$$\begin{aligned}&\hat c\int_{\Omega\times]S,T[}\eta^2\,|\nabla y|^2\,E^\mu\,dx\,dt\le \qquad (3.17)\\ &\le c\,E(0)^\mu\,E(S)+\frac12\int_S^T E^{\mu+1}\,dt+c\int_{\omega\times]S,T[}|y|^2\,E^\mu\,dx\,dt+c\int_{\omega\times]S,T[}|y'|^2\,E^\mu\,dx\,dt\ .\end{aligned}$$

Combining (3.11) and (3.17), we obtain

$$\begin{aligned}\int_S^T E^{\mu+1}\,dt \le c\,E(0)^\mu\,E(S) + c\int_{\omega\times]S,T[}|y|^2\,E^\mu\,dx\,dt\\ + c\int_{\omega\times]S,T[}|y'|^2\,E^\mu\,dx\,dt\ .\end{aligned} \tag{3.18}$$

Now, we will use a judicious multiplier to absorb the second term of the right hand side of (3.18). To this end, introduce $z(t)\in H_0^1(\Omega)$ solution of

$$\begin{cases} -\Delta z = \chi(\omega)\,y & \text{in } \Omega,\\ z = 0 & \text{on } \Gamma\ , \end{cases} \tag{3.19}$$

where $\chi(\omega)$ is the characteristic function of ω. It is easy to check that $z' = \frac{dz}{dt}$ satisfies

$$\begin{cases} -\Delta z' = \chi(\omega)\,y' & \text{in } \Omega,\\ z' = 0 & \text{on } \Gamma\ . \end{cases} \tag{3.20}$$

Some elementary calculations show that

$$\begin{cases} \displaystyle\int_\Omega |\nabla z|^2\,dx \le \frac{1}{\lambda_1^2}\int_\omega |y|^2\,dx\ ,\\ \displaystyle\int_\Omega |\nabla z'|^2\,dx \le \frac{1}{\lambda_1^2}\int_\omega |y'|^2\,dx\ ,\\ \displaystyle\int_\Omega \nabla z\cdot\nabla y\,dx = \int_\omega |y|^2\,dx\ . \end{cases} \tag{3.21}$$

Now multiply the first equation of (1.3) by zE^μ, integrate by parts on $\Omega\times]S,T[$ and use the second line of (3.21), we find

$$\begin{aligned}\int_{\omega\times]S,T[}|y|^2\,E^\mu\,dx\,dt = -\int_\Omega y'\,z\,dx\ E^\mu\Big]_S^T + \int_{\Omega\times]S,T[}E^\mu\,y'\,z'\,dx\,dt\,+\\ +\,\mu\int_{\Omega\times]S,T[}E^{\mu-1}\,E'\,y'\,z\,dx\,dt - \int_{\Omega\times]S,T[}a\,y'\,z\,E^\mu\,dx\,dt\ .\end{aligned} \tag{3.22}$$

Some elementary calculations yield

$$\left|-\int_\Omega y'\,z\,dx\ E^\mu\Big]_S^T + \mu\int_{\Omega\times]S,T[}E^{\mu-1}\,E'\,y'\,z\,dx\,dt\right| \le c\,E(0)^\mu\,E(S)\ . \tag{3.23}$$

Denoting by $\tilde c$ the constant in (3.18) and using Hölder and Young inequalities, we find

$$\tilde c\left|\int_{\Omega\times]S,T[}a\,y'\,z\,E^\mu\,dx\,dt\right| \le c\,E(0)^\mu\,E(S) + \frac14\int_S^T E^{\mu+1}\,dt \tag{3.24}$$

and

$$\tilde{c}\left|\int_{\Omega\times]S,T[} E^\mu\, y'\, z'\, dx\, dt\right| \le \frac{1}{4}\int_S^T E^{\mu+1}\, dt + c\int_{\omega\times]S,T[} E^\mu\, |y'|^2\, dx\, dt\ . \tag{3.25}$$

Reporting (3.23)–(3.25) in (3.22), we obtain

$$\begin{aligned}\tilde{c}\int_{\omega\times]S,T[} |y|^2\, E^\mu\, dx\, dt &\le c\, E(0)^\mu\, E(S) + \frac{1}{2}\int_S^T E^{\mu+1}\, dt\\ &\quad + c\int_{\omega\times]S,T[} E^\mu\, |y'|^2\, dx\, dt\ .\end{aligned} \tag{3.26}$$

The combination of (3.18) and (3.26) yields

$$\int_S^T E^{\mu+1}\, dt \le c\, E(0)^\mu\, E(S) + c\int_{\omega\times]S,T[} |y'|^2\, E^\mu\, dx\, dt\ . \tag{3.27}$$

Now, to complete the proof of Theorem 1.1, it remains to absorb the second term of the right hand side of (3.27). The proofs of (1.11) and (1.12) are distinct. In fact we need different values for the exponent μ in the two cases. Let us begin with the proof of (1.11). For this purpose we choose $\mu = \frac{N}{2mp}$. Thanks to this choice of μ and (2.7), we have

$$\begin{aligned}&c\int_{\omega\times]S,T[} |y'|^2\, E^{\frac{N}{2mp}}\, dx\, dt \le\\ &\le c\, E(0)^{\frac{N}{2mp}}\, E(S) + \frac{1}{p+1}\int_S^T E^{\frac{N}{2mp}+1}\, dt + c\, |a^{-1}|_p\, F_m^{\frac{N}{mp}}\, E(S)\ .\end{aligned} \tag{3.28}$$

Reporting (3.28) in (3.27), we find

$$\int_S^T E^{\frac{N}{2mp}+1}\, dt \le c\Big(|a^{-1}|_p\, F_m^{\frac{N}{mp}} + E(0)^{\frac{N}{2mp}}\Big)\, E(S)\ . \tag{3.29}$$

Hence taking the limit as $T \to \infty$ and applying Lemma 2.2 we obtain (1.11). Let us prove (1.12) now. To this end, we choose $\mu = \frac{N}{mp}$ and we use (2.8). It follows that

$$\begin{aligned}c\int_{\omega\times]S,T[} |y'|^2\, E^{\frac{N}{mp}}\, dx\, dt &\le c\, E(0)^{\frac{N}{mp}}\, E(S) + \frac{p+2}{2p+2}\int_S^T E^{\frac{N}{mp}+1}\, dt\\ &\quad + c\, |a^{-1}|_p^2\, F_m^{\frac{2N}{pm}}\, E(S)\ .\end{aligned} \tag{3.30}$$

Combining (3.27) and (3.30) and letting T go in infinity in the obtained result, we find

$$\int_S^\infty E^{\frac{N}{mp}+1}\, dt \le c\Big(|a^{-1}|_p^2\, F_m^{\frac{2N}{mp}} + E(0)^{\frac{N}{mp}}\Big)\, E(S)\ . \tag{3.31}$$

Applying finally Lemma 2.2, we obtain the desired estimate and this ends and the proof of Theorem 1.1. ∎

The proof of Theorem 1.2 is in a large part similar to the proof of Theorem 1.1. Therefore, we only sketch it.

Sketch of the proof of Theorem 1.2. Taking $\mu = 0$ and proceeding as in the proof of Theorem 1.1 above, we are led to

$$\int_S^T E\,dt \le c\,E(S) + c\int_{\omega\times]S,T[} |y'|^2\,dx\,dt\ . \tag{3.32}$$

But it is easy using (1.2) to check that

$$\int_{\omega\times]S,T[} |y'|^2\,dx\,dt \le c\,E(S)\ . \tag{3.33}$$

Reporting (3.33) in (3.32) and letting T go to infinity in the obtained inequality, we find

$$\int_S^\infty E\,dt \le c\,E(S)\ . \tag{3.34}$$

Finally, the application of Lemma 2.2 yields (1.13). ∎

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