PORTUGALIAE MATHEMATICA
Vol. 56 Fasc. 1 – 1999

THE EVOLUTION DAM PROBLEM FOR NONLINEAR DARCY'S LAW AND DIRICHLET BOUNDARY CONDITIONS *

A. Lyaghfouri

Abstract: In this paper, we study a time dependent dam problem modeling a nonlinear fluid flow through a homogeneous or nonhomogeneous porous medium governed by a nonlinear Darcy's law. We prove existence and uniqueness of a weak solution.

Introduction

The dam problem consists of finding the flow region and the pressure of fluid flow through a porous medium Ω under gravity. The free boundary represents the region separating the wet and the dry part of the porous medium. Assuming the flow governed by a linear Darcy's law and taking Dirichlet boundary conditions on some part of the boundary, this problem has been widely studied from several points of view both for the stationary and the evolutionary case.

For the stationary case, the first results are due to C. Baiocchi ([5], [6]) who solved the case of rectangular dams by introducing the so called Baiocchi's transformation, which leads him to consider problems of variational inequalities. Then he established existence and uniqueness results. Although this method is not adaptable for the general case, many authors ([7], [8], [16], [23]), used same techniques to treat questions related to heterogeneous or three dimensional rectangular dams.

Few years after, the steady problem has been studied in the general case by H.W. Alt ([2], [3], [4]), H. Brézis, D. Kinderlehrer and G. Stampacchia [9], J. Carrillo and M. Chipot [15]. An existence theorem has been proved and the uniqueness of the solution established up to a certain class of disturbing functions. The regularity of the free boundary was also investigated.

Received: May 21, 1997; *Revised*: June 11, 1997.
AMS Mathematical Subject Classification: 35R35, 76S05.
* Supported by the European Science Foundation Scientific Programme on the Mathematical Treatment of Free Boundary Problems.

The evolution dam problem has been solved first by A. Torelli ([24], [25]) in the case of a rectangular domain. He used a similar transformation to the Baiocchi's one, which allowed him to reduce the problem to a quasi-variational inequality problem. He obtained, in this way, results of existence, uniqueness and regularity of his solution. Unfortunately, this method is not adapted for the general case.

Later, it was G. Gilardi [19] who proved an existence theorem for a weak formulation of the evolution dam problem when the porous medium is assumed to be a general Lipschitz bounded domain of $\mathbb{R}^n$. In [18] E. Dibenedetto and A. Friedman proved an existence theorem in a general way by an other method, both for compressible or incompressible flow. Moreover they proved uniqueness for rectangular dams. The question of uniqueness of the solution, in its generality, remains open until solved by J. Carrillo [14].

In this study, we consider an incompressible fluid flow governed by a generalized nonlinear Darcy's law relating the velocity v of the fluid to its pressure p by:

$$v = -\mathcal{A}(x, \nabla(p + x_n))$$

where $\mathcal{A}$ is a function defined in $\Omega \times \mathbb{R}^n$ and $x = (x_1, ..., x_n)$ denotes points in $\mathbb{R}^n$.

The prime example of nonlinear Darcy's laws for a homogeneous porous medium (see [17]) corresponds to the q-Laplacian: $\mathcal{A}(x, \xi) = |\xi|^{q-2}\,\xi$. For heterogeneous media we have the following measurable perturbations of the q-Laplacian: $\mathcal{A}(x, \xi) = |a(x)\cdot\xi|^{q-2}\,a(x)\cdot\xi$, where $a(x)$ is a measurable positive definite matrix representing the permeability of the medium at x. Note that when $q = 2$, we rediscover the well known linear Darcy Law.

In addition, we would like to consider a model of Dirichlet boundary conditions. The paper is organized as follows: In section 1, we begin by transforming the problem usually stated in terms of the pressure function into a problem for the hydrostatic head $u = p + x_n$. The dry part is described by a bounded function g. Then we give a weaker formulation to our problem. In section 2 and 3 we prove an existence theorem by means of regularization and by using the Tychonoff fixed point theorem. In section 4, we prove some properties of the solutions. In particular for any solution (u, g), u is bounded and $\mathcal{A}$-subharmonic and g is continuous in time variable. In section 5, we assume that $q \leq n + 1$, $\mathcal{A}(x, \xi) = \mathcal{A}(\xi)$ and $\mathcal{A}(e)\cdot\nu \leq 0$ on the bottom of the dam. Then from section 4, we derive a monotonicity property for g. Making use of this result, we prove a comparison theorem and with the help of the continuity of g we deduce the uniqueness of the solution.

1 – Statement of the problem

The dam is a bounded locally Lipschitz domain Ω in $\mathbb{R}^n$ $(n \geq 2)$ (see Figure 1). The boundary Γ of Ω is divided in two parts: an impervious part Γ_1 and a pervious one Γ_2 which is assumed to be nonempty and relatively open in Γ. Let now T be a positive number, $Q = \Omega \times (0,T)$ and φ a nonnegative Lipschitz function defined in $\overline{Q}$. We define:

$$\Sigma_1 = \Gamma_1 \times (0,T)\,, \qquad \Sigma_2 = \Gamma_2 \times (0,T)\,, \qquad \Sigma_3 = (\Gamma_2 \times (0,T)) \cap \{\varphi > 0\}$$
$$\text{and} \qquad \Sigma_4 = (\Gamma_2 \times (0,T)) \cap \{\varphi = 0\}\ .$$

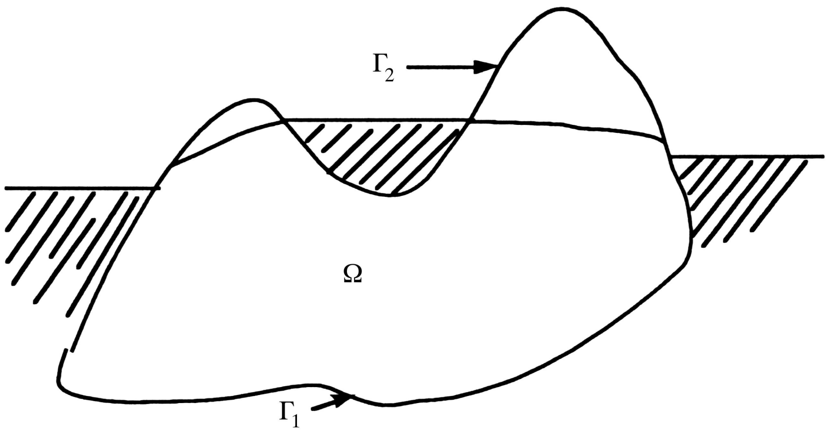

Fig. 1

We shall be interested with the problem of finding the pressure p and the saturation χ of the fluid. For convenience, we set: $\psi = \varphi + x_n$, $u = p + x_n$ and $g = 1 - \chi$. Starting from the nonlinear Darcy's law, the mass conservation law and taking a Dirichlet boundary condition on Σ_2, the flow is governed by the following equations:

$$(1.1) \qquad \begin{cases} \text{i)} & u \geq x_n, \quad 0 \leq g \leq 1, \quad g(u - x_n) = 0 \quad \text{in } Q \\ \text{ii)} & \operatorname{div}\Big(\mathcal{A}(x, \nabla u) - g\,\mathcal{A}(x,e)\Big) + g_t = 0 \quad \text{in } Q \\ \text{iii)} & u = \psi \quad \text{on } \Sigma_2 \\ \text{iv)} & g(\cdot, 0) = g_0 \quad \text{in } \Omega \\ \text{v)} & \Big(\mathcal{A}(x, \nabla u) - g\,\mathcal{A}(x,e)\Big) \cdot \nu = 0 \quad \text{on } \Sigma_1 \\ \text{vi)} & \Big(\mathcal{A}(x, \nabla u) - g\,\mathcal{A}(x,e)\Big) \cdot \nu \leq 0 \quad \text{on } \Sigma_4 \end{cases}$$

where e is the vertical unit vector of $\mathbb{R}^n$, i.e. $e = (0,1)$ with $0 \in \mathbb{R}^{n-1}$, ν denoting the outward unit normal to $\partial\Omega$, g_0 is a given function satisfying $0 \le g_0 \le 1$ and $\mathcal{A}\colon \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a mapping that satisfies the following assumptions with some constants $q > 1$ and $0 < \alpha \le \beta < \infty$:

$$(1.2) \qquad \begin{cases} \text{the function } x \longmapsto \mathcal{A}(x,\xi) \text{ is measurable } \forall\, \xi \in \mathbb{R}^n, \quad \text{and} \\ \text{the function } \xi \longmapsto \mathcal{A}(x,\xi) \text{ is continuous for a.e } x \in \Omega \ , \end{cases}$$

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$

$$(1.3) \qquad \mathcal{A}(x,\xi)\cdot\xi \ge \alpha\,|\xi|^q \ ,$$

$$(1.4) \qquad |\mathcal{A}(x,\xi)| \le \beta\,|\xi|^{q-1} \ ,$$

for all $\xi, \zeta \in \mathbb{R}^n$ such that $\xi \neq \zeta$ and a.e. $x \in \Omega$

$$(1.5) \qquad \Big(\mathcal{A}(x,\xi) - \mathcal{A}(x,\zeta)\Big)\cdot(\xi - \zeta) > 0 \ ,$$

$$(1.6) \qquad \exists\, r > 1\colon \quad \operatorname{div}(\mathcal{A}(x,e)) \in L^r(\Omega) \ .$$

The condition (1.1) i) means that we look for a nonnegative pressure p and $g(\cdot,t)$ characterises the wet region $\Omega(t)$ at time t. (1.1) iii) means that the trace pressure at the bottoms of fluid reservoirs is equal to the one of the fluid and equal to the atmospheric one when the boundary of Ω is in contact with the air. (1.1) iv) is an initial data. (1.1) v) and (1.1) vi) are due to the fact that the flux of fluid vanishes on Σ_1 (since Γ_1 is impervious) and is nonnegative on Σ_4 where the fluid is free to exit from our porous medium.

From the strong formulation (1.1), we are led to consider the following weak formulation:

$$(\mathrm{P}) \begin{cases} \text{Find } \ (u,g) \in L^q(0,T,W^{1,q}(\Omega)) \times L^\infty(Q) \ \text{ such that :} \\ \ \text{i)} \quad u \ge x_n, \quad 0 \le g \le 1, \quad g(u - x_n) = 0 \quad \text{ a.e. in } \ Q \ ; \\ \ \text{ii)} \quad u = \psi \quad \text{ on } \ \Sigma_2 \ ; \\ \ \text{iii)} \quad \displaystyle\int_Q \Big(\mathcal{A}(x,\nabla u) - g\,\mathcal{A}(x,e)\Big)\cdot\nabla\xi + g\,\xi_t\,dx\,dt + \int_\Omega g_0(x)\,\xi(x,0)\,dx \le 0 \\ \qquad\qquad \forall\,\xi \in W^{1,q}(Q), \quad \xi = 0 \text{ on } \Sigma_3, \quad \xi \ge 0 \text{ on } \Sigma_4, \quad \xi(x,T) = 0 \quad \text{ a.e. in } \ \Omega \ . \end{cases}$$

2 – A regularized problem

We first introduce the following approximated problem:

$$
(\mathrm{P}_\varepsilon)\begin{cases}
\text{Find } u_\varepsilon \in W^{1,q}(Q) \text{ such that : } \quad u_\varepsilon=\psi \text{ on } \Sigma_2 \;, \\[2mm]
\displaystyle\int_Q \mathcal{A}(x,\nabla u_\varepsilon)\cdot\nabla\xi + \varepsilon\,|u_{\varepsilon t}|^{q-2}\,u_{\varepsilon t}\cdot\xi_t + G_\varepsilon(u_\varepsilon)\,\Big(\xi_t - \mathcal{A}(x,e)\cdot\nabla\xi\Big)\,dx\,dt = \\[2mm]
\qquad = \displaystyle\int_\Omega G_\varepsilon(u_\varepsilon(x,T))\,\xi(x,T)\,dx - \int_\Omega g_0(x)\,\xi(x,0)\,dx \\[2mm]
\hfill \forall\,\xi\in W^{1,q}(Q), \quad \xi=0 \text{ on } \Sigma_2 \;,
\end{cases}
$$

where $G_\varepsilon\colon\, L^q(Q)\,(\text{resp.}\, L^q(\Omega)) \;\longrightarrow\; L^\infty(Q)\,(\text{resp.}\, L^\infty(\Omega))$ is defined by

$$
G_\varepsilon(v) \;=\; \begin{cases} 0 & \text{if} \quad v-x_n\ge\varepsilon \\ 1-(v-x_n)/\varepsilon & \text{if} \quad 0\le v-x_n\le\varepsilon \\ 1 & \text{if} \quad v-x_n\le 0\;. \end{cases} \tag{2.1}
$$

Then we have:

Theorem 2.1. *Assume that φ is a nonnegative Lipschitz continuous function and that $\mathcal{A}$ satisfies (1.2)–(1.5). Then, there exists a solution u_ε of (P_ε).*

Proof: It will be done in three steps:

Step 1: We define

$$
V=\Big\{v\in W^{1,q}(Q) \;/\; v=0 \text{ on } \Sigma_2\Big\} \quad \text{and} \quad K=\Big\{v\in W^{1,q}(Q) \;/\; v=\psi \text{ on } \Sigma_2\Big\}\;.
$$

For $u\in K$, we consider the map:

$$
A(u)\colon\, W^{1,q}(Q)\longrightarrow\mathbb{R}\,,\quad \xi\longmapsto\langle A(u),\xi\rangle=\int_Q \mathcal{A}(x,\nabla u)\cdot\nabla\xi+\varepsilon\,|u_t|^{q-2}\,u_t\,\xi_t\,dx\,dt\;.
$$

Then the operator A defined by $A\colon\, u\in K\longmapsto A(u)$, satisfies:

Lemma 2.2. *If we denote by $(W^{1,q}(Q))'$ the dual space of $W^{1,q}(Q)$, we have*

i) *For every $u\in K$, $A(u)\in(W^{1,q}(Q))'$;*

ii) *A is continuous from K into $(W^{1,q}(Q))'$;*

iii) *A is monotone and coercive.*

Proof: (see [20] for example).

Step 2: For $v \in W^{1,q}(Q)$, we consider the map: $f_v \colon W^{1,q}(Q) \longrightarrow \mathbb{R}$,

$$\xi \longmapsto \int_Q G_\varepsilon(v) \Big(\mathcal{A}(x,e)\cdot\nabla\xi - \xi_t\Big)\, dx\, dt + \int_\Omega G_\varepsilon(v(x,T))\, \xi(x,T) - g_0(x)\, \xi(x,0)\, dx\ .$$

It is clear that $f_v \in (W^{1,q}(Q))'$. Using Lemma 2.2, we deduce (see [20]) that for every $v \in W^{1,q}(Q)$ there exists a unique u_ε solution of the variational problem

$$u_\varepsilon \in K\,, \qquad \langle Au_\varepsilon, \xi\rangle = \langle f_v, \xi\rangle \qquad \forall\, \xi \in V\ . \tag{2.2}$$

Step 3: Now, let us consider the map F_ε defined by: $F_\varepsilon:\ W^{1,q}(Q) \longrightarrow K$, $v \longmapsto u_\varepsilon$. Let us denote by $\overline{B}(0,R(\varepsilon))$ the closed ball in $W^{1,q}(Q)$ of center 0 and radius $R(\varepsilon)$. Then we have:

Lemma 2.3.

i) $\exists\, R(\varepsilon) > 0\ /\ F_\varepsilon(\overline{B}(0,R(\varepsilon))) \subset \overline{B}(0,R(\varepsilon))$;

ii) $F_\varepsilon \colon \overline{B}(0,R(\varepsilon)) \longrightarrow \overline{B}(0,R(\varepsilon))$ *is weakly continuous.*

Proof: **i)** Note that $u_\varepsilon - \psi$ is a suitable test function to (2.2), so:

$$\begin{aligned} \int_Q \mathcal{A}(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon + \varepsilon\, |u_{\varepsilon t}|^q\, dx\, dt = \\ = \int_Q \mathcal{A}(x,\nabla u_\varepsilon)\cdot\nabla\psi + \varepsilon\, |u_{\varepsilon t}|^{q-2}\, u_{\varepsilon t}\cdot\psi_t\, dx\, dt \\ + \int_Q G_\varepsilon(v)\, \mathcal{A}(x,e)\cdot\nabla(u_\varepsilon-\psi)\, dx\, dt \\ - \int_Q G_\varepsilon(v)\, (u_\varepsilon-\psi)_t\, dx\, dt - \int_\Omega g_0(x)\, (u_\varepsilon-\psi)(x,0)\, dx \\ + \int_\Omega G_\varepsilon(v(x,T))\, (u_\varepsilon-\psi)(x,T)\, dx\ . \end{aligned} \tag{2.3}$$

Using (1.3), (1.4), (2.1), (2.3) and Hölder's inequality, we get for some constants c_i

$$\min(\alpha,\varepsilon) \int_Q |\nabla u_\varepsilon|^q + |u_{\varepsilon t}|^q\, dx\, dt \le c_1\, |u_\varepsilon|_{1,q}^{q-1} + c_2\, |u_\varepsilon|_{1,q} + c_3\ .$$

By Poincaré's Inequality, this leads for some constants c'_i to $|u_\varepsilon|_{1,q}^q \le c'_1 |u_\varepsilon|_{1,q}^{q-1} + c'_2 |u_\varepsilon|_{1,q} + c'_3$ from which we deduce that: $|u_\varepsilon|_{1,q} \le R(\varepsilon)$ where $R(\varepsilon)$ is some constant depending on ε. So we have: $F_\varepsilon(\overline{B}(0,R(\varepsilon))) \subset \overline{B}(0,R(\varepsilon))$. More precisely, we have proved that: $F_\varepsilon(W^{1,q}(Q)) \subset \overline{B}(0,R(\varepsilon))$.

ii) Let $(v_i)_{i\in I}$ be a generalized sequence in $C=\overline{B}(0,R(\varepsilon))$ which converges to v in C weakly.

Set $u^i_\varepsilon = F_\varepsilon(v_i)$ and $u_\varepsilon = F_\varepsilon(v)$. We want to prove that $(u^i_\varepsilon)_{i\in I}$ converges to u_ε weakly in C. Since C is compact with respect to the weak topology, it is enough to show that $(u^i_\varepsilon)_{i\in I}$ has u_ε as the unique limit point for the weak topology in C. So let u be a weak limit point for $(u^i_\varepsilon)_{i\in I}$ in C. Using the compact imbedding: $W^{1,q}(Q)\hookrightarrow L^q(Q)$, one can construct a sequence $(u^{i_k}_\varepsilon)_{k\in\mathbb{N}}$ such that $\{\, i_k/\, k\in\mathbb{N}\,\}\subset I$, $u^{i_k}_\varepsilon \rightharpoonup u$ weakly in $W^{1,q}(Q)$ and $u^{i_k}_\varepsilon \to u$ strongly in $L^q(Q)$. Choose $u^{i_k}_\varepsilon - u_\varepsilon$ as a suitable test function for (2.2) written for $u^{i_k}_\varepsilon$ and u_ε. Subtract the equations, so that

$$\begin{aligned}(2.4)\qquad &\int_Q \Big(\mathcal{A}(x,\nabla u^{i_k}_\varepsilon)-\mathcal{A}(x,\nabla u_\varepsilon)\Big)\cdot\nabla(u^{i_k}_\varepsilon-u_\varepsilon)+\\ &\qquad+\varepsilon\Big(|u^{i_k}_{\varepsilon t}|^{q-2}\,u^{i_k}_{\varepsilon t}-|u_{\varepsilon t}|^{q-2}\,u_{\varepsilon t}\Big)\cdot(u^{i_k}_\varepsilon-u_\varepsilon)_t\,dx\,dt=\\ &=\int_Q\Big(G_\varepsilon(v_{i_k})-G_\varepsilon(v)\Big)\Big(\mathcal{A}(x,e)\cdot\nabla(u^{i_k}_\varepsilon-u_\varepsilon)-(u^{i_k}_\varepsilon-u_\varepsilon)_t\Big)\,dx\,dt\\ &\quad+\int_\Omega\Big(G_\varepsilon(v_{i_k}(x,T))-G_\varepsilon(v(x,T))\Big)\,(u^{i_k}_\varepsilon-u_\varepsilon)(x,T)\,dx\ .\end{aligned}$$

Now we have by (1.4), (2.1), Hölder's inequality and the fact that $|u^{i_k}_\varepsilon-u_\varepsilon|_{1,q}\le 2R(\varepsilon)$:

$$\begin{aligned}(2.5)\qquad &\left|\int_Q\Big(G_\varepsilon(v_{i_k})-G_\varepsilon(v)\Big)\Big(\mathcal{A}(x,e)\cdot\nabla(u^{i_k}_\varepsilon-u_\varepsilon)-(u^{i_k}_\varepsilon-u_\varepsilon)_t\Big)\,dx\,dt\right|\le\\ &\qquad\le c_1(\varepsilon)\cdot|v_{i_k}-v|_q\ .\end{aligned}$$

$$\begin{aligned}(2.6)\qquad &\left|\int_\Omega\Big(G_\varepsilon(v_{i_k}(x,T))-G_\varepsilon(v(x,T))\Big)\,(u^{i_k}_\varepsilon-u_\varepsilon)(x,T)\,dx\right|\le\\ &\qquad\le c_2(\varepsilon)\cdot\Big|(v_{i_k}-v)\,(\cdot,T)\Big|_q\ .\end{aligned}$$

Now due to (1.5), (2.4)–(2.6) and the compact imbeddings: $W^{1,q}(Q)\hookrightarrow L^q(Q)$ and $W^{1-\frac{1}{q},q}(\Omega\times\{T\})\hookrightarrow L^q(\Omega\times\{T\})$, we get:

$$(2.7)\qquad \begin{aligned}&\lim_{k\to+\infty}\int_Q\Big(\mathcal{A}(x,\nabla u^{i_k}_\varepsilon)-\mathcal{A}(x,\nabla u_\varepsilon)\Big)\cdot\nabla(u^{i_k}_\varepsilon-u_\varepsilon)\,dx\,dt=0\ ,\\ &\lim_{k\to+\infty}\int_Q\Big(|u^{i_k}_{\varepsilon t}|^{q-2}\,u^{i_k}_{\varepsilon t}-|u_{\varepsilon t}|^{q-2}\,u_{\varepsilon t}\Big)\cdot(u^{i_k}_\varepsilon-u_\varepsilon)_t\,dx\,dt=0\ .\end{aligned}$$

Using (2.7), we deduce that (see [11]) there exists a subsequence of $(u^{i_k}_\varepsilon)$ also denoted by $(u^{i_k}_\varepsilon)$ such that $\nabla u^{i_k}_\varepsilon\to\nabla u_\varepsilon$ and $u^{i_k}_{\varepsilon t}\to u_{\varepsilon t}$ a.e. in Q. Taking into account the boundedness of $(u_\varepsilon^{i_k})$ in $W^{1,q}(Q)$, we get: $\nabla u_\varepsilon^{i_k} \rightharpoonup \nabla u_\varepsilon$ weakly in $\mathbb{L}^q(Q)$ and $u_{\varepsilon t}^{i_k} \rightharpoonup u_{\varepsilon t}$ weakly in $L^q(Q)$. So we have $u_\varepsilon = u$ and u_ε is the unique weak limit point of (u_ε^i) in C. Thus $u_\varepsilon^i = F_\varepsilon(v^i) \rightharpoonup u_\varepsilon = F_\varepsilon(v)$ weakly in C. Hence the continuity of F_ε holds. ∎

At this step, applying the Tychonoff fixed point theorem on $\overline{B}(0, R(\varepsilon))$ (see [22]), we derive that F_ε has a fixed point. Thus (P_ε) has at least one solution. ∎

Let us now show that our sequence (u_ε) is uniformly bounded in $L^\infty(Q)$. More precisely we have:

Proposition 2.4. *Let u_ε be a solution of (P_ε) and let $\varepsilon_0 > 0$. Then we have for any $\varepsilon \in (0, \varepsilon_0)$ and H such that $H \geq \max(\varepsilon_0 + \max\{x_n, (x', x_n) \in \overline{\Omega}\}, \max\{\psi(x,t), (x,t) \in \overline{\Sigma}_2\})$*

$$x_n \leq u_\varepsilon \leq H \quad \text{a.e. in } Q \,. \tag{2.9}$$

Proof: **i**) Since $(u_\varepsilon - H)^+$ is a suitable test function for (P_ε), we have by (2.1) and the choice of H

$$\int_Q \mathcal{A}(x, \nabla u_\varepsilon) \cdot \nabla (u_\varepsilon - H)^+ + \varepsilon |u_{\varepsilon t}|^{q-2}\, u_{\varepsilon t}\, (u_\varepsilon - H)_t^+ \, dx\, dt = \tag{2.10}$$

$$= \int_Q G_\varepsilon(u_\varepsilon) \Big(\mathcal{A}(x, e) \cdot \nabla (u_\varepsilon - H)^+ - (u_\varepsilon - H)_t^+ \Big)\, dx\, dt$$
$$- \int_\Omega g_0(x)\, (u_\varepsilon - H)^+(x, 0)\, dx + \int_\Omega G_\varepsilon(u_\varepsilon(x,T))\, (u_\varepsilon - H)^+(x, T)\, dx$$
$$= - \int_\Omega g_0(x)\, (u_\varepsilon - H)^+(x, 0)\, dx \leq 0 \,.$$

Then we deduce from (1.3) and (2.10) $\int_Q \alpha |\nabla (u_\varepsilon - H)^+|^q + \varepsilon |(u_\varepsilon - H)_t^+|^q dx\, dt \leq 0$, which leads to $|\nabla (u_\varepsilon - H)^+| = |(u_\varepsilon - H)_t^+| = 0$ a.e. in Q. Thus $(u_\varepsilon - H)^+ = 0$ and $u_\varepsilon \leq H$ a.e. in Q.

ii) We denote by $(\cdot)^-$ the negative part of a function. Then $\xi = (u_\varepsilon - x_n)^-$ is a test function for (P_ε) and one has by taking into account (2.1):

$$\int_Q \mathcal{A}(x, \nabla u_\varepsilon) \cdot \nabla (u_\varepsilon - x_n)^- + \varepsilon\, |u_{\varepsilon t}|^{q-2}\, u_{\varepsilon t} \cdot (u_\varepsilon - x_n)_t^- \, dx\, dt = \tag{2.11}$$

$$= \int_\Omega (u_\varepsilon - x_n)^-(x, T)\, dx + \int_Q \mathcal{A}(x, e) \cdot \nabla (u_\varepsilon - x_n)^- \, dx\, dt$$
$$- \int_\Omega g_0(x)\, (u_\varepsilon - x_n)^-(x, 0)\, dx - \int_Q (u_\varepsilon - x_n)_t^- \, dx\, dt \,.$$

Integrating by part the last term of (2.11), we obtain

$$\int_{[u_\varepsilon \le x_n]} \Big(\mathcal{A}(x,\nabla u_\varepsilon)-\mathcal{A}(x,\nabla x_n)\Big)\cdot(\nabla u_\varepsilon-\nabla x_n)+\varepsilon\,|(u_\varepsilon-x_n)_t|^q\,dx\,dt \;\le\; 0\,. \tag{2.12}$$

Using (1.5) and (2.12) we conclude that $u_\varepsilon \ge x_n$ a.e. in Q. ∎

Now we give an a priori estimate for ∇u_ε and $u_{\varepsilon t}$.

Proposition 2.5. *Under assumptions of Proposition 2.4, we have for any $\varepsilon \in (0,\varepsilon_0)$:*

$$\int_Q \Big(\alpha\,|\nabla u_\varepsilon|^q + \varepsilon\,|u_{\varepsilon t}|^q\Big)\,dx\,dt \le C\ , \tag{2.13}$$

where C is a constant independent of ε.

Proof: Using the fact that $u_\varepsilon - \psi$ is a suitable test function, we get

$$\begin{aligned}
&\int_Q \mathcal{A}(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon + \varepsilon\,|u_{\varepsilon t}|^q\,dx\,dt = \\
&= \int_Q \mathcal{A}(x,\nabla u_\varepsilon)\cdot\nabla\psi\,dx\,dt + \int_Q \varepsilon\,|u_{\varepsilon t}|^{q-2}\,u_{\varepsilon t}\cdot\psi_t\,dx\,dt \\
&\quad + \int_Q G_\varepsilon(u_\varepsilon)\,\mathcal{A}(x,e)\cdot\nabla(u_\varepsilon-\psi)\,dx\,dt - \int_Q G_\varepsilon(u_\varepsilon)\,(u_\varepsilon-\psi)_t\,dx\,dt \\
&\quad + \int_\Omega G_\varepsilon(u_\varepsilon(x,T))\,(u_\varepsilon-\psi)(x,T)\,dx - \int_\Omega g_0(x)\,(u_\varepsilon-\psi)(x,0)\,dx\ .
\end{aligned} \tag{2.14}$$

First let us set: $E_\varepsilon(y) = \int_0^y (1-H_\varepsilon(s))\,ds$ and $H_\varepsilon(s) = 1\wedge \dfrac{s^+}{\varepsilon}$. We have $0 \le (1-H_\varepsilon(y))\,y \le E_\varepsilon(y) \le y\ \forall\, y\ge 0$ and using (2.9), we get

$$\begin{aligned}
&\int_Q -G_\varepsilon(u_\varepsilon)\,(u_\varepsilon-\psi)_t\,dx\,dt = \\
&= \int_Q -G_\varepsilon(u_\varepsilon)\,(u_\varepsilon-x_n)_t\,dx\,dt + \int_Q G_\varepsilon(u_\varepsilon)\,\varphi_t\,dx\,dt \\
&= -\int_Q \frac{\partial}{\partial t}\,E_\varepsilon(u_\varepsilon-x_n)\,dx\,dt + \int_Q G_\varepsilon(u_\varepsilon)\,\varphi_t\,dx\,dt \\
&= \int_\Omega \Big[E_\varepsilon\Big(u_\varepsilon(x,0)-x_n\Big) - E_\varepsilon\Big(u_\varepsilon(x,T)-x_n\Big)\Big]\,dx + \int_Q G_\varepsilon(u_\varepsilon)\,\varphi_t\,dx\,dt \le C\ .
\end{aligned} \tag{2.15}$$

Next, by (2.9) the last two terms of (2.14) are bounded. So using (1.3), (2.14), (2.15) and Hölder's inequality, we derive for some constant $C > 0$: $0 \le U_\varepsilon \le C\,(1 + {U_\varepsilon}^{1/q} + {U_\varepsilon}^{1/q'})$ where $U_\varepsilon = \int_Q (\alpha\,|\nabla u_\varepsilon|^q + \varepsilon\,|u_{\varepsilon t}|^q)\,dx\,dt$. Hence we get (2.13) since $q, q' > 1$. ■

In the following proposition, we show that u_ε satisfies an inequality similar to (P) iii).

Proposition 2.6. *Let u_ε be a solution of (P_ε). Then we have:*

$$\begin{aligned}(2.16)\qquad &\int_Q \mathcal{A}(x,\nabla u_\varepsilon)\cdot\nabla\xi + \varepsilon\,|u_{\varepsilon t}|^{q-2}\,u_{\varepsilon t}\cdot\xi_t + G_\varepsilon(u_\varepsilon)\Big(\xi_t - \mathcal{A}(x,e)\cdot\nabla\xi\Big)\,dx\,dt\,+\\ &\qquad + \int_\Omega g_0(x)\,\xi(x,0)\,dx \le 0\end{aligned}$$

$$\forall\,\xi\in W^{1,q}(Q),\quad \xi=0 \ \ on\ \Sigma_3,\quad \xi\ge 0\ \ on\ \Sigma_4,\quad \xi(x,T)=0\ \text{ a.e. } x\in\Omega\ .$$

Proof: Let ξ as in (2.16). For any $\delta > 0$, $(\frac{p_\varepsilon}{\delta}\wedge\xi)$, where $p_\varepsilon = u_\varepsilon - x_n$, is a test function for (P_ε). So we can write

$$\begin{aligned}&\int_Q \mathcal{A}(x,\nabla u_\varepsilon)\cdot\nabla\Big(\frac{p_\varepsilon}{\delta}\wedge\xi\Big) - \mathcal{A}(x,e)\cdot\nabla\Big(\frac{p_\varepsilon}{\delta}\wedge\xi\Big) + \varepsilon\,|u_{\varepsilon t}|^{q-2}\,u_{\varepsilon t}\cdot\Big(\frac{p_\varepsilon}{\delta}\wedge\xi\Big)_t\,dx\,dt\,+\\ &\quad + \int_Q H_\varepsilon(u_\varepsilon - x_n)\,\mathcal{A}(x,e)\cdot\nabla\Big(\frac{p_\varepsilon}{\delta}\wedge\xi\Big)\,dx\,dt - \int_Q H_\varepsilon(u_\varepsilon - x_n)\Big(\frac{p_\varepsilon}{\delta}\wedge\xi\Big)_t\,dx\,dt =\\ &\qquad = \int_\Omega\Big(1 - g_0(x)\Big)\cdot\Big(\frac{p_\varepsilon}{\delta}\wedge\xi\Big)(x,0)\,dx\ .\end{aligned}$$

The first integral in the left side of this equality can be written as

$$\begin{aligned}&\int_{[u_\varepsilon - x_n < \delta\xi]}\Big(\mathcal{A}(x,\nabla u_\varepsilon) - \mathcal{A}(x,\nabla x_n)\Big)\cdot\nabla\frac{u_\varepsilon - x_n}{\delta} + \frac{\varepsilon}{\delta}\,|u_{\varepsilon t}|^q\,dx\,dt\,+\\ &\qquad + \int_{[u_\varepsilon - x_n \ge \delta\xi]}\Big(\mathcal{A}(x,\nabla u_\varepsilon) - \mathcal{A}(x,e)\Big)\cdot\nabla\xi + \varepsilon\,|u_{\varepsilon t}|^{q-2}\,u_{\varepsilon t}\,\xi_t\,dx\,dt\ .\end{aligned}$$

Using (1.5) and the fact that $1 - g_0(x) \ge 0$ a.e. $x\in\Omega$, we obtain

$$\begin{aligned}(2.17)\qquad &\int_{[u_\varepsilon - x_n \ge \delta\xi]}\Big(\mathcal{A}(x,\nabla u_\varepsilon) - \mathcal{A}(x,e)\Big)\cdot\nabla\xi + \varepsilon\,|u_{\varepsilon t}|^{q-2}\,u_{\varepsilon t}\,\xi_t\,dx\,dt\,+\\ &+ \int_Q H_\varepsilon(u_\varepsilon - x_n)\,\mathcal{A}(x,e)\cdot\nabla\Big(\frac{p_\varepsilon}{\delta}\wedge\xi\Big)\,dx\,dt - \int_Q H_\varepsilon(u_\varepsilon - x_n)\Big(\frac{p_\varepsilon}{\delta}\wedge\xi\Big)_t\,dx\,dt \le\\ &\qquad \le \int_\Omega (1 - g_0(x))\,\xi(x,0)\,dx\ .\end{aligned}$$

Let us show that:

$$\lim_{\delta\to 0}\int_Q H_\varepsilon(u_\varepsilon - x_n)\,\mathcal{A}(x,e)\cdot\nabla\Big(\frac{u_\varepsilon - x_n}{\delta}\wedge\xi\Big)\,dx\,dt = \\ = \int_Q H_\varepsilon(u_\varepsilon - x_n)\,\mathcal{A}(x,e)\cdot\nabla\xi\,dx\,dt\;, \tag{2.18}$$

$$\lim_{\delta\to 0}\int_Q H_\varepsilon(u_\varepsilon - x_n)\,\Big(\frac{u_\varepsilon - x_n}{\delta}\wedge\xi\Big)_t\,dx\,dt = \int_Q H_\varepsilon(u_\varepsilon - x_n)\,\xi_t\,dx\,dt\;. \tag{2.19}$$

Indeed, taking in account (1.6) we can use the divergence formula to get:

$$\int_Q H_\varepsilon(u_\varepsilon - x_n)\,\mathcal{A}(x,e)\cdot\nabla\Big(\frac{u_\varepsilon - x_n}{\delta}\wedge\xi\Big)\,dx\,dt = \\ = -\int_Q \operatorname{div}\Big(H_\varepsilon(u_\varepsilon - x_n)\,\mathcal{A}(x,e)\Big)\cdot\Big(\frac{u_\varepsilon - x_n}{\delta}\wedge\xi\Big)\,dx\,dt \\ + \int_{\partial Q} H_\varepsilon(u_\varepsilon - x_n)\,\mathcal{A}(x,e)\cdot\nu\cdot\Big(\frac{u_\varepsilon - x_n}{\delta}\wedge\xi\Big)\,d\sigma(x,t)\;.$$

Since $\xi\wedge\frac{u_\varepsilon - x_n}{\delta}\longrightarrow\xi$ a.e. on $[u_\varepsilon > x_n]$ when δ goes to 0, we obtain by the Lebesgue theorem,

$$\lim_{\delta\to 0}\int_Q H_\varepsilon(u_\varepsilon - x_n)\,\mathcal{A}(x,e)\cdot\nabla\Big(\frac{u_\varepsilon - x_n}{\delta}\wedge\xi\Big)\,dx\,dt \;= \\ = -\int_Q \operatorname{div}\Big(H_\varepsilon(u_\varepsilon - x_n)\,\mathcal{A}(x,e)\Big)\cdot\xi\,dx\,dt + \int_{\partial Q} H_\varepsilon(u_\varepsilon - x_n)\,\mathcal{A}(x,e)\cdot\nu\cdot\xi\,d\sigma(x,t) \;= \\ = \int_Q H_\varepsilon(u_\varepsilon - x_n)\,\mathcal{A}(x,e)\cdot\nabla\xi\,dx\,dt\;,$$

which proves (2.18). Similarly we establish (2.19).

So combining (2.18)–(2.19) and letting $\delta\to 0$ in (2.17) we obtain (2.16) since $\int_\Omega \xi(x,0)\,dx = -\int_Q \xi_t\,dx\,dt$. ∎

3 – Existence of a solution

Theorem 3.1. *Assume that φ is a nonnegative Lipschitz continuous function and that $\mathcal{A}$ satisfies (1.2)–(1.6). Then there exists a solution (u,g) of* (P).

The proof will consist in passing to the limit, when ε goes to 0, in (P_ε). To do this we shall need some lemmas.

First from definition (2.1) of G_ε and estimates (2.9), (2.13) and (1.4), we deduce the existence of a subsequence ε_k of ε and functions: $u \in L^q(0,T,W^{1,q}(\Omega))$, $g \in L^{q'}(Q)$, $A_0 \in \mathbb{L}^{q'}(Q)$ such that:

$$u_{\varepsilon_k} \rightharpoonup u \qquad \text{weakly in } \ L^q(0,T,W^{1,q}(\Omega)) \ , \tag{3.1}$$

$$\mathcal{A}(x,\nabla u_{\varepsilon_k}) \rightharpoonup A_0 \qquad \text{weakly in } \ \mathbb{L}^{q'}(Q) \ , \tag{3.2}$$

$$G_{\varepsilon_k}(u_{\varepsilon_k}) \rightharpoonup g \qquad \text{weakly in } \ L^{q'}(Q) \ . \tag{3.3}$$

Then we can prove:

Lemma 3.2. *Let u, g be defined by (3.1) and (3.3) respectively. Then we have:*

i) $u=\psi$ *on* Σ_2, $u \geq x_n$ *a.e. in* Q*;*

ii) $0 \leq g \leq 1$ *a.e. in* Q*.*

Proof: We consider the set $K_1 = \{v \in L^q(0,T,W^{1,q}(\Omega)) \ / \ v \geq x_n$ a.e. in Q, $v=\psi$ on $\Sigma_2\}$. K_1 is closed and convex in $L^q(0,T,W^{1,q}(\Omega))$, then it is weakly closed. Since $u_{\varepsilon_k} \in K_1$, $u \in K_1$ and i) holds. In the same way, we prove that $g \in K_2 = \{v \in L^{q'}(Q) \ / \ 0 \leq v \leq 1$ a.e. in $Q\}$ and ii) holds. ∎

Lemma 3.3. *Let u, g be defined by (3.1) and (3.3) respectively. Then we have:*

$$g(u-x_n) = 0 \qquad \text{a.e. in } \ Q \ . \tag{3.4}$$

Proof: First, note that (3.4) is not an obvious result as it is in the stationary case (see [17]), since we do not know, a priori, whether u_ε converges strongly to u in $L^q(Q)$ because the imbedding $L^q(0,T,W^{1,q}(\Omega)) \hookrightarrow L^q(Q)$ is not compact. To overcome this difficulty, we are going to prove a strong convergence of the sequence $(G_{\varepsilon_k}(u_{\varepsilon_k}))_k$ in a suitable space.

Next, we have for $\theta \in \mathfrak{D}(Q)$, $\theta \geq 0$ and $p_{\varepsilon_k} = u_{\varepsilon_k} - x_n$

$$\begin{aligned} 0 \ \leq \ &\int_Q G_{\varepsilon_k}(u_{\varepsilon_k})\,(u_{\varepsilon_k}-x_n)\,\theta\,dx\,dt \ = \\ &= \int_{Q\cap[0\leq p_{\varepsilon_k}\leq\varepsilon_k]} (1-H_{\varepsilon_k}(p_{\varepsilon_k}))\,p_{\varepsilon_k}\,\theta\,dx\,dt \ \leq \ \varepsilon_k\cdot|\theta|_\infty\cdot|Q| \ . \end{aligned}$$

So

$$\lim_{k\to+\infty} \int_Q G_{\varepsilon_k}(u_{\varepsilon_k})\,(u_{\varepsilon_k}-x_n)\,\theta\,dx\,dt = 0 \ . \tag{3.5}$$

To get (3.4), it suffices to prove that:

$$\lim_{k\to+\infty}\int_Q G_{\varepsilon_k}(u_{\varepsilon_k})\,(u_{\varepsilon_k}-x_n)\,\theta\,dx\,dt=\int_Q g(u-x_n)\,\theta\,dx\,dt\ . \tag{3.6}$$

For this purpose we introduce the function: $w_{\varepsilon_k}=\varepsilon_k\,|u_{\varepsilon_k t}|^{q-2}\,u_{\varepsilon_k t}+G_{\varepsilon_k}(u_{\varepsilon_k})$. From (2.13), since $q'>1$, we have:

$$\varepsilon_k\,|u_{\varepsilon_k t}|^{q-2}\,u_{\varepsilon_k t}\longrightarrow 0 \quad \text{strongly in} \ \ L^{q'}(Q)\ . \tag{3.7}$$

Then from (3.3) and (3.7) we deduce that

$$w_{\varepsilon_k}\rightharpoonup g \quad \text{weakly in} \ \ L^{q'}(Q)\ . \tag{3.8}$$

We are going to prove that:

$$w_{\varepsilon_k t}\rightharpoonup g_t \quad \text{weakly in} \ \ L^{q'}(0,T,W^{-1,q'}(\Omega))\ . \tag{3.9}$$

So, let us show that $g_t\in L^{q'}(0,T,W^{-1,q'}(\Omega))$. Let $\xi\in\mathfrak{D}(0,T,W_0^{1,q}(\Omega))$. Since ξ is a test function for $(\mathrm{P}_{\varepsilon_k})$, we obtain, after letting $k\to+\infty$ in $(\mathrm{P}_{\varepsilon_k})$, and taking into account (3.2), (3.3) and (3.7)

$$\int_Q g\,\xi_t\,dx\,dt=-\int_Q\Big(A_0-g\,\mathcal{A}(x,e)\Big)\cdot\nabla\xi\,dx\,dt$$

from which we deduce (see [10]) that $g_t=-\operatorname{div}(A_0-g\mathcal{A}(x,e))\in L^{q'}(0,T,W^{-1,q'}(\Omega))$.

Now we have in the distributional sense:

$$w_{\varepsilon_k t}=-\operatorname{div}\Big(\mathcal{A}(x,\nabla u_{\varepsilon_k})-G_{\varepsilon_k}(u_{\varepsilon_k})\,\mathcal{A}(x,e)\Big) \ \text{ and } \ w_{\varepsilon_k t}\rightharpoonup \operatorname{div}\Big(g\mathcal{A}(x,e)-A_0\Big)=g_t$$
$$\text{weakly in} \ \ L^{q'}(0,T,W^{-1,q'}(\Omega))\ .$$

So (3.9) holds.

At this stage let us introduce the space W defined by: $W=\{v\in L^{q'}(0,T,L^{q'}(\Omega))/ v_t\in L^{q'}(0,T,W^{-1,q'}(\Omega))\}$ which is a Banach space for the norme: $\|v\|_{L^{q'}(0,T,L^{q'}(\Omega))}+\|v_t\|_{L^{q'}(0,T,W^{-1,q'}(\Omega))}$. Since $L^{q'}(\Omega)$ and $W^{-1,q'}(\Omega)$ are reflexifs, the imbedding $L^{q'}(\Omega)\hookrightarrow W^{-1,q'}(\Omega)$ being continuous and compact (see [1]), we deduce that (see [20]), the imbedding

$$W\hookrightarrow L^{q'}(0,T,W^{-1,q'}(\Omega)) \quad \text{is compact}\ . \tag{3.10}$$

Now, the sequence $w_{\varepsilon_k}\in W$ and it is bounded in W by (3.8) and (3.9). So up to a subsequence still denoted by ε_k, we have $w_{\varepsilon_k}\rightharpoonup w$ weakly in W. But it is easy to see that $w=g$. We then deduce from (3.7) and (3.10)

$$G_{\varepsilon_k}(u_{\varepsilon_k})\to g \quad \text{strongly in} \ \ L^{q'}(0,T,W^{-1,q'}(\Omega))\ . \tag{3.11}$$

Finally, to conclude, it suffices to remark that: $(u_{\varepsilon_k}-x_n)\theta \rightharpoonup (u-x_n)\theta$ weakly in $L^q(0,T,W_0^{1,q}(\Omega))$. Then by (3.11), we get (3.6). Consequently $\int_Q g(u-x_n)\theta\, dxdt=0$ $\forall\, \theta \in \mathfrak{D}(Q)$, $\theta \geq 0$ and since $g(u-x_n) \geq 0$ a.e. in Q, we get (3.4). ∎

Remark 3.4. We have (see [20]) the imbedding $W \hookrightarrow C^0([0,T],W^{-1,q'}(\Omega))$ then $g \in C^0([0,T],W^{-1,q'}(\Omega))$. In section 4, we shall improve this regularity and prove that $g \in C^0([0,T],L^p(\Omega))$, $\forall\, p \in [1,+\infty[$.

Lemma 3.5. *Let u, A_0 and g be defined by (3.1), (3.2) and (3.3) respectively. Then we have:*

$$\int_Q \Big(A_0 - g\,\mathcal{A}(x,e)\Big)\cdot\nabla(u-\psi)\,\xi\,dx\,dt = \int_Q g(\varphi\,\xi)_t\,dx\,dt \quad \forall\,\xi \in \mathfrak{D}(0,T)\ . \tag{3.12}$$

Proof: Let ζ be a smooth function such that $d(\operatorname{supp}\zeta,\Sigma_2) > 0$ and $\operatorname{supp}\zeta \subset \mathbb{R}^n \times (\tau_0', T-\tau_0')$ for $T > \tau_0' > 0$. Then there exists $\tau_0 > 0$ such that: $\forall\,\tau \in (-\tau_0,\tau_0)$, $(x,t) \longmapsto \zeta(x,t-\tau)$ is a test function for $(\mathrm{P}_{\varepsilon_k})$. So we get, for all $\tau \in (-\tau_0,\tau_0)$, after letting k go to $+\infty$

$$\begin{aligned}\int_Q \Big(A_0(x,t) - g(x,t)\,\mathcal{A}(x,e)\Big)\cdot\nabla\zeta(x,t-\tau)\,dx\,dt\, - \\ -\frac{\partial}{\partial\tau}\Big(\int_Q g(x,t+\tau)\,\zeta(x,t)\,dx\,dt\Big) = 0\end{aligned} \tag{3.13}$$

since

$$\begin{aligned}-\int_Q g(x,t)\,\zeta_t(x,t-\tau)\,dx\,dt &= \frac{\partial}{\partial\tau}\Big(\int_Q g(x,t)\,\zeta(x,t-\tau)\,dx\,dt\Big)\\ &= \frac{\partial}{\partial\tau}\Big(\int_Q g(x,t+\tau)\,\zeta(x,t)\,dx\,dt\Big)\ .\end{aligned}$$

Now it is easy to see that (3.13) still holds for functions ζ in $L^q(0,T,W^{1,q}(\Omega))$ such that $\zeta = 0$ on Σ_2 and $\zeta = 0$ on $\Omega \times ((0,\tau_0) \cup (T-\tau_0,T))$. So if we consider $\xi \in \mathfrak{D}(\tau_0, T-\tau_0)$, $\xi \geq 0$ and set: $\zeta = (u-\psi)\,\xi$, we have $\forall\,\tau \in (-\tau_0,\tau_0)$

$$\begin{aligned}\int_Q \Big(\Big(A_0(x,t) - g(x,t)\,\mathcal{A}(x,e)\Big)\cdot\nabla\Big((u-\psi)\,\xi\Big) - g(\varphi\,\xi)_t\Big)(x,t-\tau)\,dx\,dt = \\ = \frac{\partial G}{\partial\tau}(\tau)\end{aligned} \tag{3.14}$$

with $G(\tau) = \int_Q g(x,t+\tau)\,((u-x_n)\,\xi)(x,t)\,dx\,dt$.

From (3.14) we know that $G \in C^1(-\tau_0, \tau_0)$. Moreover by Lemmas 3.2 and 3.3, we get $G(\tau) \geq 0 = G(0)\ \forall\, \tau \in (-\tau_0, \tau_0)$. So 0 is an absolute minimum for G in $(-\tau_0, \tau_0)$ and

$$\frac{\partial G}{\partial \tau}(0) = 0\ . \tag{3.15}$$

Combining (3.14) and (3.15) we get (3.12) for all $\xi \in \mathfrak{D}(\tau_0, T - \tau_0)$, $\xi \geq 0$. ∎

Thanks to Lemma 3.5, we are going to prove a result which allows us to pass to the limit in $(\mathrm{P}_{\varepsilon_k})$.

Lemma 3.6. *The sequence (u_{ε_k}) (resp. $\mathcal{A}(x, \nabla u_{\varepsilon_k})$) converges strongly to u (resp. $\mathcal{A}(x, \nabla u)$) in $L^q(0, T, W^{1,q}(\Omega))$ (resp. $\mathbb{L}^{q'}(Q)$).*

To prove Lemma 3.6, we need a lemma:

Lemma 3.7. *Let u and A_0 defined by (3.1) and (3.2) respectively. Then we have*

$$\int_Q \mathcal{A}(x, \nabla u)\cdot\nabla\xi\, dx\, dt = \int_Q A_0(x,t)\cdot\nabla\xi\, dx\, dt \quad \forall\, \xi \in L^q(0, T, W^{1,q}(\Omega))\ . \tag{3.16}$$

Proof: Let $\theta \in \mathfrak{D}(0, T)$, $\theta \geq 0$. Choose $\xi = (u_{\varepsilon_k} - \psi)\,\theta$ as a test function for $(\mathrm{P}_{\varepsilon_k})$ and write (3.12) for $\xi = \theta$. Subtract the equations, we obtain:

$$\begin{aligned}
&\int_Q \theta\, \mathcal{A}(x, \nabla u_{\varepsilon_k}) \cdot \nabla u_{\varepsilon_k}\, dx\, dt = \\
&\quad = \int_Q \theta\, A_0 \cdot \nabla u\, dx\, dt + \int_Q \Big(\mathcal{A}(x, \nabla u_{\varepsilon_k}) - A_0\Big) \cdot \nabla\psi\, \theta\, dx\, dt \\
&\quad - \int_Q \varepsilon_k\, |u_{\varepsilon_k t}|^{q-2}\, u_{\varepsilon_k t}\Big((u_{\varepsilon_k} - \psi)\,\theta\Big)_t\, dx\, dt \\
&\quad + \int_Q \mathcal{A}(x, e) \cdot \Big(G_{\varepsilon_k}(u_{\varepsilon_k})\,\nabla(u_{\varepsilon_k} - \psi) - g\,\nabla(u - \psi)\Big)\,\theta\, dx\, dt \\
&\quad - \int_Q G_{\varepsilon_k}(u_{\varepsilon_k})\,\Big((u_{\varepsilon_k} - \psi)\,\theta\Big)_t\, dx\, dt - \int_Q g(\varphi\,\theta)_t\, dx\, dt\ .
\end{aligned} \tag{3.17}$$

By (3.2) we have:

$$\lim_{k\to+\infty} \int_Q \Big(\mathcal{A}(x, \nabla u_{\varepsilon_k}) - A_0\Big) \cdot \nabla\psi\, \theta\, dx\, dt = 0\ . \tag{3.18}$$

Remark that

$$\int_Q \varepsilon_k\,|u_{\varepsilon_k t}|^{q-2}\,u_{\varepsilon_k t}\cdot\Big((u_{\varepsilon_k}-\psi)\,\theta\Big)_t\,dx\,dt=$$

$$=\int_Q \varepsilon_k\,|u_{\varepsilon_k t}|^{q}\,\theta\,dx\,dt+\int_Q \varepsilon_k\,|u_{\varepsilon_k t}|^{q-2}\,u_{\varepsilon_k t}\,u_{\varepsilon_k}\,\theta_t\,dx\,dt-\int_Q \varepsilon_k\,|u_{\varepsilon_k t}|^{q-2}\,u_{\varepsilon_k t}\cdot(\psi\,\theta)_t\,dx\,dt.$$

The first term of the above equality is nonnegative. Moreover using (3.7) and the fact that u_{ε_k} is uniformly bounded, we get:

$$\overline{\lim}\ -\int_Q \varepsilon_k\,|u_{\varepsilon_k t}|^{q-2}\,u_{\varepsilon_k t}\cdot\Big((u_{\varepsilon_k}-\psi)\,\theta\Big)_t\,dx\,dt\le 0\ . \tag{3.19}$$

From (3.4), we deduce:

$$\begin{aligned}\int_Q \mathcal{A}(x,e)\cdot\Big(G_{\varepsilon_k}(u_{\varepsilon_k})\,\nabla(u_{\varepsilon_k}-\psi)-g\,\nabla(u-\psi)\Big)\,\theta\,dx\,dt&=\\ =\int_Q \theta\,G_{\varepsilon_k}(u_{\varepsilon_k})\,\mathcal{A}(x,e)\cdot\nabla(u_{\varepsilon_k}-x_n)\,dx\,dt&\\ -\int_Q\Big(G_{\varepsilon_k}(u_{\varepsilon_k})-g\Big)\,\mathcal{A}(x,e)\cdot\nabla(\varphi\,\theta)\,dx\,dt&\ .\end{aligned} \tag{3.20}$$

By (3.3) the last term of (3.20) goes to 0. Applying the divergence formula, we get:

$$\begin{aligned}\left|\int_Q \theta\,G_{\varepsilon_k}(u_{\varepsilon_k})\,\mathcal{A}(x,e)\cdot\nabla(u_{\varepsilon_k}-x_n)\,dx\,dt\right|&=\\ =\left|\int_Q \theta\,\mathcal{A}(x,e)\cdot\nabla\Big(\int_0^{\min(u_{\varepsilon_k}-x_n,\varepsilon_k)}\Big(1-H_{\varepsilon_k}(s)\Big)\,ds\Big)\,dx\,dt\right|&\\ \le\varepsilon_k\Big(\int_Q \theta\,|\operatorname{div}(\mathcal{A}(x,e))|\,dx\,dt+\int_{\partial\Omega\times(0,T)}\theta\,|\mathcal{A}(x,e)\cdot\nu|\,d\sigma(x,t)\Big)&\ .\end{aligned} \tag{3.21}$$

We obtain from (3.20)–(3.21):

$$\lim_{k\to+\infty}\int_Q \mathcal{A}(x,e)\cdot\Big(G_{\varepsilon_k}(u_{\varepsilon_k})\,\nabla(u_{\varepsilon_k}-\psi)-g\,\nabla(u-\psi)\Big)\,\theta\,dx\,dt=0\ . \tag{3.22}$$

The last two terms of the right hand side of (3.17) can be written:

$$\begin{aligned}-\int_Q G_{\varepsilon_k}(u_{\varepsilon_k})\,\Big((u_{\varepsilon_k}-\psi)\,\theta\Big)_t\,dx\,dt-\int_Q g(\varphi\,\theta)_t\,dx\,dt&=\\ =-\int_Q G_{\varepsilon_k}(u_{\varepsilon_k})\,(u_{\varepsilon_k}-x_n)\,\theta_t\,dx\,dt-\int_Q G_{\varepsilon_k}(u_{\varepsilon_k})\,(u_{\varepsilon_k}-x_n)_t\,\theta\,dx\,dt&\\ +\int_Q\Big(G_{\varepsilon_k}(u_{\varepsilon_k})-g\Big)\,(\varphi\,\theta)_t\,dx\,dt&\ .\end{aligned} \tag{3.23}$$

Arguing as in (3.5), the first term of the right side of (3.23) converges to 0. Integrating by parts, we can see as in (3.21) that the second term goes also to 0. Using (3.3), we get

$$\lim_{k\to+\infty} -\int_Q G_{\varepsilon_k}(u_{\varepsilon_k}) \Big((u_{\varepsilon_k}-\psi)\,\theta\Big)_t\,dx\,dt - \int_Q g(\varphi\,\theta)_t\,dx\,dt = 0 \;. \tag{3.24}$$

Combining (3.17), (3.18), (3.19), (3.22) and (3.24) we conclude that:

$$\overline{\lim} \int_Q \theta\,\mathcal{A}(x,\nabla u_{\varepsilon_k})\cdot\nabla u_{\varepsilon_k}\,dx\,dt \;\le \int_Q \theta\,A_0(x,t)\cdot\nabla u\,dx\,dt \qquad \forall\,\theta\in\mathfrak{D}(0,T),\quad \theta\ge 0 \;. \tag{3.25}$$

Let now $v\in L^q(0,T,W^{1,q}(\Omega))$ and $\theta\in\mathfrak{D}(0,T)$ such that $\theta\ge 0$. Using (1.5) we have:

$$\int_Q \theta\,\Big(\mathcal{A}(x,\nabla u_{\varepsilon_k})-\mathcal{A}(x,\nabla v)\Big)\cdot(\nabla u_{\varepsilon_k}-\nabla v)\,dx\,dt \ge 0 \qquad \forall\,k\in\mathbb{N}$$

which can be written for all $k\in\mathbb{N}$:

$$\int_Q \theta\,\mathcal{A}(x,\nabla u_{\varepsilon_k})\cdot\nabla u_{\varepsilon_k}\,dx\,dt \;-\int_Q \theta\,\mathcal{A}(x,\nabla u_{\varepsilon_k})\cdot\nabla v\,dx\,dt \;- \;-\int_Q \theta\,\mathcal{A}(x,\nabla v)\cdot\nabla(u_{\varepsilon_k}-v)\,dx\,dt \;\ge\; 0 \;. \tag{3.26}$$

Passing to the limit sup in (3.26) and taking into account (3.1)–(3.2) and (3.25), we get:

$$\int_Q \theta\,\Big(A_0(x,t)-\mathcal{A}(x,\nabla v)\Big)\cdot\nabla(u-v)\,dx\,dt \ge 0 \;. \tag{3.27}$$

If we choose $v=u\pm\lambda\,\xi$ with $\xi\in L^q(0,T,W^{1,q}(\Omega))$ and $\lambda\in[0,1]$ in (3.27) we obtain, after letting λ go to 0 and taking into account (1.2) and (1.4)

$$\int_Q \theta\,\Big(A_0(x,t)-\mathcal{A}(x,\nabla u)\Big)\cdot\nabla\xi\,dx\,dt = 0 \qquad \forall\,\theta\in\mathfrak{D}(0,T),\quad \theta\ge 0, \qquad \forall\,\xi\in L^q(0,T,W^{1,q}(\Omega))$$

and by density we get (3.16). ∎

Proof of Lemma 3.6: Taking $\xi=\theta\,u$ in (3.16) with $\theta\in\mathfrak{D}(0,T)$, we get

$$\int_Q \theta\,\mathcal{A}(x,\nabla u)\cdot\nabla u\,dx\,dt = \int_Q \theta\,A_0(x,t)\cdot\nabla u\,dx\,dt \;. \tag{3.28}$$

Using (3.25) and (3.28) we obtain:

$$\overline{\lim}\int_Q \theta\,\mathcal{A}(x,\nabla u_{\varepsilon_k})\cdot\nabla u_{\varepsilon_k}\,dx\,dt \;\le \int_Q \theta\,\mathcal{A}(x,\nabla u)\cdot\nabla u\,dx\,dt\;. \tag{3.29}$$

Combining (3.1)–(3.2) and (3.28)–(3.29), one can prove easily:

$$\overline{\lim}\int_Q \theta\,\Big(\mathcal{A}(x,\nabla u_{\varepsilon_k})-\mathcal{A}(x,\nabla u)\Big)\cdot\nabla(u_{\varepsilon_k}-u)\,dx\,dt\le 0\;. \tag{3.30}$$

Now, since $\nabla u_{\varepsilon_k}\rightharpoonup\nabla u$ weakly in $\mathbb{L}^q(Q)$, we conclude by (3.30) because $\mathcal{A}$ satisfies the Browder's property (S_+) (see [12]), that $\nabla u_{\varepsilon_k}\to\nabla u$ strongly in $\mathbb{L}^q(Q)$. Moreover the mapping $\mathbb{L}^q(Q)\to\mathbb{L}^{q'}(Q)$, $v\mapsto\mathcal{A}(\cdot,v)$ being continuous, we deduce that $\mathcal{A}(x,\nabla u_{\varepsilon_k})$ converges strongly to $\mathcal{A}(x,\nabla u)$ in $\mathbb{L}^{q'}(Q)$. Now by the Poincaré Inequality one can see that u_{ε_k} converges strongly in $L^q(0,T,W^{1,q}(\Omega))$. ∎

Proof of Theorem 3.1: It is clear that (P) i) and (P) ii) follow from Lemma 3.2 and Lemma 3.3. Let $\xi\in W^{1,q}(Q)$, $\xi=0$ on Σ_3, $\xi\ge 0$ on Σ_4 and $\xi(x,T)=0$ a.e. in Ω. Letting k go to $+\infty$ in (2.16) written for ξ and using (3.3), (3.7) and Lemma 3.6, we get (P) iii). This achieves the proof of Theorem 3.1. ∎

Remark 3.8. Note that the Lemma 3.7 is sufficient for the proof of Theorem 3.1, however the result of Lemma 3.6 is more precise.

4 – Some properties

Let us first prove a technical lemma which generalizes Lemma 3.5.

Lemma 4.1. *Let (u,g) be a solution of (P), let $v\in W^{1,q}(Q)$ and $F\in W^{1,\infty}_{\rm loc}(\mathbb{R}^2)$, such that:*

- **i)** $F(u-x_n,v)\in L^q(0,T,W^{1,q}(\Omega))$;
- **ii)** $F(\psi-x_n,v)\in W^{1,q}(Q)$;
- **iii)** $F(z_1,z_2)\ge 0$ *for a.e.* $(z_1,z_2)\in\mathbb{R}^2$;
- **iv)** *either* $\dfrac{\partial F}{\partial z_1}(z_1,z_2)\ge 0$ a.e. $(z_1,z_2)\in\mathbb{R}^2$, *or* $\dfrac{\partial F}{\partial z_1}(z_1,z_2)\le 0$ a.e. $(z_1,z_2)\in\mathbb{R}^2$.

Then we have $\forall\,\xi\in\mathfrak{D}(\overline{\Omega}\times(0,T))$:

$$\begin{aligned}\int_Q\Big(\mathcal{A}(x,\nabla u)-g\mathcal{A}(x,e)\Big)\cdot\nabla\Big(F(u-x_n,v)\,\xi\Big)+g\Big(F(0,v)\,\xi\Big)_t\,dx\,dt=\\ =\int_Q\Big(\mathcal{A}(x,\nabla u)-g\,\mathcal{A}(x,e)\Big)\cdot\nabla\Big(F(\psi-x_n,v)\,\xi\Big)+g\Big(F(\psi-x_n,v)\,\xi\Big)_t\,dx\,dt\;.\end{aligned} \tag{4.1}$$

Particularly, if $F(\psi - x_n, v)\,\xi = 0$ *on* Σ_2*, then*

$$\int_Q \Big(\mathcal{A}(x,\nabla u) - g\,\mathcal{A}(x,e)\Big)\cdot\nabla\Big(F(u-x_n,v)\,\xi\Big) + g\Big(F(0,v)\,\xi\Big)_t\,dx\,dt = 0\;. \tag{4.2}$$

Proof: Arguing as in the proof of Lemma 3.5, we get for $\xi\in\mathfrak{D}(\mathbb{R}^n\times(\tau_0,T-\tau_0))$, $\xi\geq 0$, $\tau_0>0$ and $\zeta = (F(u-x_n,v) - F(\psi-x_n,v))\,\xi$

$$\begin{aligned}(4.3)\quad &\int_Q \Big(\mathcal{A}(x,\nabla u(x,t)) - g(x,t)\,\mathcal{A}(x,e)\Big)\cdot\nabla\Big(F(u-x_n,v)\,\xi\Big)(x,t-\tau) + \\ &\qquad\qquad + g(x,t)\,\Big(F(0,v)\,\xi\Big)_t(x,t-\tau)\,dx\,dt\;\; - \\ &- \int_Q \Big(\mathcal{A}(x,\nabla u(x,t)) - g(x,t)\,\mathcal{A}(x,e)\Big)\cdot\nabla\Big(F(\psi-x_n,v)\,\xi\Big)\,(x,t-\tau) - \\ &\qquad\qquad - g(x,t)\,\Big(F(\psi-x_n,v)\,\xi)_t\,(x,t-\tau)\,dx\,dt\;\; = \\ &\qquad\qquad = \frac{\partial}{\partial\tau}G(\tau)\end{aligned}$$

with $G(\tau) = \displaystyle\int_Q g(x,t+\tau)\,((F(u-x_n,v) - F(0,v))\,\xi)(x,t)\,dx\,dt$. Since the integrals on the left hand side of (4.3) are continuous functions on τ, we deduce that $G\in C^1(-\tau_0,\tau_0)$. Using the monotonicity of F and (3.4), we can see that 0 is an extremum for G in $(-\tau_0,\tau_0)$ and

$$\frac{\partial G}{\partial\tau}(0) = 0\;. \tag{4.4}$$

From (4.3) and (4.4) we deduce the Lemma. ∎

From Lemma 4.1, we have:

Corollary 4.2. *Let* (u,g) *be a solution of (P). Then:*

$$\int_Q \mathcal{A}(x,\nabla u)\cdot\nabla\Big(\min\Big(\frac{(u-x_n-k)^+}{\varepsilon},\,1\Big)\,\xi\Big)\,dx\,dt = 0$$
$$\forall\,\varepsilon>0,\quad \forall\,k\geq 0,\quad \forall\,\xi\in\mathfrak{D}(\mathbb{R}^n\times(0,T))\qquad \textit{such that}\;\; \xi\geq 0,\quad \xi=0 \textit{ on } \Sigma_3\;.$$

Proof: It suffices to choose $F(z_1,z_2) = \min(\frac{(z_1-k)^+}{\varepsilon},1)$ in Lemma 4.1 and to take in account (3.4). ∎

Corollary 4.3. *Let* (u,g) *be a solution of (P). Then we have* $u\in L^\infty(Q)$*.*

Proof: Since $u \geq x_n$ a.e. in Q, it suffices to prove that u is bounded above. So let H be a constant such that:

$$H \geq \max\Big(\max\Big\{x_n,\ (x', x_n) \in \overline{\Omega}\Big\},\ \max\Big\{\psi(x,t),\ (x,t) \in \overline{\Sigma}_2\Big\}\Big)\ .$$

Let ξ be a nonnegative function in $\mathfrak{D}(0,T)$. Then if one apply Lemma 4.1 with $F(z_1, z_2) = (z_1 - z_2)^+$ and $v = H - x_n$, we get by taking in account (3.4) and the choice of H:

$$\int_Q \xi(t)\, \mathcal{A}\Big(x, \nabla(u-H)\Big) \cdot \nabla(u-H)^+\, dx\, dt = 0\ . \tag{4.5}$$

Since $(u-H)^+ = 0$ on Σ_2, we deduce from (1.3) and (4.5) that $u \leq H$ a.e. in Q. ∎

Theorem 4.4. *Let (u, g) be a solution of (P). Then we have in the distributional sense:*

$$\operatorname{div}\Big(\mathcal{A}(x, \nabla u) - g\, \mathcal{A}(x, e)\Big) + g_t = 0\ . \tag{4.6}$$

Moreover, if $\operatorname{div}(\mathcal{A}(x,e)) \geq 0$ *in* $\mathfrak{D}'(\Omega)$*, we have:*

$$\operatorname{div}\Big(g\, \mathcal{A}(x,e)\Big) - g_t = \operatorname{div}\Big(\mathcal{A}(x, \nabla u)\Big) \geq 0\ . \tag{4.7}$$

Proof: **i**) Taking $\pm\xi \in \mathfrak{D}(Q)$ as a test function for (P), we get (4.6).

ii) Let $\xi \in \mathfrak{D}(Q)$, $\xi \geq 0$, then from Corollary 4.2, we have for $\varepsilon > 0$ and $k = 0$:

$$\int_Q \mathcal{A}(x, \nabla u) \cdot \nabla\Big(\min\Big(\frac{u - x_n}{\varepsilon}, 1\Big)\, \xi\Big)\, dx\, dt = 0\ . \tag{4.8}$$

Note that $\xi = 0$ on ∂Q, so

$$\int_Q \mathcal{A}(x, e) \cdot \nabla\Big(\Big(1 - \min\Big(\frac{u - x_n}{\varepsilon}, 1\Big)\Big)\, \xi\Big)\, dx\, dt \leq 0\ . \tag{4.9}$$

Adding (4.8) and (4.9), we get:

$$\begin{aligned} &\frac{1}{\varepsilon} \int_{Q \cap [u - x_n < \varepsilon]} \xi\Big(\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla x_n)\Big)\, (\nabla u - \nabla x_n)\, dx\, dt\, + \\ &\qquad + \int_Q \min\Big(\frac{u - x_n}{\varepsilon}, 1\Big)\, \Big(\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla x_n)\Big) \cdot \nabla\xi\, dx\, dt\ \leq \\ &\qquad\qquad \leq\ - \int_Q \mathcal{A}(x, e) \cdot \nabla\xi\, dx\, dt\ . \end{aligned}$$

Using (1.5), we get (4.7) by letting $\varepsilon \to 0$ in the last inequality. ∎

We have $g \in C^0([0,T], W^{-1,q'}(\Omega))$ (see Remark 3.4). The following theorem proves strong continuity of g:

Theorem 4.5. *Let (u,g) be a solution of (P). Assume that $\mathcal{A}(x,e) = \mathcal{A}(e)$ is a constant vector for a.e. $x \in \Omega$. Then we have:*

$$g \in C^0([0,T], L^p(\Omega)) \quad \forall\, p \in [1,+\infty[\ . \tag{4.10}$$

Proof: Using the result of Remark 3.4 and the fact that g is bounded, we deduce that, for any fixed t, we have:

$$g(x,t+h) \underset{h\to 0}{\rightharpoonup} g(x,t) \quad \text{weakly in } \ L^p(\Omega), \ \ \forall\, p \in (1,+\infty)\ . \tag{4.11}$$

Now let $p \in (1,+\infty)$ and $\Omega_h = \{x \in \Omega\, /\ d(x,\partial\Omega) > \beta h\}$, then

$$\begin{aligned} &\left|\int_\Omega \Big(g^p(x,t+h) - g^p(x,t)\Big)\,dx\right| \le \\ &\qquad \le \left|\int_\Omega g^p(x,t+h)\,dx - \int_{\Omega_h} g^p(x,t+h)\,dx\right| \\ &\qquad\quad + \left|\int_{\Omega_h} \Big(g^p(x,t+h) - g^p(x+h\,\mathcal{A}(e),t)\Big)\,dx\right| \\ &\qquad\quad + \left|\int_{\Omega_h} g^p(x+h\mathcal{A}(e),t)\,dx - \int_\Omega g^p(x,t)\,dx\right|\ . \end{aligned} \tag{4.12}$$

It is not difficult to show that, the first and the last terms in the right side of (4.12) converges to 0, when $h \to 0$. Using the following inequality: $|a^p - b^p| \le p\,|a-b|$ $\forall\, a,b \in [0,1]$ $\forall\, p > 1$ and (4.7), the second integral of (4.12) can be written:

$$\begin{aligned} &\left|\int_{\Omega_h} \Big(g^p(x,t+h) - g^p(x+h\,\mathcal{A}(e),t)\Big)\,dx\right| \le \\ &\qquad \le p\int_{\Omega_h} \Big(g(x+h\,\mathcal{A}(e),t) - g(x,t+h)\Big)\,dx\ . \end{aligned} \tag{4.13}$$

Combining (4.11) and (4.13), we get $g(x,t+h) \underset{h\to 0}{\longrightarrow} g(x,t)$ strongly in $L^p(\Omega)$ $\forall\, p \in (1,+\infty)$ and thus $g(x,t+h) \underset{h\to 0}{\longrightarrow} g(x,t)$ strongly in $L^p(\Omega)$ $\forall\, p \in [1,+\infty[$. ∎

5 – Uniqueness of the solution

In this paragraph we assume that

$$(5.1) \qquad \mathcal{A}(x,\xi) = \mathcal{A}(\xi) \quad \forall\, \xi \in \mathbb{R}^n \ \text{ a.e. } x \in \Omega \qquad \text{and} \qquad \mathcal{A}(e)\cdot\nu \le 0 \ \text{ on } \Gamma_1$$

where ν denotes the outward unit normal to Γ_1.

Let us introduce the following notations:

$$(5.2) \qquad \begin{aligned} \sigma_1 &= \overline{\Sigma}_2 \cap \Sigma_1 = (\overline{\Gamma}_2 \cap \Gamma_1) \times (0,T) \ , \\ \sigma_2 &= \overline{\Sigma}_3 \cap \Sigma_4 = (\Gamma_2 \times (0,T)) \cap \overline{\{\varphi > 0\}} \cap \{\varphi = 0\} \ . \end{aligned}$$

We shall assume that σ_1 and σ_2 are $(1,q')$ polar sets of $\overline{Q}$ (see [1]). Recall that when $q > n+1$, the only $(1,q')$ polar set of $\overline{Q}$ is the empty set. So we also assume that $q \le n+1$.

Then we first state some technical lemmas related to the entropy condition (1.1) vi) and to the monotonicity of g. Next, we derive a comparison result which allows us to prove the uniqueness of the solution of (P).

Lemma 5.1. *Let (u,g) be a solution of (P). Then we have:*

$$(5.3) \qquad \int_Q \Big(\mathcal{A}(\nabla u) - \mathcal{A}(e)\Big)\cdot\nabla\xi + (\lambda - g)^+ \Big(\mathcal{A}(e)\cdot\nabla\xi - \xi_t\Big)\, dx\, dt \le 0$$
$$\forall\, \xi \in \mathfrak{D}(\mathbb{R}^n \times (0,T)), \quad \xi \ge 0, \quad \xi = 0 \ \textit{on } \Sigma_1 \cup \Sigma_3, \quad \forall\, \lambda \in [0,1] \ .$$

Proof: First note that since ξ is a test function for (P) and $(1-g)^+ = 1-g$, we have (5.3) for $\lambda = 1$.

Next, let $\varepsilon > 0$. Applying Corollary 4.2, for $k = 0$, we get $\int_Q \mathcal{A}(\nabla u)\cdot\nabla(\min(\frac{u-x_n}{\varepsilon},1)\,\xi)\,dx\,dt = 0$. Since $\min(\frac{u-x_n}{\varepsilon},1)\,\xi = 0$ on $\partial\Omega\times(0,T)$, we have $\int_Q \mathcal{A}(e)\cdot\nabla(\min(\frac{u-x_n}{\varepsilon},1)\,\xi)\,dx\,dt = 0$. Subtracting the second equality from the first one, we obtain:

$$\int_Q \min\Big(\frac{u-x_n}{\varepsilon},1\Big)\Big(\mathcal{A}(\nabla u)-\mathcal{A}(e)\Big)\cdot\nabla\xi\,dx\,dt + {} $$
$$+ \frac{1}{\varepsilon}\int_{Q\cap[u-x_n<\varepsilon]} \xi\Big(\mathcal{A}(\nabla u)-\mathcal{A}(\nabla x_n)\Big)\cdot(\nabla u - \nabla x_n)\,dx\,dt = 0 \ .$$

But the second integral of the above equality is nonnegative by (1.5), then we deduce (5.3) for $\lambda = 0$ by letting $\varepsilon \to 0$.

Now, we can assume, without loss of generality, that $\varepsilon_0 = d(\mathrm{supp}(\xi), \Sigma_1 \cup \Sigma_3) > 0$.

Let us extend u (resp. g) outside Q by x_n (resp. 1) and still denote by u (resp. g) this function.

For $\varepsilon \in (0, \varepsilon_0/2)$, let $\rho_\varepsilon \in \mathfrak{D}(\mathbb{R}^n)$ with $\mathrm{supp}(\rho_\varepsilon) \subset B(0,\varepsilon)$ be a regularizing sequence and let $f_\varepsilon = \rho_\varepsilon * f$, the regularized of a function f. Using (5.3) for $\lambda = 1$ and $\lambda = 0$ we deduce that

$$\int_{\mathbb{R}^n\times(0,T)} (\mathcal{A}(\nabla u))_\varepsilon \cdot \nabla\xi - g_\varepsilon\Big(\mathcal{A}(e)\cdot\nabla\xi - \xi_t\Big)\,dx\,dt \le 0\ , \tag{5.4}$$

$$\int_{\mathbb{R}^n\times(0,T)} \Big((\mathcal{A}(\nabla u))_\varepsilon - \mathcal{A}(e)\Big)\cdot\nabla\xi\,dx\,dt \le 0\ . \tag{5.5}$$

For $\lambda \in [0,1]$, we deduce from (5.5):

$$\int_{\mathbb{R}^n\times(0,T)} \Big((\mathcal{A}(\nabla u))_\varepsilon - \mathcal{A}(e)\Big)\cdot\nabla\xi + (\lambda - g_\varepsilon)\,\Big(\mathcal{A}(e)\cdot\nabla\xi - \xi_t\Big)\,dx\,dt \le 0\ . \tag{5.6}$$

Note that (5.5) and (5.6) are still true for functions of the type $K\xi$ with $K \ge 0$ and $K \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n \times (0,T))$. Whence we deduce for $K = \min((\lambda - g_\varepsilon)^+/\delta,\, 1)$, $\delta > 0$:

$$\begin{aligned}
&\int_{\mathbb{R}^n\times(0,T)} \Big((\mathcal{A}(\nabla u))_\varepsilon - \mathcal{A}(e)\Big)\cdot\nabla\xi + (\lambda - g_\varepsilon)\,\Big(\mathcal{A}(e)\cdot\nabla(K\xi) - (K\xi)_t\Big)\,dx\,dt = \\
&= \int_{\mathbb{R}^n\times(0,T)} \Big((\mathcal{A}(\nabla u))_\varepsilon - \mathcal{A}(e)\Big)\cdot\nabla(K\xi) + (\lambda - g_\varepsilon)\,\Big(\mathcal{A}(e)\cdot\nabla(K\xi) - (K\xi)_t\Big)\,dx\,dt \\
&\quad + \int_{\mathbb{R}^n\times(0,T)} \Big((\mathcal{A}(\nabla u))_\varepsilon - \mathcal{A}(e)\Big)\cdot\nabla\Big((1-K)\,\xi\Big)\,dx\,dt \le 0\ .
\end{aligned} \tag{5.7}$$

Set $I_\delta = \int_{\mathbb{R}^n\times(0,T)} (\lambda - g_\varepsilon)\,\Big(\mathcal{A}(e)\cdot\nabla(K\xi) - (K\xi)_t\Big)\,dx\,dt = I^1_\delta + I^2_\delta$ with

$$I^1_\delta = \int_{\mathbb{R}^n\times(0,T)} (\lambda - g_\varepsilon)\,\min\Big((\lambda - g_\varepsilon)^+/\delta,\, 1\Big)\,\Big(\mathcal{A}(e)\cdot\nabla\xi - \xi_t\Big)\,dx\,dt\ ,$$

$$\begin{aligned}
I^2_\delta &= \int_{\mathbb{R}^n\times(0,T)} (\lambda - g_\varepsilon)\,\xi\,\Bigg(\mathcal{A}(e)\cdot\nabla\Big(\min\Big((\lambda - g_\varepsilon)^+/\delta,\, 1\Big)\Big) \\
&\qquad - \Big(\min\Big((\lambda - g_\varepsilon)^+/\delta,\, 1\Big)\Big)_t\Bigg)\,dx\,dt \\
&= -\frac{1}{2\delta}\int_{\mathbb{R}^n\times(0,T)} \Big(\min\Big((\lambda - g_\varepsilon)^+,\, \delta\Big)\Big)^2\,\Big(\mathcal{A}(e)\cdot\nabla\xi - \xi_t\Big)\,dx\,dt\ .
\end{aligned}$$

Using the above equalities and Lebesgue's theorem, we get (5.3) by letting successively $\delta \to 0$ and $\varepsilon \to 0$ in (5.7). ∎

Lemma 5.2. *Let (u,g) be a solution of (P) and let $\hat{g}\in L^\infty(Q)$ such that:*

$$0\le \hat{g}\le 1 \quad \text{and} \quad \operatorname{div}(\hat{g}\mathcal{A}(e))-\hat{g}_t=0 \quad \text{in} \ \mathfrak{D}'(Q)\ . \tag{5.8}$$

Then we have: $\forall\, k\ge 0,\ \forall\,\lambda\in 1-H(k),\ \forall\,\varepsilon>0,$

$$\int_\Omega\Big\{\mathcal{A}(\nabla u)\cdot\nabla\Big(\min\Big(\frac{(u-x_n-k)^+}{\varepsilon},1\Big)\,\xi\Big)+(\lambda-g)^+\Big(\mathcal{A}(e)\cdot\nabla\xi_1-\xi_{1t}\Big)+ \\ +(\lambda-\hat{g})^+\Big(\mathcal{A}(e)\cdot\nabla\xi_2-\xi_{2t}\Big)\Big\}\,dx\,dt\ \le\ C(u,k,\xi_1) \tag{5.9}$$

$\forall\,\xi,\xi_1,\xi_2\in\mathfrak{D}(\mathbb{R}^n\times(0,T)),\ \ \xi\ge 0,\ \ \xi_1\ge 0,\ \ \xi=\xi_1=0$ *on* $\Sigma_1\cup\Sigma_3,\ \ \xi_2=0$ *on* ∂Q ,

where H denotes the maximal monotone graph associated to the Heaviside function and

$$\begin{aligned} C(u,0,\xi_1) &= -\int_Q\Big(\mathcal{A}(\nabla u)-\mathcal{A}(e)\Big)\cdot\nabla\xi_1\,dx\,dt \\ &= \lim_{\varepsilon\to 0}\int_Q\Big\{\Big(\mathcal{A}(\nabla u)-\mathcal{A}(e)\Big)\cdot\nabla\Big(\min\Big(\frac{u-x_n}{\varepsilon},1\Big)\Big)\Big\}\,\xi_1\,dx\,dt\ , \end{aligned} \tag{5.10}$$

$$C(u,k,\xi_1)=0\,,\qquad \forall\,k>0\ .$$

Proof: From (5.8), we have immediately: $\forall\,\lambda\in\mathbb{R}$, $\forall\,\xi_2\in\mathfrak{D}(\mathbb{R}^n\times(0,T))$ such that $\xi_2=0$ on ∂Q

$$\int_Q(\lambda-\hat{g})^+\Big(\mathcal{A}(e)\cdot\nabla\xi_2-\xi_{2t}\Big)\,dx\,dt=0\ . \tag{5.11}$$

Since $\min(\frac{(u-x_n-k)^+}{\varepsilon},1)\,\xi=0$ on Σ_2 for all $k\ge 0$, we deduce from Corollary 4.2,

$$\int_Q\mathcal{A}(\nabla u)\cdot\nabla\Big(\min\Big(\frac{(u-x_n-k)^+}{\varepsilon},1\Big)\,\xi\Big)\,dx\,dt=0\ . \tag{5.12}$$

Adding (5.11) and (5.12), we deduce (5.9) for $k>0$ since in this case $\lambda=0$.

Using (5.12) for $k=0$, $\xi=\xi_1$ and the fact that $\min(\frac{u-x_n}{\varepsilon},1)\,\xi_1=0$ on $\partial\Omega\times(0,T)$, we get:

$$\int_Q\Big\{\Big(\mathcal{A}(\nabla u)-\mathcal{A}(\nabla x_n)\Big)\cdot\nabla\Big(\min\Big(\frac{u-x_n}{\varepsilon},1\Big)\Big)\Big\}\,\xi_1\,dx\,dt+ \\ +\int_Q\Big\{\Big(\mathcal{A}(\nabla u)-\mathcal{A}(\nabla x_n)\Big)\cdot\nabla\xi_1\Big\}\cdot\min\Big(\frac{u-x_n}{\varepsilon},1\Big)\,dx\,dt=0\ . \tag{5.13}$$

Now, letting $\varepsilon \to 0$, in (5.13), we obtain:

$$\int_Q \Big(\mathcal{A}(\nabla u) - \mathcal{A}(e)\Big) \cdot \nabla \xi_1 \, dx\, dt = \tag{5.14}$$
$$= - \lim_{\varepsilon \to 0} \int_Q \left\{ \Big(\mathcal{A}(\nabla u) - \mathcal{A}(e)\Big) \cdot \nabla \left(\min\Big(\frac{u - x_n}{\varepsilon}, 1 \Big)\right) \right\} \xi_1 \, dx\, dt \;.$$

Hence if we add (5.11), (5.12) and take into account (5.3) written for $\xi = \xi_1$ and (5.14), we get (5.9) for $k = 0$. ∎

Lemma 5.3. *Let (u, g) be a solution of (P) and let $\theta \in C^\infty(\mathbb{R}) \cap C^{0,1}(\mathbb{R})$ such that $\theta(0) = 0$, $\theta' \geq 0$, $\theta \leq 1$. Then we have $\forall\, k \geq 0$, $\forall\, \lambda \in 1 - H(k)$, $\forall\, \varepsilon > 0$:*

$$\int_Q \Big(\mathcal{A}(\nabla u) - \lambda \mathcal{A}(e)\Big) \cdot \nabla \left(\min\Big(\frac{(k - (u - x_n))^+}{\varepsilon}, 1 \Big) \Big(1 - \theta\,(u - x_n)\Big)\, \xi \right) dx\, dt - \tag{5.15}$$
$$- \int_Q (g - \lambda)^+ \Big(\mathcal{A}(e) \cdot \nabla \xi - \xi_t\Big)\, dx\, dt \;\geq\; 0$$
$$\forall\, \xi \in \mathfrak{D}(\mathbb{R}^n \times (0, T)), \quad \xi \geq 0, \quad \Big(1 - \theta\,(u - x_n)\Big)\, \xi = 0 \;\; on \;\Sigma_2 \;.$$

Proof: Let $B = \mathbb{R}^n \times (0, T) \backslash \overline{\Sigma_4}$. Since σ_1 and σ_2 are $(1, q\prime)$ polar sets of $\overline{Q}$, then without loss of generality one can assume that $\xi \in \mathfrak{D}(B)$ and $d(\mathrm{supp}(\xi), \overline{\Sigma_4}) = \varepsilon_0 > 0$.

For $\varepsilon > 0$, let us consider $H_\varepsilon(u - x_n) = \min(\frac{u - x_n}{\varepsilon}, 1)$. Applying Lemma 4.1 with $F(z_1, z_2) = H_\varepsilon(z_1)$ and $F(z_1, z_2) = \theta(z_1)\, H_\varepsilon(z_1)$, we get since $g = 0$ almost everywhere $H_\varepsilon(u - x_n) \neq 0$:

$$\int_Q \mathcal{A}(\nabla u) \cdot \nabla \Big(\Big(1 - \theta\,(u - x_n)\Big)\, \xi \cdot H_\varepsilon(u - x_n) \Big)\, dx\, dt = 0 \;. \tag{5.16}$$

Using (5.16) and taking into account (1.5) and (5.1), we get

$$\int_Q \left\{ \Big(\mathcal{A}(\nabla u) - \mathcal{A}(e)\Big) \cdot \nabla \Big(\Big(1 - \theta\,(u - x_n)\Big)\, \xi \Big) \right\} H_\varepsilon(u - x_n)\, dx\, dt \;\leq \tag{5.17}$$
$$\leq \int_{\Sigma_1} -(\mathcal{A}(e) \cdot \nu)\, \Big(1 - \theta\,(u - x_n)\Big)\, \xi \, d\sigma(x, t) \;.$$

Letting $\varepsilon \to 0$ in (5.17), we get

$$\int_Q \mathcal{A}(\nabla u) \cdot \nabla \Big(\Big(1 - \theta(u - x_n)\Big)\, \xi \Big)\, dx\, dt \leq 0 \;. \tag{5.18}$$

Applying Lemma 4.1 with $F(z_1, z_2) = 1 - \theta(z_1)$, we obtain since $F(0, z_2) = 1$,

$$\int_Q g\Big(\mathcal{A}(e)\cdot\nabla\xi - \xi_t\Big)\,dx\,dt = \int_Q \mathcal{A}(\nabla u)\cdot\nabla\Big(\Big(1-\theta\,(u-x_n)\Big)\,\xi\Big)\,dx\,dt \le 0 \tag{5.19}$$

for any $\xi \in \mathfrak{D}(B)$ such that $\xi \ge 0$, $(1-\theta\,(u-x_n))\,\xi = 0$ on Σ_2 and $d(\text{supp}(\xi), \Sigma_4) = \varepsilon_0 > 0$.

Let us extend g by 0 outside Q and still denote by g this function.

For $\varepsilon \in (0, \varepsilon_0/2)$, let $\rho_\varepsilon \in \mathfrak{D}(\mathbb{R}^n \times (0,T))$ be a regularizing sequence with $\text{supp}(\rho_\varepsilon) \subset B(0,\varepsilon)$. Set $g_\varepsilon = \rho_\varepsilon * g$.

From (5.19), one derives easily for any $\lambda \in \mathbb{R}$ and for $K = \min((g_\varepsilon - \lambda)^+/\delta,\, 1)$, $\delta > 0$,

$$\int_{\mathbb{R}^n\times(0,T)} (g_\varepsilon - \lambda)\,\Big(\mathcal{A}(e)\cdot\nabla(K\xi) - (K\xi)_t\Big)\,dx\,dt \le 0\ . \tag{5.20}$$

Arguing as in the proof of Lemma 5.1, by letting successively $\delta \to 0$ and $\varepsilon \to 0$ in (5.20), we get (5.15) for $k = 0$.

Assume that $k > 0$. Then $\lambda = 0$ and $(g - \lambda)^+ = g$.

Since $s \longmapsto \min(\frac{(k-s)^+}{\varepsilon},\, 1)$ is a nonincreasing function, we obtain by applying Lemma 4.1:

$$\int_Q \Big(\mathcal{A}(\nabla u) - g\mathcal{A}(e)\Big)\cdot\nabla\left(\min\Big(\frac{(k-(u-x_n))^+}{\varepsilon},\,1\Big)\cdot\Big(1-\theta\,(u-x_n)\Big)\,\xi\right) + \\ + \min\Big(\frac{k}{\varepsilon},1\Big)\,g\,\xi_t\ dx\,dt \;=\; 0$$

which can be written by taking into account (5.15) for $k = 0$ and the fact that $g\cdot(u - x_n) = 0$ a.e. in Q:

$$\int_Q \mathcal{A}(\nabla u)\cdot\nabla\left(\min\Big(\frac{(k-(u-x_n))^+}{\varepsilon},1\Big)\cdot\Big(1-\theta\,(u-x_n)\Big)\,\xi\right) - g\Big(\mathcal{A}(e)\cdot\nabla\xi - \xi_t\Big)\,dx\,dt = \\ = \left(\min\Big(\frac{k}{\varepsilon},1\Big) - 1\right)\int_Q g\Big(\mathcal{A}(e)\cdot\nabla\xi - \xi_t\Big)\,dx\,dt \;\ge\; 0\ .$$

Hence the lemma follows for $k > 0$. ∎

Then, we can prove:

Theorem 5.4. *Let (u_1, g_1) and (u_2, g_2) be two solutions of (P). Let B be a bounded open subset of $\mathbb{R}^n$ such that either $B \cap \Gamma = \emptyset$ or $B \cap \Gamma$ is a Lipschitz* *graph. Then we have for* $i=1,2$*:*

$$\int_Q \left\{ \left(\Big(\mathcal{A}(\nabla u_i) - \mathcal{A}(\nabla u_m)\Big) - (g_i - g_M)\,\mathcal{A}(e) \right) \cdot \nabla v + (g_i - g_M)\, v_t \right\} dx\,dt \;\le\; 0 \tag{5.21}$$

$$\forall\, v \in \mathfrak{D}(B\times(0,T)), \quad v\ge 0, \quad \operatorname{supp}(v)\cap(\Sigma_1\cup\Sigma_3)=\emptyset\;,$$

where $u_m=\min(u_1,u_2)$ *and* $g_M=\max(g_1,g_2)$*.*

Proof: Let us consider (u_1,g_1) and (u_2,g_2) as two pairs defined almost everywhere in $Q\times Q$ in the following way:

$$(u_1,g_1)\ (\text{resp.}\,(u_2,g_2))\ : \qquad (x,t,y,s) \longmapsto (u_1(x,t),g_1(x,t))\ (\text{resp.}\,(u_2(y,s),g_2(y,s)))\;. \tag{5.22}$$

Let $v\in\mathfrak{D}(B\times(0,T))$, $v\ge 0$, $\operatorname{supp}(v)\cap(\Sigma_1\cup\Sigma_3)=\emptyset$. Let $\rho_{1,\delta}\in\mathfrak{D}(\mathbb{R})$, $\rho_{1,\delta}\ge 0$, $\int_{\mathbb{R}}\rho_{1,\delta}(t)\,dt=1$, $\operatorname{supp}(\rho_{1,\delta})\subset(-\delta,\delta)$ and let $\rho_{2,\delta}\in\mathfrak{D}(\mathbb{R}^n)$, $\rho_{2,\delta}\ge 0$, $\int_{\mathbb{R}^n}\rho_{2,\delta}(x)\,dx=1$, $\operatorname{supp}(\rho_{2,\delta})\subset B(x_\delta,\delta)$ where $x_\delta\to 0$ when $\delta\to 0$, is such that:

$$\rho_{2,\delta}\Big(\frac{x-y}{2}\Big)=0 \qquad \forall\,(x,y)\in(B\cap\Omega)\times(B\backslash\Omega)\;. \tag{5.23}$$

If we set $\zeta(x,t,y,s)=v(\frac{x+y}{2},\frac{t+s}{2})\,\rho_{1,\delta}(\frac{t-s}{2})\,\rho_{2,\delta}(\frac{x-y}{2})$, then for δ small enough, $\zeta\in\mathfrak{D}(B\times(0,T)\times B\times(0,T))$ and satisfies for $\Sigma=\partial\Omega\times(0,T)$:

$$\zeta=0 \quad \text{on} \quad ((\Sigma_1\cup\Sigma_3)\times Q)\cup(Q\times\Sigma)\;. \tag{5.24}$$

Then, for almost every $(y,s)\in Q$ we can apply Lemma 5.2 to (u_1,g_1) with $k=u_2(y,s)-y_n$, $\lambda=g_2(y,s)$, $\xi(x,t)=\zeta(x,t,y,s)$, $\xi_1(x,t)=\zeta(x,t,y,s)$ and $\xi_2(x,t)=0$. So, denoting $\nabla_x=(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n})$, we get after integrating over Q:

$$\begin{aligned}\int_{Q\times Q}\Bigg\{\mathcal{A}(\nabla_x u_1)\cdot\nabla_x\bigg(\min\Big(\frac{((u_1-x_n)-(u_2-y_n))^+}{\varepsilon},1\Big)\,\zeta\bigg)&+\\ +(g_2-g_1)^+\Big(\mathcal{A}(e)\cdot\nabla_x\zeta-\zeta_t\Big)\Bigg\}\,dx\,dt\,dy\,ds\;&\le\\ \le\;\int_Q C(u_1,\,u_2-y_n,\,\zeta)\,dy\,ds\;.&\end{aligned} \tag{5.25}$$

Similarly, for almost every $(x,t)\in Q$, we apply Lemma 5.3 to (u_2,g_2) with $\theta=0$, $k=u_1(x,t)-x_n$, $\lambda=g_1(x,t)$, $\xi(y,s)=\zeta(x,t,y,s)$. Then we have after integrating over Q, since g_1 does not depend on y and $\zeta = 0$ on $Q \times \Sigma$:

$$(5.26)\quad \int_{Q\times Q} -\Big\{ \mathcal{A}(\nabla_y u_2)\cdot\nabla_y\Big(\min\Big(\frac{((u_1-x_n)-(u_2-y_n))^+}{\varepsilon},\,1\Big)\,\zeta\Big) - \\ -(g_2-g_1)^+\Big(\mathcal{A}(e)\cdot\nabla_y\zeta-\zeta_s\Big)\Big\}\,dx\,dt\,dy\,ds \;\le\; 0\;.$$

Since u_1 does not depend on y, u_2 does not depend on x and $\min(\frac{((u_1-x_n)-(u_2-y_n))^+}{\varepsilon},\,1)\,\zeta = 0$ on $(\Sigma\times Q)\cup(Q\times\Sigma)$, we easily derive that:

$$(5.27)\quad \int_{Q\times Q}\mathcal{A}(\nabla_x u_1)\cdot\nabla_y\Big(\min\Big(\frac{((u_1-x_n)-(u_2-y_n))^+}{\varepsilon},\,1\Big)\,\zeta\Big)\,dx\,dt\,dy\,ds \;=\; 0\;,$$

$$(5.28)\quad -\int_{Q\times Q}\mathcal{A}(\nabla_y u_2)\cdot\nabla_x\Big(\min\Big(\frac{((u_1-x_n)-(u_2-y_n))^+}{\varepsilon},\,1\Big)\,\zeta\Big)\,dx\,dt\,dy\,ds \;=\; 0\;.$$

Now since $(\nabla_x+\nabla_y)\,u_1 = \nabla_x u_1$ and $(\nabla_x+\nabla_y)\,u_2 = \nabla_y u_2$, we get by adding (5.25), (5.26), (5.27), (5.28) and taking in account (1.5)

$$(5.29)\quad \int_{Q\times Q}\Big\{\Big(\Big(\mathcal{A}\Big((\nabla_x+\nabla_y)\,u_1\Big)-\mathcal{A}\Big((\nabla_x+\nabla_y)\,u_2\Big)\Big)\cdot(\nabla_x+\nabla_y)\,\zeta\Big)\cdot \\ \cdot\min\Big(\frac{((u_1-x_n)-(u_2-y_n))^+}{\varepsilon},1\Big)+(g_2-g_1)^+\Big(\mathcal{A}(e)\cdot(\nabla_x+\nabla_y)\zeta-\zeta_t-\zeta_s\Big)\Big\}\,dx\,dt\,dy\,ds \le \\ \le \int_Q C(u_1,\,u_2-y_n,\,\zeta)\,dy\,ds\,- \\ -\int_{Q\times(Q\cap[u_2=y_n])}\Big\{\Big(\mathcal{A}(\nabla_x u_1)-\mathcal{A}(e)\Big)\cdot\nabla_x\Big(\min\Big(\frac{u_1-x_n}{\varepsilon},\,1\Big)\Big)\Big\}\,\zeta\,dx\,dt\,dy\,ds\;.$$

Using (5.10) and the Lebesgue theorem, we get

$$\int_Q C(u_1,\,u_2-y_n,\,\zeta)\,dy\,ds = \\ = \lim_{\varepsilon\to 0}\int_{Q\cap[u_2=y_n]}\Big\{\int_Q\Big\{\Big(\mathcal{A}(\nabla_x u_1)-\mathcal{A}(e)\Big)\cdot\nabla_x\Big(\min\Big(\frac{u_1-x_n}{\varepsilon},\,1\Big)\Big)\Big\}\,\zeta\,dx\,dt\Big\}\,dy\,ds$$

whence by letting $\varepsilon\to 0$ in (5.29), we obtain:

$$(5.30)\quad \int_{Q\times Q}\Big(\Big\{\Big(\mathcal{A}\Big((\nabla_x+\nabla_y)\,u_1\Big)-\mathcal{A}\Big((\nabla_x+\nabla_y)\,u_2\Big)\Big)\cdot(\nabla_x+\nabla_y)\,\zeta\Big\}\cdot \\ \cdot\chi\Big([u_1-u_2\ge x_n-y_n]\Big)+(g_2-g_1)^+\Big(\mathcal{A}(e)\cdot(\nabla_x+\nabla_y)\,\zeta-\zeta_t-\zeta_s\Big)\Big)\,dx\,dt\,dy\,ds \;\le\; 0\;.$$

At this stage, let us introduce the following change of variables:

$$\frac{x+y}{2} = z\,, \qquad \frac{t+s}{2} = \tau\,, \qquad \frac{x-y}{2} = \sigma\,, \qquad \frac{t-s}{2} = \theta\ .$$

Moreover, let:

$$\begin{aligned}
\hat u_1(z,\tau,\sigma,\theta) &= u_1(z+\sigma,\,\tau+\theta), &\qquad \hat g_1(z,\tau,\sigma,\theta) &= g_1(z+\sigma,\,\tau+\theta)\ ,\\
\hat u_2(z,\tau,\sigma,\theta) &= u_2(z-\sigma,\,\tau-\theta), &\qquad \hat g_2(z,\tau,\sigma,\theta) &= g_2(z-\sigma,\,\tau-\theta)\ ,\\
\gamma_1(z,\tau,\sigma,\theta) &= \gamma(z+\sigma,\,\tau+\theta), &\qquad \gamma_2(z,\tau,\sigma,\theta) &= \gamma(z-\sigma,\,\tau-\theta)\ ,
\end{aligned}$$

where γ is the characteristic function of Q. Then, from (5.30) we have:

$$\int_{\mathbb{R}^{2n+2}} \gamma_1\cdot\gamma_2 \biggl\{ \Bigl(\chi\Bigl([\hat u_1-\hat u_2\geq 2\sigma_n]\Bigr)\cdot\Bigl(\mathcal{A}(\nabla_z\hat u_1)-\mathcal{A}(\nabla_z\hat u_2)\Bigr)\Bigr)\cdot\nabla_z v\,+ \\ +(\hat g_2-\hat g_1)^+\Bigl(\mathcal{A}(e)\cdot\nabla_z v-v_\tau\Bigr)\biggr\}\,\rho_{1,\delta}\,\rho_{2,\delta}\;dz\,d\tau\,d\sigma\,d\theta\;\leq\;0\ .$$

Hence, by letting $\delta\to 0$, we get (5.21). ∎

Lemma 5.5. *Let (u_1,g_1) and (u_2,g_2) be two solutions of (P). Let B be a bounded open subset of $\mathbb{R}^n$ such that either $B\cap\Gamma=\emptyset$ or $B\cap\Gamma$ is a Lipschitz graph. Let $\overline g\in L^\infty(Q)$ such that:*

$$\text{(5.31)}\qquad 0\leq\overline g\leq g_i\ \text{ a.e. in } Q\,,\qquad \operatorname{div}(\overline g\mathcal{A}(e))-\overline g_t=0\ \text{ in } \mathfrak{D}'(Q)\ .$$

Then we have for $i,j\in\{1,2\}$, $i\neq j$:

$$\text{(5.32)}\qquad \int_Q\biggl\{\Bigl(\Bigl(\mathcal{A}(\nabla u_i)-\mathcal{A}(\nabla u_m)\Bigr)+(g_j-\overline g)^+\mathcal{A}(e)\Bigr)\cdot\nabla v-(g_j-\overline g)^+v_t\biggr\}\,dx\,dt\;\leq\;0 \\ \forall\, v\in\mathfrak{D}(B\times(0,T)),\quad v\geq 0,\quad \operatorname{supp}(v)\cap(\sigma_1\cup\Sigma_4)=\emptyset\ .$$

Proof: We consider (u_1,g_1) and (u_2,g_2) as two pairs of functions defined in $Q\times Q$ like in (5.22). Let v be as in (5.32), let $\rho_{1,\delta_1}\in\mathfrak{D}(\mathbb{R})$, $\rho_{1,\delta_1}\geq 0$, $\int_{\mathbb{R}}\rho_{1,\delta_1}(t)\,dt=1$, $\operatorname{supp}(\rho_{1,\delta_1})\subset(-\delta_1,\delta_1)$ and let $\rho_{2,\delta_2}\in\mathfrak{D}(\mathbb{R}^n)$, $\rho_{2,\delta_2}\geq 0$, $\int_{\mathbb{R}^n}\rho_{2,\delta_2}(x)\,dx=1$, $\operatorname{supp}(\rho_{2,\delta_2})\subset B(x_{\delta_2},\delta_2)$ where $x_{\delta_2}\to 0$ when $\delta_2\to 0$, is such that: $\rho_{2,\delta_2}(\frac{x-y}{2})=0$ $\forall\,(x,y)\in(B\backslash\Omega)\times(B\cap\Omega)$. Then for δ_1 and δ_2 small enough, let us define $\zeta\in\mathfrak{D}(B\times(0,T)\times B\times(0,T))$ by: $\zeta(x,t,y,s)=v(\frac{x+y}{2},\,\frac{t+s}{2})\,\rho_{1,\delta_1}(\frac{t-s}{2})\,\rho_{2,\delta_2}(\frac{x-y}{2})$. Then we have:

$$\text{(5.33)}\qquad \zeta=0\quad\text{ on }\ (\Sigma\times Q)\cup(Q\times(\sigma_1\cup\Sigma_4))\ .$$

Now, since $\text{supp}(v)\cap(\sigma_1\cup\Sigma_4)=\emptyset$, we can find η such that $0<\eta<\min\limits_{\text{supp}(v)\cap\Sigma_3}\varphi$, if $\text{supp}(v)\cap\Sigma_3\neq\emptyset$. Then there exists $\theta\in C^\infty(\mathbb{R})\cap C^{0,1}(\mathbb{R})$ such that $\theta'\geq 0$ and $\theta(r)=0$ if $r\leq 0$, $\theta(r)=1$ if $r\geq\eta$. If $\text{supp}(v)\cap\Sigma_3=\emptyset$, we take $\theta=0$. Consequently, for δ_1 and δ_2 small enough, we have

$$\Big(1-\theta\left(u_2-y_n\right)\Big)\,\zeta=0 \quad \text{on} \;\; (\Sigma\times Q)\cup(Q\times\Sigma_2)\ . \tag{5.34}$$

Using (5.34), for a.e. $(y,s)\in Q$, we can apply Lemma 5.2 to (u_1,g_1) with $k=u_2(y,s)-y_n$, $\lambda=g_2(y,s)$, $\xi(x,t)=\xi_2(x,t)=\zeta(x,t,y,s)$, $\xi_1(x,t)=0$ and $\hat{g}=\overline{g}$. So we get after integrating over Q:

$$\int_{Q\times Q}\Big\{\mathcal{A}(\nabla_x u_1)\cdot\nabla_x\Big(F_\varepsilon(u_1,u_2)\,\zeta\Big)+ + (g_2-\overline{g})^+\Big(\mathcal{A}(e)\cdot\nabla_x\zeta-\zeta_t\Big)\Big\}\,dx\,dt\,dy\,ds\;\leq\;0 \tag{5.35}$$

where $F_\varepsilon(u_1,u_2)=\min(\frac{((u_1-x_n)-(u_2-y_n))^+}{\varepsilon},1)$. Similarly, for a.e. $(x,t)\in Q$, we apply Lemma 5.3 to (u_2,g_2), with $k=u_1(x,t)-x_n$, $\lambda=\overline{g}(x,t)$, $\xi(y,s)=\zeta(x,t,y,s)$. Then we have:

$$\int_{Q\times Q}-\Big\{\Big(\mathcal{A}(\nabla_y u_2)-\overline{g}\mathcal{A}(e)\Big)\cdot\nabla_y\Big(F_\varepsilon(u_1,u_2)\,\zeta\Big(1-\theta(u_2-y_n)\Big)\Big)- - (g_2-\overline{g})^+\Big(\mathcal{A}(e)\cdot\nabla_y\zeta-\zeta_s\Big)\Big\}\,dx\,dt\,dy\,ds\;\leq\;0\ . \tag{5.36}$$

By Corollary 4.2, we get:

$$\int_{Q\times Q}\mathcal{A}(\nabla_x u_1)\cdot\nabla_x\Big(F_\varepsilon(u_1,u_2)\,\zeta\Big)\,dx\,dt\,dy\,ds=0\ . \tag{5.37}$$

Then we get by adding (5.35) and (5.36):

$$\begin{aligned}
&\int_{Q\times Q}\Big\{\Big(\mathcal{A}(\nabla_x u_1)-\mathcal{A}(\nabla_y u_2)\Big)\cdot(\nabla_x+\nabla_y)\,\Big(F_\varepsilon(u_1,u_2)\,\zeta\Big)+\\
&\qquad + (g_2-\overline{g})^+\Big(\mathcal{A}(e)\cdot(\nabla_x+\nabla_y)\,\zeta-\zeta_t-\zeta_s\Big)\Big\}\,dx\,dt\,dy\,ds\;-\\
&\qquad -\int_{Q\times Q}\mathcal{A}(\nabla_x u_1)\cdot\nabla_y\Big(F_\varepsilon(u_1,u_2)\,\zeta\Big)\,dx\,dt\,dy\,ds\\
&\qquad +\int_{Q\times Q}\mathcal{A}(\nabla_y u_2)\cdot\nabla_x\Big(F_\varepsilon(u_1,u_2)\,\zeta\Big)\,dx\,dt\,dy\,ds\\
&\qquad +\int_{Q\times Q}\mathcal{A}(\nabla_y u_2)\cdot\nabla_y\Big(F_\varepsilon(u_1,u_2)\,\theta(u_2-y_n)\,\zeta\Big)\,dx\,dt\,dy\,ds\\
&\qquad +\int_{Q\times Q}\overline{g}\,\mathcal{A}(e)\cdot\nabla_y\Big(F_\varepsilon(u_1,u_2)\,\Big(1-\theta\left(u_2-y_n\right)\Big)\,\zeta\Big)\,dx\,dt\,dy\,ds\;\;\leq\;\;0
\end{aligned}$$

which leads by taking into account (5.31), (5.34), (5.37)

$$\begin{aligned}(5.38)\quad &\int_{Q\times Q}\Big\{\Big(\mathcal{A}\Big((\nabla_x+\nabla_y)\,u_1\Big)-\mathcal{A}\Big((\nabla_x+\nabla_y)\,u_2\Big)\Big)\cdot(\nabla_x+\nabla_y)\Big(F_\varepsilon(u_1,u_2)\,\zeta\Big)\Big\}+\\ &\qquad+(g_2-\overline{g})^+\Big(\mathcal{A}(e)\cdot(\nabla_x+\nabla_y)\,\zeta-\zeta_t-\zeta_s\Big)\,dx\,dt\,dy\,ds\ +\\ &+\int_{Q\times Q}\Big(\mathcal{A}\Big((\nabla_x+\nabla_y)\,u_2\Big)-\overline{g}\mathcal{A}(e)\Big)\cdot(\nabla_x+\nabla_y)\,\Big(F_\varepsilon(u_1,u_2)\,\theta(u_2-y_n)\,\zeta\Big)-\\ &\quad-\Big(\mathcal{A}\Big((\nabla_x+\nabla_y)\,u_1\Big)-\overline{g}\mathcal{A}(e)\Big)\cdot(\nabla_x+\nabla_y)\,\Big(F_\varepsilon(u_1,u_2)\,\zeta\Big)\,dx\,dt\,dy\,ds\ \le\ 0\ .\end{aligned}$$

Let us denote by $I_1(\varepsilon,\delta_1,\delta_2)$ the first integral in (5.38) and by $I_2(\varepsilon,\delta_1,\delta_2)$ the second one. Using (1.5) we get by letting successively $\delta_2\to 0$, $\delta_1\to 0$, $\varepsilon\to 0$

$$\begin{aligned}(5.39)\quad &\int_Q\Big(\mathcal{A}(\nabla u_1)-\mathcal{A}(\nabla u_m)\Big)\cdot\nabla v+(g_2-\overline{g})^+\Big(\mathcal{A}(e)\cdot\nabla v-v_t\Big)\,dx\,dt\ \le\\ &\qquad\le\ \liminf_{\varepsilon\to 0}\Big(\lim_{\delta_1\to 0}\Big(\lim_{\delta_2\to 0}I_1(\varepsilon,\delta_1,\delta_2)\Big)\Big)\ .\end{aligned}$$

Now letting $\delta_2\to 0$, we get:

$$\begin{aligned}(5.40)\qquad &I_2(\varepsilon,\delta_1)=\lim_{\delta_2\to 0}I_2(\varepsilon,\delta_1,\delta_2)=\\ &=\int_0^T\!\!\int_Q\Big\{\Big(\mathcal{A}(\nabla u_2)-\overline{g}\,\mathcal{A}(e)\Big)\cdot\nabla\Big(\min\Big(\frac{(u_1-u_2)^+}{\varepsilon},\,1\Big)\,\theta(u_2-x_n)\,v\,\rho_{1,\delta_1}\Big)\\ &\quad-\Big(\mathcal{A}(\nabla u_1)-\overline{g}\mathcal{A}(e)\Big)\cdot\nabla\Big(\min\Big(\frac{(u_1-u_2)^+}{\varepsilon},\,1\Big)\,v\,\rho_{1,\delta_1}\Big)\Big\}\,dx\,dt\,ds\ ,\end{aligned}$$

where $u_1=u_1(x,t)$, $\overline{g}=\overline{g}(x,t)$, $u_2=u_2(x,s)$, $g_2=g_2(x,s)$, $v=v(x,\frac{t+s}{2})$, $\rho_{1,\delta_1}=\rho_{1,\delta_1}(\frac{t-s}{2})$. By taking into account that $(1-\theta(u_2(x,s)-x_n))\,v(x,t,s)\,\rho_{1,\delta_1}(\frac{t-s}{2})=0$ $\forall\,(x,t)\in\Sigma_2$, $\forall\,s\in(0,T)$, we deduce from Lemma 4.1 with $F(z_1,z_2)=\min(\frac{(z_1-z_2)^+}{\varepsilon},1)\,(1-\theta(z_2))$ and $v=u_2-x_n$:

$$\begin{aligned}&\int_0^T\!\!\int_Q\Big(\mathcal{A}(\nabla u_1)-g_1\,\mathcal{A}(e)\Big)\cdot\\ &\qquad\cdot\nabla\Big(\min\Big(\frac{(u_1-u_2)^+}{\varepsilon},\,1\Big)\Big(1-\theta(u_2-x_n)\Big)\,v\,\rho_{1,\delta_1}\Big)\,dx\,dt\,ds\ =\ 0\end{aligned}$$

and therefore, from (5.40), we get by taking into account that $\overline{g} \leq g_1$ and that $\theta(0) = 0$:

$$(5.41) \qquad I_2(\varepsilon, \delta_1) =$$

$$= \int_0^T\!\!\int_Q \Big(\mathcal{A}(\nabla u_2) - g_2\, \mathcal{A}(e)\Big)\cdot\nabla\left(\min\Big(\frac{(u_1-u_2)^+}{\varepsilon}, 1\Big)\, \theta(u_2 - x_n)\, v\, \rho_{1,\delta_1}\right) dx\, dt\, ds$$
$$- \int_0^T\!\!\int_Q \Big(\mathcal{A}(\nabla u_1) - g_1\, \mathcal{A}(e)\Big)\cdot\nabla\left(\min\Big(\frac{(u_1-u_2)^+}{\varepsilon}, 1\Big)\, \theta(u_2 - x_n)\, v\, \rho_{1,\delta_1}\right) dx\, dt\, ds\ .$$

Apply Lemma 4.1 to u_2 with $v = u_1 - x_n$, $F(z_1, z_2) = \min(\frac{(z_2-z_1)^+}{\varepsilon}, 1)\,(1-\theta(z_1))$ and $F(z_1, z_2) = \min(\frac{(z_2-z_1)^+}{\varepsilon}, 1)$. Subtract the equations, we get by taking in account the fact that $\theta(0) = 0$, $g_2 \cdot (u_2 - x_n) = 0$ a.e. in Q and since

$$\pm\left(\min\Big(\frac{(u_1-\psi(x,s))^+}{\varepsilon}, 1\Big) - \min\Big(\frac{(\psi(x,t)-\psi(x,s))^+}{\varepsilon}, 1\Big)\right)\theta(\psi(x,s)-x_n)\, v\, \rho_{1,\delta_1}$$

is a test function for (P):

$$(5.42) \quad \int_0^T\!\!\int_Q \Big(\mathcal{A}(\nabla u_2) - g_2\, \mathcal{A}(e)\Big)\cdot\nabla\left(\min\Big(\frac{(u_1-u_2)^+}{\varepsilon}, 1\Big)\, \theta(u_2 - x_n)\, v\, \rho_{1,\delta_1}\right) dx\, dt\, ds =$$

$$= \int_0^T\!\!\int_Q \Big(\mathcal{A}(\nabla u_2) - g_2\, \mathcal{A}(e)\Big)\cdot\nabla\left(\min\Big(\frac{(\psi(x,t)-\psi(x,s))^+}{\varepsilon}, 1\Big)\, \theta(\psi(x,s) - x_n)\, v\rho_{1,\delta_1}\right)$$
$$+ g_2\, \frac{\partial}{\partial s}\left(\min\Big(\frac{(\psi(x,t) - \psi(x,s))^+}{\varepsilon}, 1\Big)\, \theta(\psi(x,s) - x_n)\, v\rho_{1,\delta_1}\right) dx\, dt\, ds\ .$$

In the same way, we have by applying Lemma 4.1 to u_1 with $v = u_2 - x_n$, $F(z_1, z_2) = \min(\frac{(z_1-z_2)^+}{\varepsilon}, 1)\,\theta(z_2)$

$$(5.43) \quad \int_0^T\!\!\int_Q \Big(\mathcal{A}(\nabla u_1) - g_1\, \mathcal{A}(e)\Big)\cdot\nabla\left(\min\Big(\frac{(u_1-u_2)^+}{\varepsilon}, 1\Big)\, \theta(u_2 - x_n)\, v\, \rho_{1,\delta_1}\right) dx\, dt\, ds =$$

$$= \int_0^T\!\!\int_Q \Big(\mathcal{A}(\nabla u_1) - g_1\mathcal{A}(e)\Big)\cdot\nabla\left(\min\Big(\frac{(\psi(x,t)-\psi(x,s))^+}{\varepsilon}, 1\Big)\, \theta\Big(\psi(x,s) - x_n\Big)\, v\, \rho_{1,\delta_1}\right)$$
$$+ g_1\, \frac{\partial}{\partial t}\left(\min\Big(\frac{(\psi(x,t) - \psi(x,s))^+}{\varepsilon}, 1\Big)\, \theta\Big(\psi(x,s) - x_n\Big)\, v\, \rho_{1,\delta_1}\right) dx\, dt\, ds\ .$$

Combining (5.41)–(5.43) we get since g_1 does not depend on s and g_2 does not depend on t:

$$I_2(\varepsilon,\delta_1) = \int_{(0,T)^2}\int_\Omega \Big\{ \Big(\mathcal{A}(\nabla u_2) - \mathcal{A}(\nabla u_1)\Big) + \tag{5.44}$$

$$+ (g_1 - g_2)\,\mathcal{A}(e)\cdot\nabla\left(\min\Big(\frac{(\psi(x,t)-\psi(x,s))^+}{\varepsilon},1\Big)\,\theta\Big(\psi(x,s)-x_n\Big)\,v\right)$$

$$+ (g_2-g_1)\Big(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\Big)\left(\min\Big(\frac{(\psi(x,t)-\psi(x,s))^+}{\varepsilon},1\Big)\,\theta\Big(\psi(x,s)-x_n\Big)\,v\right)\Big\}\,\rho_{1,\delta_1}\,dx\,dt\,ds.$$

Then by letting $\delta_1\to 0$ in (5.44) we obtain obviously $I_2(\varepsilon) = \lim_{\delta_1\to 0} I_2(\varepsilon,\delta_1) = 0$ and therefore from (5.38)–(5.40) we deduce (5.32). ∎

From the above lemma we can prove the following result:

Lemma 5.6. *Let (u_1,g_1) and (u_2,g_2) be two solutions of (P). Let B be a bounded open subset of $\mathbb{R}^n$ such that $B\cap\Gamma=\emptyset$ or $B\cap\Gamma$ is a Lipschitz graph. Then we have for $i=1,2$:*

$$\int_Q \Big\{\Big(\Big(\mathcal{A}(\nabla u_i)-\mathcal{A}(\nabla u_m)\Big)-(g_i-g_M)\,\mathcal{A}(e)\Big)\cdot\nabla v+(g_i-g_M)\,v_t\Big\}\,dx\,dt \;\le\; 0 \tag{5.45}$$

$$\forall\, v\in\mathfrak{D}(B\times(0,T)),\quad v\ge 0,\quad \operatorname{supp}(v)\cap(\sigma_1\cup\Sigma_2)=\emptyset\ .$$

Proof: Let $v\in\mathfrak{D}(B\times(0,T))$, $v\ge 0$ and $\operatorname{supp}(v)\cap(\sigma_1\cup\Sigma_2)=\emptyset$. Let $\theta_\varepsilon\in W^{1,\infty}(\Omega)$ defined by $\theta_\varepsilon(x)=(1-\frac{d(x,\partial\Omega)}{\varepsilon})^+$, set $\Omega_\varepsilon=\{x\in\Omega\,/\,d(x,\partial\Omega)<\varepsilon\}$ and let $\overline{g}$ as in (5.31). Then, from Lemma 5.4, we have for $i=1,2$

$$\int_Q \Big(\Big(\mathcal{A}(\nabla u_i)-\mathcal{A}(\nabla u_m)\Big)-(g_i-g_M)\,\mathcal{A}(e)\Big)\cdot\nabla v+(g_i-g_M)\,v_t\,dx\,dt \;\le$$

$$\le \int_Q \Big(\Big(\mathcal{A}(\nabla u_i)-\mathcal{A}(\nabla u_m)\Big)-(g_i-g_M)\,\mathcal{A}(e)\Big)\cdot\nabla(v\theta_\varepsilon)+(g_i-g_M)\,(v\theta_\varepsilon)_t\,dx\,dt \;=$$

$$= \int_Q \Big(\Big(\mathcal{A}(\nabla u_i)-\mathcal{A}(\nabla u_m)\Big)+(g_j-\overline{g})^+\mathcal{A}(e)\Big)\cdot\nabla(v\theta_\varepsilon)-(g_j-\overline{g})^+(v\theta_\varepsilon)_t\,dx\,dt$$

$$+\int_Q \Big((g_j-g_i)^+-(g_j-\overline{g})^+\Big)\Big(\mathcal{A}(e)\cdot\nabla(v\,\theta_\varepsilon)-(v\,\theta_\varepsilon)_t\Big)\,dx\,dt$$

$$= \int_{\Omega_\varepsilon\times(0,T)} \Big((g_j-g_i)^+-(g_j-\overline{g})^+\Big)\Big(\mathcal{A}(e)\cdot\nabla(v\theta_\varepsilon)-(v\theta_\varepsilon)_t\Big)\,dx\,dt \quad \text{by Lemma 5.5}$$

$$\leq \max(1,\beta)\left(|\Omega_\varepsilon\times(0,T)|^{1/2}\left(\Big(\int_{\Omega_\varepsilon\times(0,T)}|\nabla v|^2\,dx\,dt\Big)^{1/2}+\Big(\int_{\Omega_\varepsilon\times(0,T)}|v_t|^2\,dx\,dt\Big)^{1/2}\right)\right.$$
$$\left.+\frac{1}{\varepsilon}\int_{\Omega_\varepsilon\times(0,T)} v\,dx\,dt\right)\,.$$

From Hölder and Poincaré inequalities, we deduce the existence of a constant C such that:

$$\int_Q\left\{\Big(\Big(\mathcal{A}(\nabla u_i)-\mathcal{A}(\nabla u_m)\Big)-(g_i-g_M)\,\mathcal{A}(e)\Big)\cdot\nabla v+(g_i-g_M)\,v_t\right\}dx\,dt\;\leq$$
$$\leq \max(1,\beta)\left(\Big(T^{1/2}\,|\Omega_\varepsilon|^{1/2}+\varepsilon^{1/2}\,C\Big)\cdot\right.$$
$$\left.\cdot\left(\Big(\int_{\Omega_\varepsilon\times(0,T)}|\nabla v|^2\,dx\,dt\Big)^{1/2}+\Big(\int_{\Omega_\varepsilon\times(0,T)}|v_t|^2\,dx\,dt\Big)^{1/2}\right)\right)$$

which leads to (5.45) by letting ε go to 0. ∎

Lemma 5.7. *Let (u_1,g_1) and (u_2,g_2) be two solutions of (P). Let B be a bounded open subset of $\mathbb{R}^n$ such that $B\cap\Gamma=\emptyset$ or $B\cap\Gamma$ is a Lipschitz graph. Then we have for $i=1,2$:*

$$\int_Q\left\{\Big(\Big(\mathcal{A}(\nabla u_i)-\mathcal{A}(\nabla u_m)\Big)-(g_i-g_M)\,\mathcal{A}(e)\Big)\cdot\nabla v+(g_i-g_M)\,v_t\right\}dx\,dt\;\leq\;0 \tag{5.46}$$
$$\forall\,v\in\mathfrak{D}(B\times(0,T)),\quad v\geq 0,\quad \mathrm{supp}(v)\cap(\Sigma_1\cup\Sigma_4)=\emptyset\,.$$

Proof: Let v be as in Lemma 5.7. Let θ be a smooth function such that $\theta(0)=0$, $0\leq\theta\leq 1$ and $v(1-\theta(\psi-x_n))=0$ on Σ. Let θ_ε, Ω_ε and $\overline{g}$ as in the precedent proof, one derives for $i,j=1,2$:

$$\int_Q\Big(\Big(\mathcal{A}(\nabla u_i)-\mathcal{A}(\nabla u_m)\Big)-(g_i-g_M)\,\mathcal{A}(e)\Big)\cdot\nabla v+(g_i-g_M)\,v_t\,dx\,dt\leq$$
$$\leq\int_{\Omega_\varepsilon\times(0,T)}\Big((g_j-g_i)^+-(g_j-\overline{g})^+\Big)\left(\mathcal{A}(e)\cdot\nabla\Big(v\,\theta_\varepsilon\Big(1-\theta(u_j-x_n)\Big)\Big)-(v\theta_\varepsilon)_t\right)dx\,dt\,.$$

We conclude by using Hölder and Poincaré inequalities and letting ε go to 0. ∎

From the previous lemmas, we can easily state the following theorem:

Theorem 5.8. *Let (u_1,g_1) and (u_2,g_2) be two solutions of (P). Then we have for $i=1,2$:*

$$\int_Q\left\{\Big(\Big(\mathcal{A}(\nabla u_i)-\mathcal{A}(\nabla u_m)\Big)-(g_i-g_M)\,\mathcal{A}(e)\Big)\cdot\nabla v+(g_i-g_M)\,v_t\right\}dx\,dt\;\leq\;0 \tag{5.47}$$
$$\forall\,v\in\mathfrak{D}(B\times(0,T)),\quad v\geq 0,\quad v(x,0)=v(x,T)=0\;\text{ a.e. in }\Omega\,.$$

Proof: Let v like in (5.47). By taking into account that σ_1 and σ_2 are $(1,q')$ polar sets of $\overline{Q}$, there exists a sequence (v_ε) such that: $v_\varepsilon \in \mathfrak{D}(\overline{\Omega}\times(0,T)\backslash(\sigma_1\cup\sigma_2))$, $v_\varepsilon \geq 0$, $v_\varepsilon \longrightarrow v$ in $W^{1,q}(Q)$. By means of partition of the unit we can write: $v_\varepsilon = u_{\varepsilon,1} + v_{\varepsilon,1} + w_{\varepsilon,1}$ where $u_{\varepsilon,1}$, $v_{\varepsilon,1}$ and $w_{\varepsilon,1}$ satisfy respectively Lemma 5.4, Lemma 5.6 and Lemma 5.7, whence we get for $i = 1, 2$

$$\int_\Omega \Big(\Big(\mathcal{A}(\nabla u_i) - \mathcal{A}(\nabla u_m) \Big) - (g_i - g_M)\, \mathcal{A}(e) \Big) \cdot \nabla v_\varepsilon + (g_i - g_M)\, v_{\varepsilon t}\, dx\, dt \;\leq\; 0$$

and by letting $\varepsilon \to 0$, we obtain (5.47). ∎

Now we have the following uniqueness result:

Theorem 5.9. *Assume that (5.1)–(5.2) satisfied, then there exists one and only one solution of (P).*

Proof: Apply (5.47) to a function $v \in \mathfrak{D}(0,T)$, $v \geq 0$, we get:

$$\int_Q (g_i - g_M)\, v_t\, dx\, dt \leq 0 \;\text{ which leads to }\; \frac{d}{dt} \int_\Omega (g_M - g_i)\, dx \leq 0 \;\text{ in } \mathfrak{D}'(0,T),\; i=1,2.$$

Taking into account (4.10), we deduce that $g_1 = g_2 = g_M$ a.e. in Q. From Theorem 5.8 we have now for $i = 1, 2$

$$\text{(5.48)} \qquad \int_Q \Big(\mathcal{A}(\nabla u_i) - \mathcal{A}(\nabla u_m) \Big) \cdot \nabla v\, dx\, dt \leq 0 \qquad \forall\, v \in \mathfrak{D}(\overline{\Omega} \times (0,T)), \quad v \geq 0\,.$$

By approximation, (5.48) still holds for $v = u_i - u_m$ and then we get by (1.5), since $u_1 - u_2 = 0$ on Σ_2, $u_1 = u_2 = u_m$ a.e. in Q. This achieves the proof. ∎

ACKNOWLEDGEMENTS – I would like to thank ESF for supporting my visit at the Universidad Complutense de Madrid where this work was done and I want to thank Professor J. Carrillo for interesting and stimulating conversations.

REFERENCES

[1] Adams, R.A. – *Sobolev Spaces*, Academic Press, New-York, 1975.

[2] Alt, H.W. – Strömungen durch inhomogene poröse, *Medien mit freiem Rand. Journal für die Reine und Angewandte Mathematik,* 305 (1979), 89–115.

[3] Alt, H.W. – The fluid flow through porous media. Regularity of the free surface, *Manuscripta Math.,* 21 (1977), 255–272.

[4] Alt, H.W. – A free boundary problem associated with the flow of ground water, *Arch. Rat. Mech. Anal.,* 64 (1977), 111–126.

[5] Baiocchi, C. – Free boundary problems in the theory of fluid flow through porous media, *Proceedings of the International Congress of Mathematicians – Vancouver,* (1974), 237–243.

[6] Baiocchi, C. – *Free boundary problems in fluid flows through porous media and variational inequalities*, in "Free Boundary Problems" – Proceedings of a seminar held in Pavia Sept–Oct (1979), Vol. 1, Roma, 1980, 175–191.

[7] Baiocchi, C., Comincioli, V., Magenes, E. and Pozzi, G.A. – Free boundary problems in the theory of fluid flow through porous media: existence and uniqueness theorems, *Ann. Mat. Pura Appl.,* 96 (1973), 1–82.

[8] Baiocchi, C. and Friedman, A. – A filtration problem in a porous medium with variable permeability, *Ann. Mat. Pura Appl.,* 4, 114, (1977), 377–393.

[9] Brézis, H., Kinderlehrer, D. and Stampacchia, G. – Sur une nouvelle formulation du problème de l'écoulement à travers une digue, *C.R. Acad. Sci Paris Serie A,* 287 (1978), 711–714.

[10] Brézis, H. – *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North Holland, 1973.

[11] Browder, F.E. – *Problèmes non-linéaires*, Séminaire de Mathématiques supérieures – été 1965. Les presses de l'Université de Montréal.

[12] Browder, F.E. – *Existence theorems for nonlinear partial differential equations*, in "Symposia in Pure Mathematik", Vol. XVI (S.S. Chern and S. Smale, Eds.), pp. 1–60, Amer. Math. Soc., Providence, R.I., 1970.

[13] Bear, J. and Verruijt, A. – *Modeling Groundwater Flow and Pollution*, D. Reidel Publishing Company, Holland, 1992.

[14] Carrillo, J. – On the uniqueness of the solution of the evolution dam problem, *Nonlinear Analysis, Theory, Methods & Applications,* 22(5) (1994), 573–607.

[15] Carrillo, J. and Chipot, M. – On the dam problem, *J. Differential Equations,* 45 (1982), 234–271.

[16] Caffarelli, L.A. and Friedman, A. – The dam problem with two layers, *Arch. Ration. Mech. Anal.,* 68 (1978), 125–154.

[17] Chipot, M. and Lyaghfouri, A. – The dam problem for nonlinear Darcy's law and leaky boundary conditions, *Mathematical Methods in the Applied Sciences,* 20 (1997), 1045–1068.

[18] Dibenedetto, E. and Friedman, A. – Periodic behaviour for the evolutionary dam problem and related free boundary problems, *Communs Partial Diff. Eqns.,* 11 (1986), 1297–1377.

[19] Gilardi, G. – A new approach to evolution free boundary problems, *Communs Partial Diff. Eqns.,* 4 (1979), 1099–1123; 5 (1980), 983–984.

[20] Lions, J.L. – *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod/Gauthier-Villars, Paris, 1969.

[21] Rodrigues, J.F. – On the dam problem with leaky boundary condition, *Portugaliae Mathematica,* 39(1–4) (1980), 399–411.

[22] Smart, D.R. – *Fixed Point Theorems*, Cambridge University Press, 1974.

[23] Stampacchia, G. – On the filtration of a fluid through a porous medium with variable cross section, *Russian Math. Surveys,* 29(4) (1974), 89–102.

[24] Torelli, A. – Existence and uniqueness of the solution of a non steady free boundary problem, *Boll. U.M.I.,* 14-B(5) (1977), 423–466.

[25] Torelli, A. – On a free boundary value problem connected with a non steady filtration phenomenon, *Ann. Sc. Norm. Sup. Pisa,* 4(4) (1977), 33–59.

A. Lyaghfouri,
Institut für Angewandte Mathematik der Universität Zürich-Irchel, Winterthurerstr. 190,
CH-8057 Zürich – SWITZERLAND