PORTUGALIAE MATHEMATICA
Vol. 56 Fasc. 2 – 1999

LINEAR FLOW IN POROUS MEDIA WITH DOUBLE PERIODICITY

R. Bunoiu and J. Saint Jean Paulin

Abstract: We study the classical steady Stokes equations with homogeneous Dirichlet boundary conditions. We work in a 3-D domain which contains solid obstacles, two-periodically distributed with period ε (respectively ε^2), where ε is a small parameter. Our aim is to study the asymptotic behaviour, as $\varepsilon \to 0$. We use the 3-scale convergence for getting the 3-scale limit problem. The problem obtained is a three-pressures system.

Résumé: On étudie le problème de Stokes stationnaire classique, avec des conditions de Dirichlet homogènes au bord. Le problème est posé dans un domaine qui contient des inclusions solides réparties périodiquement, avec périodicité de l'ordre d'un petit paramètre ε et de l'ordre de ε^2. Pour le passsage à la limite en ε, on utilise la méthode de convergence 3-échelle. Le problème limite 3-échelle obtenu est un problème à trois pressions.

Introduction

We study here the homogenization of the Stokes steady flow in double periodic media. We will apply the multi-scale convergence method, introduced by G. Allaire, M. Briane [1]. This method generalizes the two-scale convergence method introduced by G. Nguetseng [10] for the simply periodic domains.

The problem presented here was first treated by J.-L. Lions [8]. The method used for getting the limit problem was the formal expansion of the velocity and of the pressure. The results we present here justify the expansions.

In §1 we give the mathematical model of the problem. We define the domain which has two parts: the fluid part and the solid part. The solid part is made

Received: July 15, 1997; *Revised*: November 14, 1997.

by solid obstacles two-periodically distributed, with period ε (respectively ε^2), where ε is a small parameter.

In §2 we give a priori estimates and convergence results for the velocity. Next we recall and prove some results related on the three-scale convergence.

In §3 we construct the extension of the pressure to the whole of Ω and we give a convergence result. The difficulty is here the construction of an extension operator for the pressure to the solid part of the domain. We already know some methods for constructing such an extension (cf. L. Tartar [12], R. Lipton, M. Avellaneda [9], C. Conca [2], I.-A. Ene, J. Saint Jean Paulin [5]). The last two methods are applied for a problem with Neumann type boundary conditions at the fluid-solid interface. The extension presented here is a generalization of the method presented in L. Tartar [12].

In §4 we pass to the limit as $\varepsilon \to 0$ in the initial problem. We obtain the 3-scale limit system, which represents a three-pressures problem.

The Stokes problem in double periodic media was already studied by T. Lévy [7] and P. Donato, J. Saint Jean Paulin [3], but the domain presented here is a different one. The solid obstacles periodically distributed with period ε in the domain presented here are replaced in [7] and [3] by the fluid, which corresponds to a porous fissured rock.

An analogous result for the Poisson equation in porous fissured rocks was studied by P. Donato, J. Saint Jean Paulin [4].

1 – Positionning of the problem

Let Ω be a bounded open domain of boundary $\partial\Omega$ in $\mathbb{R}^N$, $N \geq 2$.

Let us consider two sets $Y = \prod_{i=1}^N]0,1[$ and $Z = \prod_{i=1}^N]0,1[$ and two closed subsets $Y_s \subset Y$, $Z_s \subset Z$, with non-empty interior, contained in Y (respectively Z). We define:

$$Y^* = Y \setminus Y_s\,, \qquad Z^* = Z \setminus Z_s\ .$$

Let ε be a small positive parameter. Let us suppose that there exists an ε such that the domain $\overline{Y}$ is exactly covered by a finite number of cells $\overline{\varepsilon Z}$. Moreover, let us suppose that Y_s is exactly covered by a finite number of cells εZ. This last hypothesis implies some restrictions for the geometry of Y_s (see an example in Figure 1.1). We deduce that there is no intersection between the solid obstacles Y_s and εZ_s in the cell Y, as we can see in Figure 1.3. If we consider all the small parameters $\dfrac{\varepsilon}{2^n}$, the above assumptions are still true.

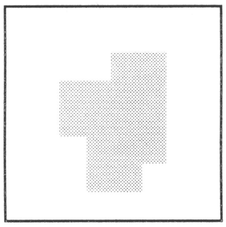

Fig. 1.1 — Domain Y.

Fig. 1.2 – Domain Z.

We multiply the new cell (Figure 1.3) by ε and we repeat it in the domain Ω. We assume (for simplicity), that $\overline{\Omega}$ is exactly covered by a finite number of cells $\overline{\varepsilon Y}$. We define Ω_ε by taking out of Ω the domains εY_s and $\varepsilon^2 Z_s$. Let us notice that there is no intersection between the solid obstacles εY_s and $\varepsilon^2 Z_s$ in Ω_ε, because there is no intersection between the solid ostacles Y_s and εZ_s in the cell Y. The domain Ω_ε (which corresponds to the fluid) is connected, but the union of solid obstacles is not connected.

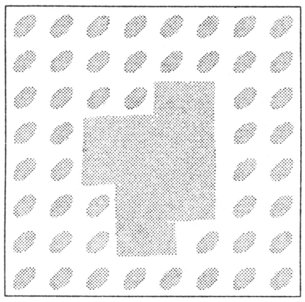

Fig. 1.1 – Domain Y with obstacle Y_s and obstacles εZ_s.

Let χ_{Y^*} and χ_{Z^*} be the characteristic functions of the domains Y^* and Z^*, defined by:

$$\chi_{Y^*}(y) = \begin{cases} 1 & \text{in } Y^*, \\ 0 & \text{in } Y \setminus Y^* \,, \end{cases} \qquad \chi_{Z^*}(z) = \begin{cases} 1 & \text{in } Z^*, \\ 0 & \text{in } Z \setminus Z^* \,. \end{cases}$$

We extend the characteristic functions χ_{Y^*} (respectively χ_{Z^*}) by periodicity, with period 1 in y_i and in z_i, for $i = 1, ..., N$. The domain Ω_ε, defined as above is described by:

$$\Omega_\varepsilon = \left\{ x \,|\, x \in \Omega,\ \chi_{Y^*}\left(\frac{x}{\varepsilon}\right) \chi_{Z^*}\left(\frac{x}{\varepsilon^2}\right) = 1 \right\} . \tag{1.1}$$

The domain Ω_ε presents a double periodicity, with small solid obstacles of order ε and with very small obstacles of order ε^2. This domain modelizes a rigid porous medium with double periodicity.

We define the boundary of Ω_ε, denoted by $\partial\Omega_\varepsilon$ and composed by three parts:

– the boundary of obstacles εY_s,

– the boundary of obstacles $\varepsilon^2 Z_s$,

– the boundary of Ω.

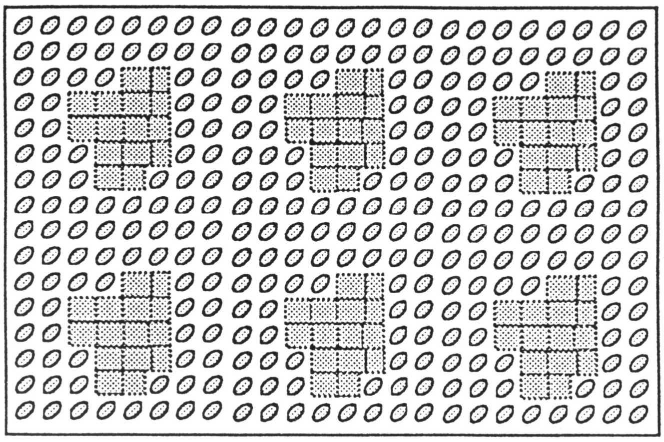

Fig. 1.4 – A porous medium with double periodicity.

In Ω_ε defined as above, we consider the following Stokes problem:

$$\begin{cases} -\varepsilon^2 \Delta u^\varepsilon + \nabla p^\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \operatorname{div} u^\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \ . \end{cases} \tag{1.2}$$

The first relation in (1.2) represents the classical steady Stokes equation. The term ε^2 represents the order of fluid's viscosity. This assumption is not essential for a linear problem, because we can always rescale. The second relation is the incompressibility condition of the fluid. On the boundary of Ω_ε we consider Dirichlet homogeneous conditions. We recall that for a domain D we define the spaces $\mathbb{L}^2(D)$ and $\mathbb{H}_0^1(D)$ by:

$$\mathbb{L}^2(D) = (L^2(D))^N \ ,$$

$$\mathbb{H}_0^1(D) = \left\{ \psi \in \mathbb{L}^2(D),\ \frac{\partial \psi}{\partial x_i} \in \mathbb{L}^2(D),\ \psi = 0 \text{ on } \partial D \right\} .$$

The exterior body forces are denoted by f. The function $f = (f_i)_{i=1,...,N}$ belongs to $\mathbb{L}^2(\Omega)$ and the right hand side of relation (1.2) represents the restriction of f to Ω_ε. The existence and the uniqueness of a solution $(u^\varepsilon, p^\varepsilon) \in \mathbb{H}_0^1(\Omega_\varepsilon) \times \mathbb{L}^2(\Omega)/\mathbb{R}$ for (1.2) is classical (see R. Temam [13]).

2 – A priori estimates and convergence results for the velocity

Our aim is to study the asymptotic behaviour, as $\varepsilon \to 0$, of the solution of problem (1.2). For passing to the limit we need extensions of velocity and pressure to the whole of Ω. We first give a Poincaré's type lemma, adapted at the domain presented in §1:

Lemma 2.1. *For any function $\phi \in \mathbb{H}_0^1(\Omega_\varepsilon)$, we have:*

$$|\phi|_{\mathbb{L}^2(\Omega_\varepsilon)} \leq c\,\varepsilon^2\, |\nabla\phi|_{[\mathbb{L}^2(\Omega_\varepsilon)]^N} \ . \tag{2.1}$$

In the following, c denotes a constant independent of ε.

Let $\widetilde{u^\varepsilon}$ be the extension of u^ε by zero to the whole of Ω. For the function $\widetilde{u^\varepsilon}$ we can easily prove the following a priori estimates:

Proposition 2.2. *If u^ε is solution of (1.2), then we have:*

$$|\nabla\widetilde{u^\varepsilon}|_{[\mathbb{L}^2(\Omega_\varepsilon)]^N} \leq c \ , \tag{2.2}$$

$$|\widetilde{u^\varepsilon}|_{\mathbb{L}^2(\Omega_\varepsilon)} \leq c\,\varepsilon^2 \ . \tag{2.3}$$

Before establishing convergence of the velocity, we first recall and prove some general results adapted below to our case.

Let us denote by $C_p^\infty(Y \times Z)$ the space of C^∞ functions, Y-periodic and Z-periodic. We have the following lemma:

Lemma 2.3 (G. Allaire, M. Briane [1])**.** *Let v^ε be a sequence of bounded functions in $\mathbb{L}^2(\Omega)$. Then there exists a subsequence still denoted v^ε and a function $v \in \mathbb{L}^2(\Omega \times Y \times Z)$ such that:*

$$\lim_{\varepsilon\to 0} \int_\Omega v^\varepsilon(x)\,\varphi\Big(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\Big)\,dx = \iiint_{\Omega\,Y\,Z} v(x,y,z)\,\varphi(x,y,z)\,dx\,dy\,dz \ , \tag{2.4}$$

for every function $\varphi(x,y,z) \in \mathbb{L}^2(\Omega, C_p^\infty(Y\times Z))$. *We say that* v^ε *3-scale converges to* v. *Moreover,*

$$v^\varepsilon \rightharpoonup v_0 = \int_Y\int_Z v\,dy\,dz \quad \textit{weakly in } \mathbb{L}^2(\Omega)\ .$$

As in G. Allaire, M. Briane [1], we prove the result:

Proposition 2.4. *Let* v^ε *be a bounded sequence in* $\mathbb{L}^2(\Omega)$ *which 3-scale converges to* v *and such that*

$$\operatorname{div} v^\varepsilon = 0 \quad \textit{in } \Omega\ . \tag{2.5}$$

Then the limit v *satisfies the following relations:*

$$\operatorname{div}_x \int_Y\int_Z v\,dz\,dy = 0\ , \tag{2.6}$$

$$\operatorname{div}_y \int_Z v\,dz = 0\ , \tag{2.7}$$

$$\operatorname{div}_z v = 0\ . \tag{2.8}$$

Proof: Let φ be a function in $\mathcal{D}(\Omega)$. We multiply relation (2.5) by φ and integrating by parts, we get:

$$0 = \lim_{\varepsilon\to 0}\int_\Omega (\operatorname{div} v^\varepsilon(x))\,\varphi(x)\,dx = -\lim_{\varepsilon\to 0}\int_\Omega v^\varepsilon\,\nabla\varphi\,dx\ .$$

But

$$\lim_{\varepsilon\to 0}\int_\Omega v^\varepsilon\,\nabla\varphi\,dx = \int_\Omega\int_Y\int_Z v(x,y,z)\,\nabla\varphi(x)\,dx\,dy\,dz$$

since v^ε 3-scale converges to v.

We deduce

$$\int_\Omega \operatorname{div}_x\Big(\int_Y\int_Z v\,dz\,dy\Big)\,\varphi(x)\,dx = 0\,, \qquad \forall\,\varphi\in\mathcal{D}(\Omega)\ ,$$

which implies (2.6). Multiplying (2.5) by particular functions $\varphi \in \mathcal{D}(\Omega, C_p^\infty(Y))$ and $\varphi \in \mathcal{D}(\Omega, C_p^\infty(Y\times Z))$, we obtain the relations (2.7)–(2.8). ∎

Remark 2.5. For a set D, let $\mathbb{H}_p^1(D)$ be the space of functions $\mathbb{H}_{\mathrm{loc}}^1(\mathbb{R}^N)$ which are D-periodic.

Choosing particular test functions $\varphi \in \mathbb{H}^1_p(Y)$ (respectively $\varphi \in \mathbb{H}^1_p(Z)$) in relation (2.7) (respectively (2.8)), we obtain the following periodicity condition:

$$\left[\int_Z v(x,y,z,)\,dz\right]\nu_Y \text{ takes opposite values on opposite faces of } Y\ ,$$

$$v\,\nu_Z \text{ takes opposite values on opposite faces of } Z\ ,$$

where ν_Y (resp. ν_Z) represents the unit outward normal to Y (resp. Z).

For the velocity's extension $\widetilde{u^\varepsilon}$ we prove the following results:

Proposition 2.6. *Let $\widetilde{u^\varepsilon}$ be defined as before. Then there exists $u\in\mathbb{L}^2(\Omega\times Y\times Z)$ such that, up to a subsequence, we have:*

$$\varepsilon^{-2}\,\widetilde{u^\varepsilon} \to u \quad \textit{3-scale}\ , \tag{2.9}$$

$$u(x,y,z) = 0 \quad \textit{in}\ \ \Omega\times Y_s\times Z_s\ , \tag{2.10}$$

$$\varepsilon^{-2}\,\widetilde{u^\varepsilon} \rightharpoonup u_0 = \int_{Y^*}\int_{Z^*} u\,dz\,dy \quad \textit{weakly in}\ \ \mathbb{L}^2(\Omega)\ , \tag{2.11}$$

$$\nabla\widetilde{u^\varepsilon} \to \nabla_z u \quad \textit{3-scale}\ . \tag{2.12}$$

Proof: Relation (2.9) is a direct consequence of Lemma 2.3 applied for $v=\varepsilon^{-2}\,\widetilde{u^\varepsilon}$. This is possible according to estimate (2.3).

For proving (2.10) we note that, for $v^\varepsilon=\varepsilon^{-2}\,\widetilde{u^\varepsilon}$, relation (2.4) becomes:

$$\lim_{\varepsilon\to 0}\int_\Omega \varepsilon^{-2}\,\widetilde{u^\varepsilon}(x)\,\varphi\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big)\,dx \;=\int_\Omega\int_Y\int_Z u(x,y,z)\,\varphi(x,y,z)\,dz\,dy\,dx\ .$$

We choose a test function φ such that $\varphi=0$ in $\Omega\times Y^*\times Z^*$. Using $\widetilde{u^\varepsilon}=0$ in $\Omega\setminus\Omega_\varepsilon$, we deduce:

$$0 \;=\int_\Omega\int_{Y_s}\int_{Z_s} u(x,y,z)\,\varphi(x,y,z)\,dz\,dy\,dx\ ,$$

wich implies (2.10).

Relation (2.11) is a direct consequence of Lemma 2.3 and of relation (2.10).

For proving relation (2.12) we note that relation (2.2) and Lemma 2.3 imply the existence of a function $\xi\in[\mathbb{L}^2(\Omega\times Y\times Z)]^N$ such that:

$$\lim_{\varepsilon\to 0}\int_\Omega \nabla\widetilde{u^\varepsilon}\,\varphi\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big)\,dx \;=\int_\Omega\int_Y\int_Z \xi(x,y,z,)\,\varphi(x,y,z)\,dz\,dy\,dx\ .$$

Integrating the left hand term by parts, we get:

$$\begin{aligned}\int_\Omega \nabla\widetilde{u^\varepsilon}\,\varphi\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big)\,dx &= -\int_\Omega \widetilde{u^\varepsilon}\,\mathrm{div}\,\varphi\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big)\,dx\\ &= -\int_\Omega \widetilde{u^\varepsilon}\Big(\mathrm{div}_x\,\varphi+\frac{1}{\varepsilon}\,\mathrm{div}_y\,\varphi+\frac{1}{\varepsilon^2}\,\mathrm{div}_z\,\varphi\Big)\,dx\\ &= -\int_\Omega \varepsilon^{-2}\Big(\varepsilon^2\,\widetilde{u^\varepsilon}\,\mathrm{div}_x\,\varphi+\varepsilon\,\widetilde{u^\varepsilon}\,\mathrm{div}_y\,\varphi+\widetilde{u^\varepsilon}\,\mathrm{div}_z\,\varphi\Big)\,dx\ .\end{aligned}$$

Passing to the limit in ε, we derive:

$$-\int_\Omega\int_Y\int_Z u(x,y,z)\,\mathrm{div}_z\,\varphi(x,y,z)\,dz\,dy\,dx \;=\; \int_\Omega\int_Y\int_Z \xi(x,y,z)\,\varphi(x,y,z)\,dz\,dy\,dx\ .$$

Integrating the left hand side of the previous relation by parts we deduce:

$$\int_\Omega\int_Y\int_Z \Big[\xi(x,y,z)-\nabla_z u(x,y,z)\Big]\,\varphi(x,y,z)\,dz\,dy\,dx \;=\; 0\,, \qquad \forall\,\varphi\in\mathcal{D}(\Omega,C_p^\infty(Y\times Z))\ ,$$

consequently $\xi(x,y,z)=\nabla_z u(x,y,z)$, which ends the proof. ∎

Proposition 2.7. *Let u be the function defined by Proposition 2.6. Then we have:*

$$\mathrm{div}_x \int_{Y^*}\int_{Z^*} u\,dz\,dy = 0\ , \tag{2.13}$$

$$\mathrm{div}_y \int_{Z^*} u\,dz = 0\ , \tag{2.14}$$

$$\mathrm{div}_z\, u = 0\ , \tag{2.15}$$

$$\Big[\int_{Y^*}\int_{Z^*} u\,dy\,dz\Big]\cdot\nu = 0 \quad \textit{on}\ \ \partial\Omega\ , \tag{2.16}$$

$$u\cdot\nu_Z \ \ \textit{takes opposite values on opposite faces of}\ Z\ , \tag{2.17}$$

$$\Big[\int_{Z^*} u\,dz\Big]\cdot\nu_Y \ \ \textit{takes opposite values on opposite faces of}\ Y\ , \tag{2.18}$$

$$u\cdot\nu_Z = 0 \quad \textit{on}\ \ \partial Z_s\ , \tag{2.19}$$

$$\Big[\int_{Z^*} u\,dz\Big]\cdot\nu_Y = 0 \quad \textit{on}\ \ \partial Y_s\ . \tag{2.20}$$

Proof: The relations (2.13)–(2.15) are a consequence of Proposition 2.4 applied to the sequence $\varepsilon^{-2}\,\widetilde{u^\varepsilon}$ (which 3-scale converges to u) and of relation (2.10).

In order to obtain relation (2.16), we use the linearity and the continuity of the normal trace application from

$$\mathbb{H}(\mathrm{div},\Omega) = \Big\{\psi \in \mathbb{L}^2(\Omega) \mid \mathrm{div}\,\psi \in L^2(\Omega)\Big\}$$

into $\mathbb{H}^{-\frac{1}{2}}(\partial\Omega)$, then we use the relations (2.10)–(2.11).

The relations (2.17)–(2.18) are a consequence of Remark 2.5 and of relation (2.10).

Multiplying (2.15) by a test function $\psi \in \mathbb{H}^1_p(Z)$ and using (2.17) we obtain relation (2.19). We get relation (2.20) by multiplying (2.14) by $\psi \in \mathbb{H}^1_p(Y)$ and using (2.18). ∎

3 – Extension of the pressure and convergence result

We now construct a restriction operator S_{ε^2} from $\mathbb{H}^1_0(\Omega)$ into $\mathbb{H}^1_0(\Omega_\varepsilon)$. Using this operator we will define an extension for the pressure to the whole of Ω.

Let Y^ε_f be the domain defined by:

$$Y^\varepsilon_f = Y \setminus \Big(Y_s \cup \Big(\bigcup \varepsilon Z_s\Big)\Big)\ .$$

We define the space $\mathbb{H}^1_s(Y^\varepsilon_f)$ by:

$$\mathbb{H}^1_s(Y^\varepsilon_f) = \Big\{\phi \in \mathbb{H}^1(Y^\varepsilon_f) \mid\ \phi = 0 \ \text{ on } \partial Y_s \ \text{ and on } \partial\Big(\bigcup(\varepsilon Z_s)\Big)\Big\}\ .$$

We define the space $\mathbb{H}^1_s(Y^*)$ by:

$$\mathbb{H}^1_s(Y^*) = \Big\{\phi \in \mathbb{H}^1(Y^*) \mid\ \phi = 0 \ \text{ on } \partial Y_s\Big\}\ .$$

To prove the claimed result, we first construct a restriction operator R from the space $\mathbb{H}^1(Y)$ into the space $\mathbb{H}^1_s(Y^*)$ and next we construct the operator W_ε from the space $\mathbb{H}^1_s(Y^*)$ into the space $\mathbb{H}^1_s(Y^\varepsilon_s)$. Using operators R and W_ε, we construct the operator

$$S_\varepsilon\colon\ \mathbb{H}^1(Y) \to \mathbb{H}^1(Y^\varepsilon_f)$$

and next we define S_{ε^2} by applying S_ε to each period εY of Ω.

So we construct S_{ε^2} in three steps, corresponding to the three following lemmas.

Lemma 3.1. *There exists a restriction operator*

$$R\colon\ \mathbb{H}^1(Y) \to \mathbb{H}^1_s(Y^*)$$

such that for $v \in \mathbb{H}^1(Y)$ *we have:*

$$Rv = v \quad \text{if } \ v = 0 \ \text{ in } Y_s\ , \tag{3.1}$$

$$\operatorname{div} Rv = 0 \ \text{ in } Y^* \quad \text{if } \ \operatorname{div} v = 0 \ \text{ in } Y\ , \tag{3.2}$$

$$|Rv|_{\mathbb{H}^1(Y)} \le c\,|v|_{\mathbb{H}^1(Y)}\ . \tag{3.3}$$

Proof: Let us consider a smooth surface γ strictly contained in Y, enclosing Y_S. We denote by $\widetilde{Y_s}$ the domain between γ and ∂Y_s.

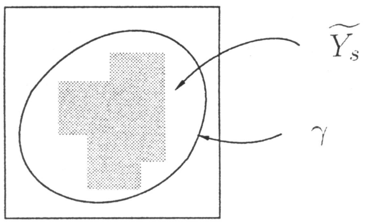

Fig. 3.1 – Domain Y.

As in Lemma 3 of L. Tartar [12], we have the following result:
If $v \in \mathbb{H}^1(Y)$, there exist $w \in \mathbb{H}^1(\widetilde{Y_s})$, $q \in L^2(\widetilde{Y_s})/\mathbb{R}$ such that:

$$\begin{cases} -\Delta w = -\Delta v + \nabla q & \text{in } \widetilde{Y_s}\,,\\ \operatorname{div} w = \operatorname{div} v + \dfrac{1}{|\widetilde{Y_s}|}\displaystyle\int_{Y_s} \operatorname{div} v\,dy & \text{in } \widetilde{Y_s}\,,\\ w|_\gamma = v|_\gamma\,, \quad w|_{\partial Y_s} = 0\ , \end{cases}$$

where $|\widetilde{Y_s}|$ represents the measure of Y_s. Moreover, there exists a constant c independent of v such that:

$$|w|_{\mathbb{H}^1(\widetilde{Y_s})} \le c\,|v|_{\mathbb{H}^1(Y)}\ .$$

Let us notice that: $Y = Y^* \cup Y_s = (Y^* \setminus \widetilde{Y_s}) \cup \widetilde{Y_s} \cup Y_s$.

We define the operator R by:

$$Rv(y) = \begin{cases} v(y) & \text{if } y \in Y^* \setminus \widetilde{Y_s}\,, \\ w(y) & \text{if } y \in \widetilde{Y_s}\,, \\ 0 & \text{if } y \in Y_s\ . \end{cases}$$

This definition and properties satisfied by the function v imply the relations (3.1)–(3.3). ∎

Lemma 3.2. *There exists a restriction operator*

$$W_\varepsilon\colon\, \mathbb{H}^1_s(Y^*) \to \mathbb{H}^1_s(Y^\varepsilon_f)$$

such that for $Rv \in \mathbb{H}^1_s(Y^*)$ we have:

$$W_\varepsilon(Rv) = Rv \quad \text{if } \ Rv = 0 \ \text{ in } \bigcup(\varepsilon Z_s)\ , \tag{3.4}$$

$$\operatorname{div} W_\varepsilon(Rv) = 0 \ \text{ in } Y^\varepsilon_f \quad \text{if } \ \operatorname{div} Rv = 0 \ \text{ in } Y^*\ , \tag{3.5}$$

$$\varepsilon^2\, |\nabla W_\varepsilon(Rv)|^2_{[\mathbb{L}^2(Y^\varepsilon_f)]^N} + |W_\varepsilon(Rv)|^2_{\mathbb{L}^2(Y^\varepsilon_f)} \le c\, |v|^2_{\mathbb{H}^1(Y)}\ . \tag{3.6}$$

Proof: Let $\mathbb{H}^1_s(Z^*)$ be the space defined by:

$$\mathbb{H}^1_s(Z^*) = \Big\{\phi \in \mathbb{H}^1(Z^*)\,|\ \phi = 0 \ \text{ in } \partial Z_s\Big\}\ .$$

In the fixed cell Z, let us consider a smooth surface $\widetilde{\gamma}$ strictly contained in Z. We denote by $\widetilde{Z_s}$ the domain between $\widetilde{\gamma}$ and ∂Z_s. The domain $\widetilde{Z_s}$ is independent of the parameter ε.

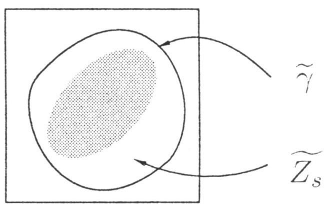

Fig. 3.2 – Domain Z.

As in lemma 3 of L. Tartar [12], we have the following result. If $\bar{u} \in \mathbb{H}^1(Z)$, there exist $\bar{w} \in \mathbb{H}^1(\widetilde{Z_s})$, $\bar{q} \in L^2(\widetilde{Z_s})/\mathbb{R}$ such that:

$$\begin{cases} -\Delta \bar{w} = -\Delta \bar{u} + \nabla \bar{q} & \text{in } \widetilde{Z_s}\,, \\ \operatorname{div} \bar{w} = \operatorname{div} \bar{u} + \dfrac{1}{|\widetilde{Z_s}|} \displaystyle\int_{Y_s} \operatorname{div} \bar{u}\, dy & \text{in } \widetilde{Z_s}\,, \\ \bar{w}|_{\widetilde{\gamma}} = \widetilde{u}|_{\widetilde{\gamma}}\,, \qquad \bar{w}|_{\partial Z_s} = 0\ . \end{cases}$$

Moreover, there exists a constant c independent of $\bar{u}$ such that:

$$|\bar{w}|_{\mathbb{H}^1(\widetilde{Z_s})} \leq c\, |\bar{u}|_{\mathbb{H}^1(Z)}\ .$$

Let us notice that $Z = Z^* \cup Z_s = (Z^* \setminus \widetilde{Z_s}) \cup \widetilde{Z_s} \cup Z_s$.

For every function $\bar{u} \in \mathbb{H}^1(Z)$ we construct an application

$$\overline{W}\colon\ \mathbb{H}^1(Z) \to \mathbb{H}^1_s(Z^*)$$

defined by:

$$\overline{W}(\bar{u})(z) = \begin{cases} \bar{u}(z) & \text{if } z \in Z^* \setminus \widetilde{Z_s}\,, \\ \overline{w}(z) & \text{if } z \in \widetilde{Z_s}\,, \\ 0 & \text{if } z \in Z_s\ , \end{cases} \tag{3.7}$$

and satisfying:

$$\begin{aligned} &\overline{W}(\bar{u}) = \bar{u} \quad \text{if } \ \bar{u} = 0 \ \text{ in } Z_s\ , \\ &\operatorname{div} \overline{W}(\bar{u}) = 0 \quad \text{if } \ \operatorname{div} \bar{u} = 0\ , \\ &|\overline{W}(\bar{u})|_{\mathbb{H}^1(Z)} \leq c\, |\bar{u}|_{\mathbb{H}^1(Z)}\ . \end{aligned} \tag{3.8}$$

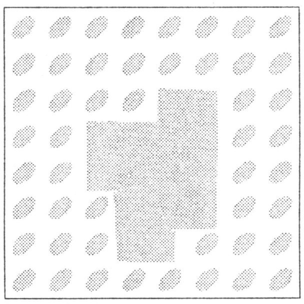

Fig. 3.3 – Cell Y with solid obstacle Y_s and obstacles εZ_s.

Next we apply $\overline{W}$ to every period εZ of $Y \setminus Y_s$ and we obtain a function W_ε,

$$W_\varepsilon \colon \mathbb{H}^1_s(Y^*) \to \mathbb{H}^1_s(Y^\varepsilon_f) \ ,$$

satisfying the relations (3.4) and (3.5). We apply W_ε to $Rv \in \mathbb{H}^1_s(Y^*)$ and using relation (3.8) we get:

$$\varepsilon^2 \, |\nabla W_\varepsilon(Rv)|^2_{[\mathbb{L}^2(Y^\varepsilon_f)]^N} + |W_\varepsilon(Rv)|^2_{\mathbb{L}^2(Y^\varepsilon_f)} \ \le\ \varepsilon^2 \, |\nabla Rv|^2_{[\mathbb{L}^2(Y^*)]^N} + |Rv|^2_{\mathbb{L}^2(Y^*)} \ .$$

Since $|Rv|_{\mathbb{H}^1(Y)} \le c \, |v|_{\mathbb{H}^1(Y)}$, we deduce:

$$\varepsilon^2 \, |\nabla W(Rv)|^2_{[\mathbb{L}^2(Y^\varepsilon_f)]^N} + |W(Rv)|^2_{\mathbb{L}^2(Y^\varepsilon_f)} \ \le\ c \, |v|^2_{\mathbb{H}^1(Y)},$$

which is exactly (3.6). ∎

Lemma 3.3. *There exists a restriction operator*

$$S_{\varepsilon^2} \colon \mathbb{H}^1_0(\Omega) \to \mathbb{H}^1_0(\Omega_\varepsilon)$$

such that:

$$S_{\varepsilon^2}(v) = v \quad \text{in } \ \Omega_\varepsilon, \ \ \forall\, v \in \mathbb{H}^1_0(\Omega_\varepsilon) \ , \tag{3.9}$$

$$\operatorname{div} S_{\varepsilon^2} v = 0 \ \ \text{in } \Omega_\varepsilon \quad \text{ if } \ \operatorname{div} v = 0 \ \ \text{in } \Omega \ , \tag{3.10}$$

$$|\nabla S_{\varepsilon^2} v|_{[\mathbb{L}^2(\Omega_\varepsilon)]^N} \ \le\ c\Big(\frac{1}{\varepsilon^2} \, |v|_{\mathbb{L}^2(\Omega)} + \frac{1}{\varepsilon} \, |\nabla v|_{[\mathbb{L}^2(\Omega)]^N} \Big) \ , \tag{3.11}$$

$$|S_{\varepsilon^2} v|_{\mathbb{L}^2(\Omega_\varepsilon)} \ \le\ c\Big(|v|_{\mathbb{L}^2(\Omega)} + \varepsilon \, |\nabla v|_{[\mathbb{L}^2(\Omega)]^N} \Big) \ . \tag{3.12}$$

Proof: Let $S_\varepsilon \colon \mathbb{H}^1(Y) \to \mathbb{H}^1(Y^\varepsilon_f)$ be the application defined by:

$$S_\varepsilon v(y) = \begin{cases} W_\varepsilon(Rv)(y) & \text{if } y \in Y^* \,, \\ 0 & \text{if } y \in Y_s \ . \end{cases}$$

Using the construction of W_ε we also have:

$$S_\varepsilon v(y) = \begin{cases} W_\varepsilon(Rv)(y) & \text{if } y \in Y^\varepsilon_f \,, \\ 0 & \text{otherwise} \ , \end{cases}$$

and S_ε satisfies (3.9)–(3.10).

Due to (3.6), the application S_ε satisfies:

$$\varepsilon^2 \, |\nabla S_\varepsilon v|^2_{[\mathbb{L}^2(Y^\varepsilon_f)]^N} + |S_\varepsilon v|^2_{\mathbb{L}^2(Y^\varepsilon_f)} \ \le\ c \, |v|^2_{\mathbb{H}^1(Y)} \ . \tag{3.13}$$

We define S_{ε^2} by applying S_ε to each period εY. The relation (3.13) then implies:

$$\varepsilon^4 \, |\nabla S_{\varepsilon^2} v|^2_{[\mathbb{L}^2(\Omega_\varepsilon)]^N} + |S_{\varepsilon^2} v|^2_{\mathbb{L}^2(\Omega_\varepsilon)} \le c\Big(|v|^2_{\mathbb{L}^2(\Omega)} + \varepsilon^2 \, |\nabla v|^2_{[\mathbb{L}^2(\Omega)]^N}\Big) ,$$

and we deduce relations (3.11)–(3.12). ∎

Let v be a functin of $\mathbb{H}^1_0(\Omega)$. As $\nabla p^\varepsilon \in \mathbb{H}^{-1}(\Omega_\varepsilon)$, we define the application F^ε by:

$$\langle F^\varepsilon, v\rangle_\Omega \;=\; \langle \nabla p^\varepsilon, S_{\varepsilon^2} v\rangle_{\Omega_\varepsilon} \;, \tag{3.14}$$

where S_{ε^2} is the operator defined by Lemma 3.3. The following proposition gives us the extension of the pressure p^ε to the whole Ω. Moreover, we establish a strong convergence result for this extension. Following the ideas of L. Tartar [12], we can prove:

Proposition 3.4. *Let p^ε be as in (1.2). Then, for each ε there exists an extension P^ε of p^ε defined on Ω such that:*

$$P^\varepsilon = p^\varepsilon \quad \textit{in} \;\; \Omega_\varepsilon \;.$$

Moreover, up to a subsequence, we have:

$$P^\varepsilon \to p_0 \quad \textit{strongly in} \;\; L^2(\Omega)/\mathbb{R} \;. \tag{3.15}$$

The function F^ε and the pressure p^ε are linked by:

$$F^\varepsilon = \nabla P^\varepsilon \;. \tag{3.16}$$

4 – Passage to the limit and 3-scale limit problem

We recall that as in I.-A. Ene [6] we have the following "de Rham"-type result:

Lemma 4.1. *Let $w \in \mathbb{L}^2(\Omega \times Y \times Z)$ be a function satisfying:*

$$\int_\Omega\!\int_Y\!\int_Z w(x,y,z)\,\phi(x,y,z)\,dx\,dy\,dz \;=\; 0 \;, \tag{4.1}$$

for all function ϕ belonging to $\mathcal{D}(\Omega, C^\infty_p(Y\times Z))$ such that:

$$\mathrm{div}_y\; \phi(x,y,z) = 0 \,, \qquad \mathrm{div}_z\; \phi(x,y,z) = 0 \;. \tag{4.2}$$

Then there exist two functions $q_1 \in \mathbb{L}^2(\Omega, \mathbb{H}^1_p(Y)/\mathbb{R})$ *and* $q_2 \in \mathbb{L}^2(\Omega \times Y, \mathbb{H}^1_p(Z)/\mathbb{R})$ *such that:*

$$w(x,y,z) = \nabla_y\, q_1(x,y) + \nabla_z\, q_2(x,y,z)\ . \tag{4.3}$$

Let us recall that we denoted by u the 3-scale limit of $\varepsilon^{-2}\,\widetilde{u^\varepsilon}$ (see relation (2.9)) and that p_0 (defined in relation (3.15)) represents the strong limit of the pressure's extension in $\mathbb{L}^2(\Omega)$. Using Lemma 4.1 and 3-scale convergence results of §2–§3, we prove the main result of this paper:

Theorem 4.2. *Let u and p_0 be as before. Then there exist* $p_1 \in \mathbb{L}^2(\Omega, \mathbb{H}^1_p(Y^*)/\mathbb{R})$ *and* $p_2 \in \mathbb{L}^2(\Omega \times Y, \mathbb{H}^1_p(Z^*)/\mathbb{R})$ *such that:*

$$-\Delta_z u + \nabla_x\, p_0 + \nabla_y\, p_1 + \nabla_z\, p_2 \;=\; f \quad \textit{in}\ \ \Omega \times Y^* \times Z^*\ . \tag{4.4}$$

Proof: We recall the first equation of (1.2):

$$-\varepsilon^2\, \Delta u^\varepsilon + \nabla p^\varepsilon = f \quad \text{in}\ \ \Omega_\varepsilon\ .$$

We multiply it by a function $\varphi \in \mathcal{D}(\Omega, C^\infty_p(Y{\times}Z))$ such that $\operatorname{div}_y \varphi(x,y,z) = 0$ and $\operatorname{div}_z \varphi(x,y,z) = 0$. Integrating the first term of the left hand side by parts we get:

$$\varepsilon^2 \int\limits_{\Omega_\varepsilon} \nabla u^\varepsilon(x)\, \nabla\varphi\Big(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\Big)\, dx + \int\limits_{\Omega_\varepsilon} \nabla p^\varepsilon(x)\, \varphi\Big(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\Big)\, dx \;=\; = \int\limits_{\Omega_\varepsilon} f(x)\, \varphi\Big(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\Big)\, dx\ .$$

Using the definition of the extension $\widetilde{u^\varepsilon}$, relations (3.14) and (3.16) and making the additional assumption $\varphi(x,y,z){=}0$ in $\Omega \times Y_s \times Z_s$ (i.e. $\varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}) \in \mathbb{H}^1_0(\Omega_\varepsilon)$) we obtain:

$$\varepsilon^2 \int\limits_{\Omega} \nabla\widetilde{u^\varepsilon}(x)\, \nabla\varphi\Big(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\Big)\, dx - \int\limits_{\Omega} P^\varepsilon(x)\, \operatorname{div}_x\varphi\Big(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\Big)\, dx = = \int\limits_{\Omega} f(x)\, \varphi\Big(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\Big)\, dx\ . \tag{4.5}$$

Passage to the 3-scale limit in (4.5) implies:

$$\int\limits_{\Omega}\int\limits_{Y^*}\int\limits_{Z^*} \nabla_z u(x,y,z)\, \nabla_z\varphi(x,y,z)\, dx\, dy\, dz - \int\limits_{\Omega}\int\limits_{Y^*}\int\limits_{Z^*} p_0(x)\, \operatorname{div}_x \varphi(x,y,z)\, dx\, dy\, dz = = \int\limits_{\Omega}\int\limits_{Y^*}\int\limits_{Z^*} f(x)\, \varphi(x,y,z)\, dx\, dy\, dz\ .$$

Hence,

$$\int_{\Omega}\int_{Y^*}\int_{Z^*}\Big[-\Delta_z u(x,y,z)+\nabla_x p_0(x)-f(x)\Big]\,\varphi(x,y,z)\,dx\,dy\,dz \;=\; 0\;.$$

Using the particular form of φ, Lemma 4.1 then implies relation (4.4). ∎

Conclusions

Theorem 4.2 and the results of §2 imply the following three-scale system (4.6).

$$(4.6)\qquad \begin{cases} -\Delta_z u+\nabla_x p_0+\nabla_y p_1+\nabla_z p_2 \;=\; f & \text{in } \;\Omega\times Y^*\times Z^*,\\ \operatorname{div}_x \displaystyle\int_{Y^*}\int_{Z^*} u\,dz\,dy \;=\; 0 & \text{in } \;\Omega,\\ \operatorname{div}_y \displaystyle\int_{Z^*} u\,dz = 0 & \text{in } \;\Omega\times Y^*,\\ \operatorname{div}_z u = 0 & \text{in } \;\Omega\times Y^*\times Z^*,\\ \Big[\displaystyle\int_{Y^*}\int_{Z^*} u\,dy\,dz\Big]\cdot\nu = 0 & \text{on } \;\partial\Omega,\\ u\cdot\nu_Z \;\text{ takes opposite values on opposite faces of } Z, & \\ \Big[\displaystyle\int_{Z^*} u\,dz\Big]\cdot\nu_Y \text{ takes opposite values on opposite faces of } Y, & \\ u\cdot\nu_Z = 0 & \text{on } \;\partial Z_s\,,\\ \Big[\displaystyle\int_{Z^*} u\,dz\Big]\cdot\nu_Y = 0 & \text{on } \;\partial Y_s\;. \end{cases}$$

Remark 4.3. System (4.6) is obtained in J.-L. Lions [8, Chapter 2, Section 3], with the method of asymptotic expansion on the velocity and of the pressure. The first equation is a three-pressure equation. The three pressures p_0, p_1, p_2 are the three first terms in the asymptotic expansion of the pressure p^ε. We recall here that, as in J.-L. Lions [8], we may write the function u in two different ways:

(**i**) The function u satisfies the homogenized equation (a Darcy-law type):

$$u(x,y,z)=\phi(y,z)\,\Big(f(x)-\nabla_x p_0(x)\Big)\;,$$

where the function ϕ is solution of a local problem in $Y^*\times Z^*$.

With the notation $\mathcal{M}(\phi) = \dfrac{1}{|Y^*|\,|Z^*|} \displaystyle\int_{Y^*}\int_{Z^*} \phi(y,z)\,dy\,dz$, the function p_0 is solution of the following Neumann problem:

$$\Big(\mathcal{M}(\phi)\,(f - \nabla_x p_0),\, \nabla q\Big) = 0\,, \qquad \forall\, q \in H^1(\Omega)\ .$$

(ii) We can also express u by a relation depending of both pressures p_0 and p_1. We have:

$$u(x,y,z) \;=\; \phi_1(z)\,\Big(f - \nabla_x p_0(x) - \nabla_y p_1(x,y)\Big)\ ,$$

where ϕ_1 is solution of a local problem in Z^* and the pressure $p_1(x,y)$ is solution of the following Neumann problem:

$$\Big(\mathcal{M}(\phi_1)\,\Big(f - \nabla_x p_0(x) - \nabla_y p_1(x,y)\Big),\, \nabla_y q_1\Big)_{Y^*} \;=\; 0\,,$$

$$\forall\, q_1 \in H^1(Y^*), \quad q_1\ \ Y\text{-periodic}\ .$$

Remark 4.4. The results presented here may be generalized. Let r_ε be a parameter depending on ε such that:

$$\frac{r_\varepsilon}{\varepsilon} \to 0 \quad \text{ if }\ \varepsilon \to 0\ .$$

In the domain Ω we replace the very small obstacles of order ε^2 by obstacles of order r_ε, periodically distributed with periodicity r_ε. We consider the problem:

$$\begin{cases} -r_\varepsilon\,\Delta u^\varepsilon + \nabla p^\varepsilon = f & \text{in } \Omega_\varepsilon\,, \\ \operatorname{div} u^\varepsilon = 0 & \text{in } \Omega_\varepsilon\,, \\ u^\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon\ . \end{cases} \tag{4.7}$$

The case already treated corresponds to $r_\varepsilon = \varepsilon^2$.

For the extension of the velocity, solution of (4.7), we can prove the convergences:

$$r_\varepsilon^{-1}\,\widetilde{u^\varepsilon} \to u \quad \text{ 3-scale}\ ,$$

$$r_\varepsilon^{-1}\,\widetilde{u^\varepsilon} \rightharpoonup u_0 = \int_{Y^*}\int_{Z^*} u\,dz\,dy \quad \text{ weakly in }\ \mathbb{L}^2(\Omega)\ ,$$

$$\widetilde{\nabla u^\varepsilon} \to \nabla_z u \quad \text{ 3-scale}\ .$$

For the strong convergence of pressure's extension we have Proposition 3.4, which still holds. We can prove that velocity and pressure limits satisfy the system (4.6).

REFERENCES

[1] Allaire, G. and Briane, M. – Multi-scale convergence and reiterated homogenization, *Proc. Roy. Soc. Edinburgh,* 126A (1996), 297–342.

[2] Conca, C. – On the application of the homogenization theory to a class of problems arising in fluid mechanics, *J. Math. Pures et Appliquées,* 64 (1985), 31–75.

[3] Donato, P. and Saint Jean Paulin, J. – *Stokes flow in a porous medium with double periodicity*, "Progress in Partial Differential Equations: the Metz surveys" (M. Chipot, J. Saint Jean Paulin and I. Shafrir, Eds.), Pitman, p. 116–129, 1994.

[4] Donato, P. and Saint Jean Paulin, J. – Homogenization of the Poisson equation in a porous medium with double periodicity, *Japan J. of Ind. and Appl. Mathematics,* 10(2) (1993), 333–349.

[5] Ene, I.-A. and Saint Jean Paulin, J. – Homogenization and two-scale convergence for a Stokes or Navier–Stokes flow in an elastic thin porous medium, *M3AS,* 6(7) (1996), 941–955.

[6] Ene, I.-A. – *Etude de Quelques Problèmes D'écoulement dans les Milieux Poreux*, Thèse de l'Université de Metz, Juin 1995.

[7] Levy, T. – Filtration in a porous fissured rock: influence of the fissures connexity, *Eur. J. Mech. B/Fluids,* 9(4) (1990), 309–327.

[8] Lions, J.-L. – *Some Methods in the Mathematical Analysis of Systems and Their Control*, Gordon and Breach, Science Press, Beijing, 1981.

[9] Lipton, R. and Avellaneda, M. – Darcy's law for slow viscous flow past a stationary array of bubbles, *Proc. Roy. Soc. Edinburgh,* 114A (1990), 71–79.

[10] Nguetseng, G. – A general convergence result for a functional related to the theory of homogenization, *Siam. J. Math. Anal.,* 20(3) (1989), 608–623.

[11] Sanchez-Palencia, E. – *Non Homogeneous Media and Vibration Theory*, Lecture Notes in Physics, 127, Springer-Verlag, 1980.

[12] Tartar, L. – *Convergence of the Homogenization Process*, Appendix of [11].

[13] Temam, R. – *Navier–Stokes Equations*, North-Holland, Amsterdam, 1978.

Renata Bunoiu and Jeannine Saint Jean Paulin,
Département de Mathématiques, Ile du Saulcy,
BP 80794, 57012 Metz, cedex 1 – FRANCE