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SOLVABILITY IN SOBOLEV SPACES OF A PROBLEM FOR A SECOND ORDER PARABOLIC EQUATION WITH TIME DERIVATIVE IN THE BOUNDARY CONDITION

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1 – Introduction

In this paper we consider the initial boundary value problem for a second order parabolic equation with both time and spatial derivatives in the boundary condition. The unique solvability in Hölder function spaces, in weighted Hölder function spaces, and in Sobolev function spaces in the case $p = 2$ for problems with such a boundary condition in domains with smooth boundaries was obtained in the papers [1–5]. Here we obtain the unique solvability of the problem in the Sobolev function spaces in the case $p > n+2$. Condition (20) in Theorem 5 is a usual assumption on the combination of signs in the boundary condition. In the case of another combination the problem is ill-posed, it was shown in [6].

In §2 we consider the model problem for the heat equation in a half-space and obtain L_p-estimates for its solution using some results on Fourier multipliers [7, 8]. This allows us to prove the solvability of the problem in a domain with smooth boundary by the construction of a regularizer in the same way as in [9, 3].

In §3 we make this for a periodic case. Periodic problem (19) arose in studying the periodic one-phase Stefan problem [10].

In §4 we reduce the assumptions on the smoothness of the boundary which is usually made in the construction of a regularizer. Assuming that the boundary belongs to $W_p^{2-1/p}$, we prove the solvability of the problem with time derivative in the boundary condition.

We also consider in §2 a model problem with time derivative in the *conjugation* condition which is useful in studying the two-phase Stefan problem in the Sobolev function spaces.

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2 – Model problems(*)

We introduce the following notations:

$$x=(x',x_n)\in\mathbb{R}^n\ ,$$
$$R_1=\mathbb{R}^n_-=\Big\{x\in\mathbb{R}^n\mid x_n<0\Big\}\,,\qquad R_2=\mathbb{R}^n_+=\Big\{x\in\mathbb{R}^n\mid x_n>0\Big\}\ ,$$
$$\Gamma=\Big\{x\in\mathbb{R}^n\mid x_n=0\Big\}\,,\qquad R_{i,T}=R_i\times[0,T),\quad i=1,2\ ,$$
$$\Gamma_T=\Gamma\times[0,T)\ .$$

We consider the following model problem in the half-space:

$$\begin{gathered}v_t-\Delta v=f\,,\qquad (x,t)\in R_{2,T}\ ,\\ \left[\beta\,v_t-\frac{\partial v}{\partial x_n}+\sum_{l=1}^{n-1}b_l\,\frac{\partial v}{\partial x_l}\right]\bigg|_{\Gamma_T}=\varphi(x',t)\,,\qquad v|_{t=0}=0\ ,\end{gathered}\tag{1}$$

where $\beta>0$, $b_l\in\mathbb{R}$.

The specific character of this problem consists in the fact that the boundary condition contains both time and normal derivatives. Boundary operator in problem (1) does not satisfy the complementing condition for $n>1$, therefore this problem can not be included into the general theory of parabolic problems for second order equations constructed in [9].

Problem (1) arises in studying the one-phase Stefan problem and some problems of the heat conduction theory. If we consider the two-phase Stefan problem then it is useful to investigate the following model problem with time derivative in the conjugation condition:

$$\begin{aligned}&\frac{\partial v_k}{\partial t}-a_k^2\,\Delta v_k=0\,,\qquad (x,t)\in R_{k,T}\ ,\\ &\frac{\partial v_1}{\partial t}+\boldsymbol{b}\cdot\nabla v_1-\boldsymbol{c}\cdot\nabla v_2|_{\Gamma_T}=\psi(x',t)\ ,\\ &v_1-v_2|_{\Gamma_T}=0\,,\qquad v_k|_{t=0}=0\ ,\end{aligned}\tag{2}$$

where $k=1,2$; $\boldsymbol{b},\boldsymbol{c}\in\mathbb{R}^n$; $a_1,a_2\in\mathbb{R}$; $b_n>0$, $c_n>0$.

Problems (1) and (2) were studied in Hölder function spaces [1, 2, 13], in weighted Hölder spaces [3, 14], and in Sobolev spaces with $p=2$ [4, 15] (only

(*) This part of the paper has been published in Russian [11]; the L_p-estimates for such model problems also has been obtained by H. Koch [12].

problem (1)). Here we obtain L_p-estimates of the solutions for problems (1) and (2) in the case $p > 1$.

Let us put $T = +\infty$ and make the Fourier transform with respect to the space variables x' and the Laplace transform with respect to t:

$$Fv(\xi, x_n, s) = \widetilde{v}(\xi, x_n, s) = \frac{1}{(\sqrt{2\pi})^{n-1}} \int_0^{+\infty} dt \int_{\mathbb{R}^{n-1}} v(x', x_n, t)\, e^{-ts+ix'\cdot\xi}\, dx' \ , \quad \xi \in \mathbb{R}^{n-1}, \quad \operatorname{Re} s > 0 \ . \tag{3}$$

Then, from problem (1) with $f = 0$ and problem (2) we pass to the following problems for ordinary differential equations

$$\begin{aligned} & -\frac{d^2\widetilde{v}}{dx_n^2} + (\xi^2 + s)\,\widetilde{v} = 0\,, \qquad x_n > 0\ , \\ & \Big[-\frac{d\widetilde{v}}{dx_n} + \Big(\beta\, s - i\sum_{j=1}^{n-1} \xi_j\, b_j\Big)\,\widetilde{v}\Big]\Big|_{x_n=0} = \widetilde{\varphi}(\xi, s) \end{aligned} \tag{4}$$

and

$$\begin{aligned} & -a_k^2\,\frac{d^2\widetilde{v}_k}{dx_n^2} + (a_k^2\,\xi^2 + s)\,\widetilde{v}_k = 0\ , \\ & \widetilde{v}_1|_{x_n=0} = \widetilde{v}_2|_{x_n=0}\ , \\ & s\,\widetilde{v}_1 - i(\boldsymbol{b}' - \boldsymbol{c}')\cdot\xi + b_n\,\frac{d\widetilde{v}_1}{dx_n} - c_n\,\frac{d\widetilde{v}_2}{dx_n}\Big|_{x_n=0} = \widetilde{\psi}(\xi, s)\ , \end{aligned} \tag{5}$$

where $\boldsymbol{b}' = (b_1, ..., b_{n-1})^T$, $\boldsymbol{c}' = (c_1, ..., c_{n-1})^T$.

Solutions of problems (4) and (5) can be found in the explicit form:

$$\widetilde{v}(\xi, x_n, s) = \frac{1}{r + \beta\, s - i\displaystyle\sum_{j=1}^{n-1} \xi_j\, b_j}\, e^{-rx_n}\, \widetilde{\varphi}(\xi, s)\ , \tag{6}$$

where $r = \sqrt{s + \xi^2}$, $\operatorname{Re} r > 0$, and

$$\widetilde{v}_k(\xi, x_n, s) = \frac{1}{s + \frac{b_n}{a_1}\, r_1 + \frac{c_n}{a_2}\, r_2 - i(\boldsymbol{b}' - \boldsymbol{c}')\cdot\xi}\, e^{-\frac{r_k}{a_k}|x_n|}\, \widetilde{\psi}(\xi, s)\ , \tag{7}$$

where $r_k = \sqrt{s + a_k^2\,\xi^2}$, $\operatorname{Re} r_k > 0$, $k = 1, 2$.

To obtain the solutions of problems (1) with $f = 0$ and (2) we have to use the integral transform inverse to (3).

Let us begin the formulation of the results by reminding some definitions.

The norm in $L_p(R_{k,T})$ is defined by the formula

$$\|u\|_{p,R_{k,T}} = \left(\int_0^T dt \int_{R_k} |u(x,t)|^p \, dx\right)^{1/p} .$$

By $W_p^{m,m/2}(R_{k,T})$ $(m/2 \in \mathbb{N})$ we mean the closure of the set of smooth functions in the norm

$$\|u\|^{(m)}_{p,R_{k,T}} = \sum_{0\le|\alpha|+2a\le m} \|\mathcal{D}_t^a \mathcal{D}_x^\alpha u\|_{p,R_{k,T}} \ ,$$

$$\alpha = (\alpha_1, ..., \alpha_n) \, , \qquad |\alpha| = \sum_{i=1}^n \alpha_i, \quad \alpha_i \in \mathbb{N} \cup \{0\} \ .$$

The norm in the space of traces $W_p^{m-1/p,\, m/2-1/2p}(\Gamma_T)$ is defined by the formula

$$\|u\|^{(m-1/p)}_{p,\Gamma_T} = \sum_{0\le|\alpha|+2a<m} \|\mathcal{D}_t^a \mathcal{D}_x^\alpha u\|_{p,\Gamma_T} + \langle\!\langle u \rangle\!\rangle^{(m-1/p)}_{p,\Gamma_T} \ ,$$

where

$$\langle\!\langle u \rangle\!\rangle^{(m-1/p)}_{p,\Gamma_T} =$$

$$= \sum_{|\alpha|+2a=m-1} \left(\int_0^T\! dt \int_\Gamma\! dx \int_\Gamma \Big|\mathcal{D}_t^a\, \mathcal{D}_x^\alpha u(x,t) - \mathcal{D}_t^a \mathcal{D}_y^\alpha u(y,t)\Big|^p \, \frac{dy}{|x-y|^{n-1+p(1-1/p)}}\right)^{1/p} +$$

$$+ \sum_{0<m-1/p-2a-|\alpha|<2} \left(\int_\Gamma\! dx \int_0^T\!\!\int_0^T \Big|\mathcal{D}_t^a \mathcal{D}_x^\alpha u(x,t) - \mathcal{D}_\tau^a \mathcal{D}_x^\alpha u(x,\tau)\Big|^p \, \frac{dt\, d\tau}{|t-\tau|^{1+p(1-1/p)}}\right)^{1/p},$$

$\langle\!\langle\cdot\rangle\!\rangle_p^{(m)}$ and $\langle\!\langle\cdot\rangle\!\rangle_p^{(m-1/p)}$ are the main parts of the norms $\|\cdot\|_p^{(m)}$ and $\|\cdot\|_p^{(m-1/p)}$ respectively.

The subspace of $W_p^{m,m/2}(R_{k,T})$ $(W_p^{m-1/p,\, m/2-1/2p}(\Gamma_T))$ which consists of functions satisfying zero initial conditions we denote by $\overset{\circ}{W}{}_p^{m,m/2}(R_{k,T})$ $(\overset{\circ}{W}{}_p^{m-1/p,\, m/2-1/2p}(\Gamma_T))$.

Theorem 1. *For any* $\varphi \in \overset{\circ}{W}{}_p^{m+1-1/p,\, m/2+1/2-1/2p}(\Gamma_T)$, $f \in \overset{\circ}{W}{}_p^{m,m/2}(R_{2,T})$, *there exists a unique solution of problem (1)* $v \in \overset{\circ}{W}{}_p^{m+2,\, m/2+1}(R_{2,T})$, *and there holds the estimate*

$$\text{(8)} \qquad \langle\!\langle v \rangle\!\rangle^{(m+2)}_{p,R_{2,T}} \le c\Big(\langle\!\langle \varphi \rangle\!\rangle^{(m+1-1/p)}_{p,\Gamma_T} + \langle\!\langle f \rangle\!\rangle^{(m)}_{p,R_{2,T}}\Big) , \qquad \frac{m}{2} \in \mathbb{N} \cup \{0\} \ .$$

Theorem 2. *For any* $\psi \in \overset{\circ}{W}{}_p^{m+1-1/p,\, m/2+1/2-1/2p}$ *there exists a unique solution of problem (2)* $v_k \in \overset{\circ}{W}{}_p^{m+2,\, m/2+1}(R_{k,T})$ *and there holds the estimate*

$$\sum_{k=1,2} \langle\!\langle v_k \rangle\!\rangle_{p,R_{k,T}}^{(m+2)} \le c\, \langle\!\langle \psi \rangle\!\rangle_{p,\Gamma_T}^{(m+1-1/p)} \ .$$

We prove theorems 1 and 2 by using the following results on Fourier multipliers [7, 8, 16].

Definition. Function g is called Fourier multiplier of class (p,p) if for any $v \in C_0^\infty$ the estimate

$$\|F^{-1} g\, Fv\|_p \le c\|v\|_p\,, \qquad 1 \le p < \infty \ .$$

holds.

We set

$$\mathcal{M}_p^p(g) = \sup_{v \in C_0^\infty} \frac{\|F^{-1} g\, Fv\|_p}{\|v\|_p} \ .$$

Theorem 3. *Let* $\xi \in \mathbb{R}^n$. *Assume that function* $g(\xi)$ *and all its mixed derivatives*

$$\frac{\partial^l g(\xi)}{\partial \xi_{k_1} \cdots \partial \xi_{k_l}}, \qquad \text{where } \ l \ge 1, \quad k_i \ne k_j \ \text{ for } i \ne j, \quad i,j = 1,...,l \ ,$$

are continuous for $|\xi_j| > 0$, $j = 1,...,n$, *and that there exists a positive constant* M *such that*

$$\left| \xi_{k_1} \cdots \xi_{k_l} \frac{\partial^l g(\xi)}{\partial \xi_{k_1} \cdots \partial \xi_{k_l}} \right| \le M \ . \tag{9}$$

Then g *is a Fourier multiplier of class* (p,p) *and* $\mathcal{M}_p^p(g) \le c\, M$.

Theorem 4. *Let functions* $H(x', x_n, t)$ *and* $\eta(x', x_n, t)$ *be defined in* $\mathbb{R}^n_{+,\infty}$, $\eta(x,t) \in L_p(\mathbb{R}^n_{+,\infty})$, *let* $FH(\xi, x_n, s)$ *be a Fourier multiplier of class* (p,p), *and* $\mathcal{M}_p^p(H(\cdot, x_n, \cdot)) \le \frac{c}{x_n}$. *Let*

$$\mathcal{H}\eta = \int_0^{+\infty} F^{-1}\Big[H(\xi, x_n + y, s)\, F\eta(\xi, y, s) \Big]\, dy \ .$$

Then $\mathcal{H}\eta \in L_p(\mathbb{R}^n_{+,\infty})$ *and there holds the estimate*

$$\|\mathcal{H}\eta\|_{p,\mathbb{R}^n_{+,\infty}} \le c\|\eta\|_{p,\mathbb{R}^n_{+,\infty}} \ .$$

Proof of Theorem 1: We consider the following auxiliary problem:

$$w_t - \Delta w = f(x,t)\,, \qquad (x,t) \in R_{2,T}\ ,$$
$$w\big|_{\Gamma_T} = 0\,, \qquad w\big|_{t=0} = 0\ .$$

There exists the unique solution $w \in \underset{\circ}{W}{}_p^{m+2,\,m/2+1}(R_{2,T})$ of this problem [9], and there holds the estimate

$$\langle\!\langle w\rangle\!\rangle^{(m+2)}_{p,R_{2,T}} \le c\langle\!\langle f\rangle\!\rangle^{(m)}_{p,R_{2,T}}\ . \tag{10}$$

Looking for the solution of problem (1) in the form $v = w + u$, for the new unknown function u we have problem (1) with $f = 0$:

$$u_t - \Delta u = 0\,, \qquad (x,t) \in R_{2,T}\ ,$$
$$\left(\beta\, u_t - \frac{\partial u}{\partial x_n} + \sum_{l=1}^{n-1} b_l\, \frac{\partial u}{\partial x_l}\right)\bigg|_{\Gamma_T} = \varphi(x',t) + \frac{\partial w}{\partial x_n} = \varphi^*(x',t)\,, \qquad u\big|_{t=0} = 0\ . \tag{1a}$$

At first we find $\varphi_1 \in \underset{\circ}{W}{}_p^{m+1-1/p,\, m/2+1/2-1/2p}(\Gamma_\infty)$ such that

$$\varphi_1\big|_{t\le T} = \varphi^*\,, \qquad \varphi_1\big|_{t\ge T+\varepsilon} = 0, \quad \varepsilon > 0\ ,$$
$$\langle\!\langle \varphi_1\rangle\!\rangle^{(m+1-1/p)}_{p,\Gamma_\infty} \le c\langle\!\langle \varphi^*\rangle\!\rangle^{(m+1-1/p)}_{p,\Gamma_T} \le c\Big(\langle\!\langle \varphi\rangle\!\rangle^{(m+1-1/p)}_{p,\Gamma_T} + \langle\!\langle f\rangle\!\rangle^{(m)}_{p,R_{2,T}}\Big)\ , \tag{11}$$

and, making the Fourier–Laplace transform (3), we arrive at problem (4). Then we extend the function φ_1 into the half-space $R_{2,\infty}$, finding $\varpi \in \underset{\circ}{W}{}_p^{m+1,\, m/2+1/2}(R_{2,\infty})$ such that $\varpi|_{\Gamma_\infty} = \varphi_1$ and the estimate

$$\langle\!\langle \varpi\rangle\!\rangle^{(m+1)}_{p,R_{2,\infty}} \le c\,\langle\!\langle \varphi_1\rangle\!\rangle^{(m+1-1/p)}_{p,\Gamma_\infty} \tag{12}$$

holds. With the help of the Newton–Leibnitz formula we obtain from (6) the following integral representation for the solution $\widetilde{u}$ of problem (4) with $\varphi = \varphi_1$

$$\widetilde{u}(\xi, x_n, s) = -\int_0^{+\infty} \frac{\partial}{\partial y} \left[\frac{1}{r + \beta\, s - i \sum_{j=1}^{n-1} b_j\, \xi_j}\, e^{-r(x_n+y)}\, \widetilde{\varpi}(\xi, y, s) \right] dy$$

$$= \int_0^{+\infty} \frac{r}{r + \beta\, s - i \sum_{j=1}^{n-1} \xi_j\, b_j}\, e^{-r(x_n+y)}\, \widetilde{\varpi}(\xi, y, s)\, dy$$

$$- \int_0^{+\infty} \frac{1}{r + \beta\, s - i \sum_{j=1}^{n-1} \xi_j\, b_j}\, e^{-r(x_n+y)}\, \frac{\partial}{\partial y} \widetilde{\varpi}(\xi, y, s)\, dy \ .$$

Hence, for $j = 2, ..., m+2$,

$$F^{-1}(r^j\, Fu) =$$

(13)
$$= \int_0^{+\infty} F^{-1}\Big[H(\xi, x_n{+}y, s) \Big(r^{j-1}\, \widetilde{\varpi}(\xi, y, s) - r^{j-2}\, \frac{\partial \widetilde{\varpi}(\xi, y, s)}{\partial y} \Big) \Big]\, dy \ ,$$

where

$$H(\xi, x_n, s) = \frac{r^2}{r + \beta\, s - i \sum_{j=1}^{n-1} \xi_j\, b_j}\, e^{-r x_n} \ .$$

Let us show that $H(\xi, x_n, s)$ is a Fourier multiplier of class (p,p) and

$$\mathcal{M}_p^p\Big(H(\cdot, x_n, \cdot) \Big) \le \frac{c}{x_n} \ .$$

It is obvious that the function H and its mixed derivatives are continuous for $\operatorname{Re} s > 0$, $|\xi_j| > 0$, $j = 1, ..., n-1$. Let us check that for H and its derivatives the estimate (9) holds with $\mathcal{M} = \frac{c}{x_n}$.

It is easy to see that

(14)
$$\frac{\operatorname{Re} r}{\left| r + \beta\, s - i \sum_{l=1}^{n-1} \xi_l\, \beta_l \right|} \le c \ .$$

So,

$$
(15)\qquad
\begin{aligned}
&\Big|H(\xi,x_n,s)\Big| \le \frac{|r|^2}{\operatorname{Re} r}\, e^{-\operatorname{Re} r\, x_n}\;,\\
&\left|\xi_j\,\frac{\partial H(\xi,x_n,s)}{\partial \xi_j}\right| = \left|\frac{2\xi_j^2 - x_n\,\xi_j^2\, r}{r+\beta\, s - i\displaystyle\sum_{l=1}^{n-1}\xi_l\, b_l}\, e^{-r x_n} - \frac{\xi_j^2\, r - i\, b_j\,\xi_j\, r^2}{\Big(r+\beta\, s - i\displaystyle\sum_{l=1}^{n-1}\xi_l\, b_l\Big)^2}\, e^{-r x_n}\right|\\
&\qquad\le c\Big(|r| + x_n |r^2|\Big)\, e^{-\operatorname{Re} r\, x_n}\;.
\end{aligned}
$$

Using the inequality

$$
(16)\qquad y^j\, e^{-yz} \le \frac{c}{z^j}\,,\qquad \forall\, y>0,\quad z>0,\quad j\in\mathbb{N}\;,
$$

with $j=1,2$, we obtain

$$
(17)\qquad \operatorname{Re} r\, e^{-\operatorname{Re} r\, x_n} \le \frac{c}{x_n}\,,\qquad (\operatorname{Re} r)^2\, x_n\, e^{-\operatorname{Re} r\, x_n} \le \frac{c}{x_n}\;.
$$

So, (15) and (17) imply

$$
\begin{aligned}
&\Big|H(\xi,x_n,s)\Big| \le \frac{|r|^2}{(\operatorname{Re} r)^2}\cdot\frac{1}{x_n} \le \frac{c}{x_n}\;,\\
&\left|\xi_j\,\frac{\partial H(\xi,x_n,s)}{\partial \xi_j}\right| \le c\Big(\frac{|r|}{\operatorname{Re} r} + \frac{|r|^2}{(\operatorname{Re} r)^2}\Big)\frac{1}{x_n} \le \frac{c}{x_n}\;.
\end{aligned}
$$

Here we take into account the fact that the function $r = \sqrt{\xi^2+s}$ with $\operatorname{Re} r>0$, $\operatorname{Re}(\xi^2+s)>0$ satisfies the inequality $\frac{|r|}{\operatorname{Re} r}\le c$.

Further, we have

$$
\begin{aligned}
\left|s\,\frac{\partial H(\xi,x_n,s)}{\partial s}\right| &= \left|\frac{s(1-\frac12\, r\, x_n)}{r+\beta\, s - i\displaystyle\sum_{l=1}^{n-1} b_l\,\xi_l}\, e^{-r x_n} - \frac{s(\frac12 r + \beta\, r)}{\Big(r+\beta\, s - i\displaystyle\sum_{l=1}^{n-1} b_l\,\xi_l\Big)^2}\, e^{-r x_n}\right|\\
&\le c\left(1 + \frac{|s|}{\Big|r+\beta\, s - i\displaystyle\sum_{l=1}^{n-1}\xi_l\, b_l\Big|}\right)\frac{1}{x_n}\;.
\end{aligned}
$$

Let us prove that

$$\mathcal{P} = \frac{|s|}{\left|r + \beta\, s - i \sum\limits_{l=1}^{n-1} \xi_l\, b_l\right|} \le c\ . \tag{18}$$

If $|\sum_{l=1}^{n-1} \xi_l\, b_l| \le \frac{1}{2}(\beta|\operatorname{Im} s|+|\operatorname{Im} r|)$ then, taking into account that $\beta > 0$, $\operatorname{Re} s > 0$ and $\operatorname{sign}(\operatorname{Im} s) = \operatorname{sign}(\operatorname{Im} r)$, we have

$$\mathcal{P} \le \frac{|s|}{\sqrt{(\operatorname{Re} r + \beta \operatorname{Re} s)^2 + \frac{\beta^2}{4}(\operatorname{Im} s)^2}} \le c\ .$$

If $|\sum_{l=1}^{n-1} \xi_l\, b_l| \ge \frac{1}{2}(\beta|\operatorname{Im} s| + |\operatorname{Im} r|)$ then $|\operatorname{Im} s| \le c|\xi|$ and

$$\mathcal{P} \le c\, \frac{\operatorname{Re} s + |\xi|}{\operatorname{Re} \sqrt{\xi^2 + s}} \le c\ .$$

So, for the all possible values of ξ and s, the inequality (18) holds and $|s \frac{\partial H(\xi, x_n, s)}{\partial s}|$ can be estimated by $\frac{c}{x_n}$.

Thus, we obtain the required estimates (9) for the function H and its derivatives of the first order. The estimates (9) for the higher order derivatives of the function H are obtained in a similar way.

Taking into account (14) and (18), we have

$$\left|\xi_{k_1} \cdots \xi_{k_l}\, \frac{\partial^l H(\xi, x_n, s)}{\partial \xi_{k_1} \cdots \partial \xi_{k_l}}\right| + \left|s\, \xi_{k_1} \cdots \xi_{k_{l-1}}\, \frac{\partial^l H(\xi, x_n, s)}{\partial \xi_{k_1} \cdots \partial \xi_{k_{l-1}}\, \partial s}\right| \le$$
$$\le c\Big(|r| + x_n|r|^2 + ... + x_n^l |r|^{l+1}\Big)\, e^{-\operatorname{Re} r\, x_n}\ ,$$

and, with the help of inequality (16) with $j = 1, 2, ..., l+1$, we arrive at the required estimates.

Hence, due to Theorem 3, we make a conclusion that $H(\xi, x_n, s)$ is a Fourier multiplier of class (p, p).

According to Theorem 4, it follows from the representation formula (13) that

$$\|F^{-1}(r^j\, \widetilde{u})\|_p \le c\bigg(\Big\|F^{-1}(r^{j-1}\, F\varpi)\Big\|_{p,R_{2,\infty}} + \Big\|F^{-1}\Big(r^{j-2}\, \frac{\partial}{\partial x_n}\, F\,\varpi\Big)\Big\|_{p.R_{2,\infty}}\bigg)\ ,$$
$$j = 2, ..., m{+}2\ .$$

Making the integral transform inverse to (3), we find the solution of problem (1a) $u = F^{-1}(\widetilde{u})$.

From the explicit form (6), for this solution we obtain

$$\frac{\partial^j u}{\partial x_n^j} = F^{-1}\Big(\frac{\partial^j}{\partial x_n^j}\, Fu\Big) = F^{-1}\Big((-1)^j\, r^j\, \widetilde{u}\Big)\ .$$

Thus,

$$\begin{aligned}\langle\!\langle u\rangle\!\rangle^{(m+2)}_{p,R_{2,T}} \le \langle\!\langle u\rangle\!\rangle^{(m+2)}_{p,R_{2,\infty}} &\le c\Big(\|F^{-1} r^{m+2}\, Fu\|_{p,R_{2,\infty}}\, \Big\|\frac{\partial^{m+2}u}{\partial x_n^{m+2}}\Big\|_{\rho,R_{2,\infty}}\Big)\\ &\le c\Big(\Big\|F^{-1}(r^{m+1}\, F\varpi)\Big\|_{p,R_{2,\infty}} + \Big\|F^{-1}\Big(r^m\, \frac{\partial}{\partial x_n} F\varpi\Big)\Big\|_{p,R_{2,\infty}}\Big)\\ &\le c\langle\!\langle \varpi\, e^{t\,\mathrm{Re}\, s}\rangle\!\rangle^{(m+1)}_{p,R_{2,\infty}}\ .\end{aligned}$$

From here, taking into account estimates (11) and (12), we obtain the desired estimate

$$\langle\!\langle u\rangle\!\rangle^{(m+2)}_{p,R_{2,T}} \le c\Big(\langle\!\langle \varphi\rangle\!\rangle^{(m+1-1/p)}_{p,\Gamma_T} + \langle\!\langle f\rangle\!\rangle^{(m)}_{p,R_{2,T}}\Big)\ .$$

Together with estimate (10) this estimate implies (8) for the solution of problem (1) $v = w + u$. ∎

Remark. There holds also the estimate

$$\|v\|^{(m+2)}_{p,R_{2,T}} \le c\Big(\|\varphi\|^{(m+1-1/p)}_{p,\Gamma_T} + \|f\|^{(m)}_{p,R_{2,T}}\Big)\ .$$

The uniqueness of the solution follows from the fact that if we consider homogeneous problem corresponding to problem (1) and make the Laplace–Fourier transform (3) we arrive at the homogeneous problem (4) which has only zero solution.

Proof of Theorem 2: At first we extend the function ψ from Γ_T into $R_{k,\infty}$, constructing

$$\psi_k \in \overset{\circ}{W}{}_p^{m+1,\frac{m}{2}+\frac{1}{2}}(R_{k,\infty})\,, \qquad k = 1, 2\ ,$$

such that

$$\psi_k\Big|_{\substack{x_n=0\\ t\le T}} = \psi\,, \qquad \psi_k\Big|_{\substack{x_n=0\\ t\ge T+\varepsilon}} = 0, \quad \varepsilon > 0\ ,$$

and

$$\langle\!\langle \psi_k\rangle\!\rangle^{(m+1)}_{p,R_{k,\infty}} \le c\, \langle\!\langle \psi\rangle\!\rangle^{(m+1-1/p)}_{p,\Gamma_T}\ .$$

For $\widetilde{v}_k$ which is given by formula (7) we use an integral representation similar to (13). As in the proof of Theorem 1, we should check that the function

$$H_k(\xi, x_n, s) = \frac{r_k^2\, e^{-|x_n|\, r_k}}{s + \frac{b_n}{a_1}\, r_1 + \frac{c_n}{a_2}\, r_2 - i(\boldsymbol{b}' - \boldsymbol{c}')\cdot \xi}$$

is a Fourier multiplier of class (p,p) and that

$$\mathcal{M}_p^p\Big(H_k(\cdot,x_n,\cdot)\Big) \le \frac{c}{x_n}\ .$$

Calculating the derivatives of the function H_k and using the same arguments as in the proof of the estimates for the mixed derivatives of the function H, we make sure that estimates (9) with $M=\frac{c}{x_n}$ hold true.

For example,

$$\left| s\,\frac{\partial H_k}{\partial s}\right| =$$

$$= \left| \frac{s-\frac{1}{2}\,r_k\,|x_n|\,s}{s+\frac{b_n}{a_1}\,r_1+\frac{c_n}{a_2}\,r_2-i(\boldsymbol{b}'-\boldsymbol{c}')\cdot\xi} - \frac{s\,r_k^2\Big(1+\frac{b_k}{2\,a_1\,r_1}+\frac{c_n}{2\,a_2\,r_2}\Big)}{\Big(s+\frac{b_n}{a_1}\,r_1+\frac{c_n}{a_2}\,r_2-i(\boldsymbol{b}'-\boldsymbol{c}')\cdot\xi\Big)^2}\right| \cdot \left|e^{-r_k|x_n|}\right|$$

$$\le \left(\frac{|s\,r_k\,x_n|}{\Big|s+\frac{b_n}{a_1}\,r_1+\frac{c_n}{a_2}-i(\boldsymbol{b}'-\boldsymbol{c}')\cdot\xi\Big|} + \frac{|s|\,|r_k|^2}{\Big|s+\frac{b_n}{a_1}\,r_1+\frac{c_n}{a_2}\,r_2-i(\boldsymbol{b}'-\boldsymbol{c}')\cdot\xi\Big|^2}\right) e^{-\operatorname{Re} r_k\,|x_n|}$$

$$\le \frac{c}{|x_n|}\,\frac{|r_k|}{\operatorname{Re} r_k} \le \frac{c}{|x_n|}\ .$$

Then, we use Theorem 4 and complete the proof of Theorem 2 in the same way as it has been done above in the proof of Theorem 1. ∎

3 – Periodic problem

In this section we assume that all the given functions and surfaces are periodic with respect to the space variables x_i $(i=1,...,n-1)$ with the period 1 and call such functions simply periodic.

Let $\Omega\subset\mathbb{R}^n$ be a domain which lies between two mutually disjoint smooth periodic surfaces S_1 and S_2.

We denote by $B_1(a)$, $a\in\mathbb{R}^{n-1}$, the cube in the space $\mathbb{R}^{n-1}$

$$B_1(a) = \left\{x'\in\mathbb{R}^{n-1}\mid |x_i-a_i|<\frac{1}{2},\ i{=}1,...,n{-}1\right\}\ .$$

By $\prod_{1,a}$ we denote the strip $B_1(a)\times\mathbb{R}=\{x\in\mathbb{R}^n\mid x'\in B_1(a)\}$.

For an arbitrary set $E\subset\mathbb{R}^n$ we set $E'(a)=E\cap\prod_{1,a}$.

By $\widetilde{L}_p(\Omega_T)$ we mean the space of periodic functions with a finite norm in $L_p(\Omega'_T(a))$, $\forall\,a\in\mathbb{R}^{n-1}$.

We denote by $\widetilde{W}_p^{2,1}(\Omega_T)$ the space of periodic functions which are defined in $\Omega_T = \Omega \times [0,T)$, have generalized derivatives $\mathcal{D}_x^\alpha u(x,t)$, $|\alpha| \le 2$, $\alpha = (\alpha_1, ..., \alpha_n)$, $\mathcal{D}_t u(x,t)$ and a finite norm

$$\|u\|_{p,\Omega_T}^{(2)} = \|u\|_{\widetilde{W}_p^{2,1}(\Omega_T)} = \|u\|_{W_p^{2,1}(\Omega'_T(a))}, \qquad \forall\, a \in \mathbb{R}^{n-1} .$$

$\widetilde{W}_p^{2-1/p,\,1-1/2p}(S_T)$ is the space of traces of functions from $\widetilde{W}_p^{2,1}(\Omega_T)$ on the periodic surface $S \in \overline{\Omega}$. The norm in this space is determined by the formula

$$\|u\|_{p,S_T}^{(2-1/p)} = \|u\|_{\widetilde{W}_p^{2-1/p,\,1-1/2p}(S_T)} = \|u\|_{W_p^{2-1/p,\,1-1/2p}(S'_T(a))}, \qquad \forall\, a \in \mathbb{R}^{n-1} .$$

$\overset{\circ}{\widetilde{W}}{}_p^{2,1}(\Omega_T)$ ($\overset{\circ}{\widetilde{W}}{}_p^{2-1/p,\,1-1/2p}(S_T)$) is a subspace of the space $\widetilde{W}_p^{2,1}(\Omega_T)$ ($\widetilde{W}_p^{2-1/p,\,1-1/2p}(S_T)$) containing the functions satisfying zero initial conditions:

$$u|_{t=0} = 0 .$$

By $\widetilde{W}_p^{2-\gamma}(\mathbb{R}^{n-1})$, $\gamma \in (0,1)$, we denote the space of periodic functions having first order generalized derivatives and a finite norm

$$\|u\|_{\widetilde{W}_p^{2-\gamma}(\mathbb{R}^{n-1})} = \|u\|_{W_p^{2-\gamma}(B_1(a))}, \qquad \forall\, a \in \mathbb{R}^{n-1} .$$

We consider the following problem

$$\begin{aligned} & Lu \equiv u_t - \sum_{i,j=1}^n a_{ij}(x,t)\, u_{x_i x_j} - \sum_{i=1}^{n-1} a_i(x,t)\, u_{x_i} - a(x,t)\, u = f, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \Omega_T , \\ & Bu = \mu\, u_t + \sum_{i=1}^n b_i(x,t)\, u_{x_i}\, |_{x \in S_1} = \varphi_1 , \\ & Cu = \sum_{i=1}^n c_i(x,t)\, u_{x_i} + c(x,t)\, u\, |_{x\in S_2} = \varphi_2 , \\ & u|_{t=0} = 0 , \end{aligned} \tag{19}$$

where

$$\gamma_1\, \xi^2 \le a_{ij}(x,t)\, \xi_i\, \xi_j \le \gamma_2\, \xi^2 , \qquad \gamma_1, \gamma_2 > 0, \;\; \forall\, \xi \in \mathbb{R}^n, \;\; t \in [0,T) .$$

We assume that the coefficients a_{ij}, a_i, a, b_i, c_i, c and the given functions f, φ_1, φ_2 are periodic.

The surfaces S_1 and S_2 are supposed from the class C^2. It means that there exists such a positive number d that in the ball K_d with the radius d and with the center in any $x_0 \in S_j$ $(j=1,2)$, the equation of the surface $S_j \cap K_d$ in local coordinates with the center in the point x_0 has the form $y_n = F(y_1, ..., y_{n-1})$, where the function F belongs to $C^2(\mathbb{R}^{n-1})$.

Theorem 5. *Let in problem (19)* $a_{ij} \in C(\Omega_T)$, $a_i, a \in \widetilde{L}_p(\Omega_T)$, $b_i \in \widetilde{W}_p^{1-1/p,\,1/2-1/2p}(S_{1,T})$, $c_i \in \widetilde{W}_p^{1-1/p,\,1/2-1/2p}(S_{2,T})$ *with a certain* $p > n+2$, *and there hold the inequalities:*

$$\mu > 0\,, \qquad \sum_{j=1}^{n} b_j(x,t)\,\nu_j \geq \delta > 0, \quad (x,t) \in \Omega_T \ , \tag{20}$$

where $\boldsymbol{\nu} = (\nu_1, ..., \nu_n)^T$ *is the outward normal to the surface* S_1.

Then, for any $f \in \widetilde{L}_p(D_T)$,

$$\varphi_1 \in \underset{\circ}{\widetilde{W}}{}_p^{1-1/p,\,1/2-1/2p}(S_{1,T})\,, \qquad \varphi_2 \in \underset{\circ}{\widetilde{W}}{}_p^{1-1/p,\,1/2-1/2p}(S_{2,T}) \ ,$$

problem (19) has a unique solution $u \in \underset{\circ}{\widetilde{W}}{}_p^{2,1}(\Omega_T)$, *this solution has the additional smoothness on the surface* S_1: $u_t|_{x\in S_1} \in \underset{\circ}{\widetilde{W}}{}_p^{1-1/p,\,1/2-1/2p}(S_{1,T})$ *and the estimate*

$$\|u\|^{(2)}_{p,\Omega_T} + \|u_t\|^{(1-1/p)}_{p,S_{1,T}} \leq c\Big(\|f\|_{p,\Omega_T} + \|\varphi_1\|^{(1-1/p)}_{p,S_{1,T}} + \|\varphi_2\|^{(1-1/p)}_{p,S_{2,T}}\Big) \tag{21}$$

holds.

Proof: We are going to prove this result with the help of constructing a regularizer in the same way as it is usually done in a nonperiodic case [9, 17].

We rewrite problem (19) in the form

$$Au = h \ , \tag{22}$$

where

$$Au = \Big[Lu, Bu, Cu\Big]^T \ ,$$

$$h = [f, \varphi_1, \varphi_2]^T \ .$$

For any small $\lambda > 0$ we construct the systems of subdomains $\omega^{(k)} \subset \Omega^{(k)} \subset \Omega$ with the following properties:

1) $\bigcup_k \omega^{(k)} = \bigcup_k \Omega^{(k)} = \Omega$.

2) For any point $x \in \Omega$ there can be found such $\omega^{(k)}$ that $x \in \omega^{(k)}$ and $\operatorname{dist}(x, \Omega\backslash\omega^{(k)})$
$\geq d\lambda$, $d > 0$.

3) There exists $N_0 \in \mathbb{N}$ such that $\bigcap_{j=1}^{N_0+1} \Omega^{(k_j)} = \emptyset$ for any $k_1, ..., k_{N_0+1}$, $k_i \neq k_j$ for $i \neq j$.

4) The systems of subdomains $\omega^{(k)}$, $\Omega^{(k)}$ $(k \in \mathbb{N})$ are periodic with respect to the space variables x_i, $i = 1, ..., n-1$, with the period 1, $\operatorname{diam}\Omega^k < \frac{1}{2}$.

We identify the domains $\omega^{(k)}$, $\Omega^{(k)}$ which can be obtained from each other by displacement by the period. Then, we obtain a finite system of periodic sets $\widetilde{\omega}^{(j)}$ and $\widetilde{\Omega}^{(j)}$ $(j = 1, ..., M)$.

Let $\xi^{(j)}(x)$ $(j = 1, ..., M)$ be smooth periodic functions such that

$$\xi^{(j)}(x) = \begin{cases} 1, & x \in \widetilde{\omega}^{(j)}, \\ 0, & x \in \Omega\backslash\widetilde{\Omega}^{(j)} \end{cases},$$

$$0 \leq \xi^{(j)} \leq 1\,, \qquad |D_x^s \xi^{(j)}| \leq \frac{c}{\lambda^{|s|}}\ .$$

By virtue of property 3 of the domains $\Omega^{(k)}$

$$1 \leq \sum_{k=1}^{M} \xi^k(x) \leq N_0 \qquad \text{for any} \quad x \in \Omega\ ,$$

and, consequently, the periodic functions

$$\eta^{(j)}(x) = \frac{\xi^{(j)}(x)}{\sum\limits_j \xi^{(j)2}(x)}$$

have the following properties:

$$\eta^{(j)}(x) = 0, \quad x \in \Omega\backslash\widetilde{\Omega}^{(j)}\,, \qquad |D_x^s \eta^{(j)}(x)| \leq \frac{c}{\lambda^{|s|}}$$

and

$$\sum_{j=1}^{M} \eta^{(j)}(x)\,\xi^{(j)}(x) = 1 \qquad \text{for any} \quad x \in \Omega\ .$$

We fix in every $\widetilde{\omega}^{(k)}$ the set of points $\widetilde{x}_k$ which can be obtained from each other by displacement by the period. Without loss of generality we can assume that if $\overline{\widetilde{\Omega}}^{(k)} \cap S_j \neq \emptyset$ then $\overline{\widetilde{\omega}}^{(k)} \cap S_j \neq \emptyset$, in this case we fix points $\widetilde{x}_k$ on $\overline{\widetilde{\omega}}^{(k)} \cap S_j$.

If we freeze the coefficients in problem (19) at the point $x_k \in \widetilde{x}_k$, take the principal part of the operator A, and multiply the given functions on $\xi^{(k)}$, then, by virtue of the periodicity of the coefficients and the given functions, we arrive at one of the following periodic problems:

1) in the case when $\overline{\widetilde{\Omega}}^{(k)} \subset \Omega$:

$$(23)\quad Lu^{(k)} = u_t - \sum_{i,j=1}^{n} a_{ij}(x_k, 0)\, u_{x_i x_j} = f\,\xi^{(k)} = f^{(k)}\,, \quad u|_{t=0} = 0,\ \ k = 1, ..., M_1\ ;$$

2) in the case when $\overline{\widetilde{\Omega}}^{(k)} \cap S_2 \neq \emptyset$:

$$(24)\qquad L^{(k)}u = f^{(k)},\ \ C^{(k)}u = \sum_{l=1}^{n} C_l(x_k, 0)\, u_{x_l}|_{x \in S_2} = \varphi_2\,\xi^{(k)} = \varphi_2^{(k)}\ ,$$
$$u|_{t=0} = 0,\quad k = M_1 + 1, ..., M_2\ ;$$

3) in the case when $\overline{\widetilde{\Omega}}^{(k)} \cap S_1 \neq \emptyset$:

$$(25)\qquad \begin{aligned} &L^{(k)}u = f^{(k)}\ ,\\ &B^{(k)}u = \mu\, u_t + \sum_{l=1}^{n} b_l(x_k, 0)\, u_{x_l}|_{x \in S_1} = \varphi_1\,\xi^{(k)} = \varphi_1^{(k)}\ ,\\ &u|_{t=0} = 0\,, \qquad k = M_2 + 1, ..., M\ . \end{aligned}$$

Let us choose the domain Ω^k from the periodic set $\widetilde{\Omega}^k$ $(k = 1, ..., M_1)$ and consider problem (23) in this domain. Making the zero extension of the given functions from Ω^k to $\mathbb{R}^n$ we arrive at the Cauchy problem in $\mathbb{R}^n$. The unique solvability of the Cauchy problem for second order parabolic equations with constant coefficients in Sobolev function spaces was established in [9]. We multiply the solution by the function $\eta^{(k)}|_{\Omega^k}$ and do the periodic extension from Ω^k to $\widetilde{\Omega}^{(k)}$. By virtue of the periodicity of the given functions and uniqueness of the solution of the Cauchy problem, the obtained periodic function does not depend on the choice of the domain $\Omega^{(k)}$ from the set $\widetilde{\Omega}^k$.

If $\widetilde{\omega}^{(k)}$ is closed to the boundary and in neighbourhoods of the points $\widetilde{x}^{(k)}$ the boundary surface can be given in a local coordinate system by the equation $y_n = F^{(k)}(y')$, we make in every $\Omega^{(k)} \in \widetilde{\Omega}^{(k)}$ the coordinate transformation $Z^{(k)}$:

$$z_i = y_i,\ \ i = 1, ..., n-1\,, \qquad z_n = y_n - F^{(k)}(y')\ ,$$

which straightens the boundary. After this transformation we arrive at a periodic problem of type (24) or (25) in a half-space in which the operators $Z^{(k)}$, $C^{(k)}$ and $B^{(k)}$ are recalculated in the new coordinates and take the form $\widetilde{Z}^k$, $\widetilde{C}^k$, $\widetilde{B}^k$.

If $\widetilde{\omega}^{(k)}$ is closed to S_2 we have

$$
\begin{aligned}
&\widetilde{Z}^{(k)} u^{(k)} = \widetilde{f}^{(k)} , \qquad x \in \mathbb{R}^n_+, \quad t \in [0,T) \ , \\
&\widetilde{C}^{(k)} u^{(k)}|_{x_n=0} = \widetilde{\varphi}_2^{(k)} , \qquad u^{(k)}|_{t=0} = 0 \ ,
\end{aligned} \tag{26}
$$

where

$$
\widetilde{f}^{(k)} = P_k^{-1} f^{(k)} , \qquad \widetilde{\varphi}_2^{(k)} = P_k^{-1} \varphi_2^{(k)} \ ,
$$

P_k is an operator which makes correspond to a function in local coordinates the same function in the original coordinates.

We consider problem (26) in the image of the domain $\Omega^k \in \widetilde{\Omega}^k$ ($k = M_1+1, ..., M_2$) under the transform $Z^{(k)}$ and do the zero extension of the given functions into $\mathbb{R}^n_+$. We obtain the *second boundary value problem* in a half-space, the unique solvability of which in Sobolev function spaces is known [9]. We do the transform inverse to $Z^{(k)}$, multiply the solution by $\eta^{(k)}|_{\Omega^k}$, and do the periodic extension to $\widetilde{\Omega}^k$.

If $\widetilde{\omega}^{(k)}$ is closed to S_1 then, after the transformation $Z^{(k)}$, we arrive at the problem

$$
\begin{aligned}
&\widetilde{Z}^{(k)} u^{(k)} = \widetilde{f}^{(k)} , \qquad x \in \mathbb{R}^n_-, \quad t \in [0,T) \ , \\
&\widetilde{B}^{(k)} u^{(k)}|_{x_n=0} = \widetilde{\varphi}_1^{(k)} , \qquad u^{(k)}|_{t=0} = 0 \ , \\
&\widetilde{\varphi}_1^{(k)} = P_k^{-1} \varphi_1^{(k)} \ .
\end{aligned} \tag{27}
$$

Let us consider the problem in a half-space for the second order parabolic equation with constant coefficients:

$$
\begin{aligned}
&u_t - \sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j} = g \, , \qquad x \in \mathbb{R}^n_-, \quad t \in [0,T) \ , \\
&\mu\, u_t + \sum_{i=1}^{n} b_i\, u_{x_i}|_{x_n=0} = \varphi \ , \\
&u|_{t=0} = 0 \ ,
\end{aligned} \tag{28}
$$

where

$$
a_{ij}, b_i \in \mathbb{R} \, , \qquad b_n < 0, \quad \mu > 0 \ .
$$

We make the orthogonal coordinate transform which brings the matrix of the coefficients in the equation to a diagonal form. Under this transform the boundary condition in problem (28) transforms to the boundary condition of the same type. After additional linear change of variables we obtain the problem for the heat equation. Making once more orthogonal coordinate transform we have in the new variables z_i the boundary condition on the plane $z_n = 0$. So, we can reduce problem (28) to the corresponding problem for the heat equation, the unique solvability for this problem in the Sobolev function spaces is established by Theorem 1.

We consider problem (27) in the image of the domain $\Omega^k \in \widetilde{\Omega}^{(k)}$ ($k = M_2+1, ..., M$) under the transform $Z^{(k)}$ and do the zero extension of the given functions into $\mathbb{R}^n_+$, then we arrive at the problem of type (28). We do the transform inverse to $Z^{(k)}$, multiply the solution of the obtained problem by $\eta^{(k)}|_{\Omega^k}$, and do the periodic extension to $\widetilde{\Omega}^{(k)}$.

Note, that by virtue of the periodicity of the given functions and uniqueness of the solution of the second boundary value problem and of problem (28), the obtained periodic functions are independent on the choice of the domain Ω^k from the set $\widetilde{\Omega}^{(k)}$.

We put

$$\begin{aligned} Rh = & \sum_{k=1}^{M_1} \sqcap\, \eta^{(k)}(x)\, R^{(k)}(\xi^{(k)} f) \\ & + \sum_{k=M_1+1}^{M_2} \sqcap\, \eta^{(k)}(x)\, P_k R^{(k)}(P_k^{-1}\xi^{(k)} f,\, P_k^{-1}\xi^{(k)}\, \varphi_2) \\ & + \sum_{k=M_2+1}^{M} \sqcap\, \eta^{(k)}(x)\, P_k R^{k}(P_k^{-1}\xi^{(k)} f,\, P_k^{-1}\xi^{(k)}\, \varphi_1)\ , \end{aligned}$$

where by R^k we mean the operator which assigns to the given functions the solution of

1) problem (23) for $k = 1, ..., M_1$,

2) problem (26) for $k = M_1 + 1, ..., M_2$,

3) problem (27) for $k = M_2 + 1, ..., M$.

in the domain $\Omega^k \in \widetilde{\Omega}^k$, by $\sqcap$ we mean the operator which makes the periodic extension.

In just the same way as it was done in [9] for nonperiodic case, making all the estimates in the norms of periodic Sobolev spaces, it can be shown that for a sufficiently small $\tau > 0$

$$\|AR - I\| < 1 \quad \text{and} \quad \|RA - I\| < 1\ .$$

The term $\mu\, u_t$ in the boundary condition on the surface S_1 does not complicate the estimates because it is presented also in problems (25), μ is a constant, and the functions $\xi^{(k)}$, $\eta^{(k)}$ and $F^{(k)}$ do not depend on the variable t. So in the operators $AR - I$ and $RA - I$ there is no additional terms in comparison with the case of the second boundary condition.

Consequently, the operator R is a regularizer and, for the sufficiently small τ the operator A in problem (22) has a bounded inverse operator. It means that problem (19) has a unique solution $u \in \underset{\circ}{\widetilde{W}}{}_p^{2,1}(\Omega_\tau)$, which has the additional smoothness on $S_{1\tau}$ and for which there holds estimate (21). To prove the existence of the solution on the whole time interval $[0, T)$ we use the same reasoning as in [15]. Namely, we choose $t_0 \in (\frac{3\tau}{4}, \tau)$ and extend $u|_{\Omega\times[0,t_0]}$ to $\Omega \times [0, 2t_0]$ using the Hestence–Whitney method:

$$\widehat{u}(x,t) = \begin{cases} u(x,t), & t \in [0,t_0], \\ \sum\limits_{j=1}^{N} u\Big(x, \dfrac{2t_0 - t}{j}\Big)\lambda_j, & t \in [t_0, 2t_0] \end{cases} ,$$

$$\sum_{j=1}^{N} \lambda_j \Big(-\frac{1}{j}\Big)^s = 1 \,, \qquad s = 0, 1, ..., N-1 \ .$$

For a sufficiently large N we have

$$\widehat{u} \in \underset{\circ}{\widetilde{W}}{}_p^{2,1}(\Omega_{2,t_0}) \,, \qquad \widetilde{u}_t|_{S_1} \in \underset{\circ}{\widetilde{W}}{}_p^{1-1/p,\,1/2-1/2p}(S_{1,2t_0}) \ .$$

We find a solution $w(x,t)$ of the problem

$$\begin{aligned} & Lw = f - L\widehat{u} = \widehat{f} \,, \qquad x \in \Omega, \quad t \in [t_0, 2t_0) \ , \\ & Cw|_{x\in S_2} = \varphi_2 - C\widehat{u}|_{x \in S_2} = \widehat{\varphi}_2 \ , \\ & Bw|_{x\in S_1} = \varphi_1 - \mu\, \widehat{u}_t - \sum_{i=1}^{n} b_i(x,t)\, \widehat{u}_{x_i}|_{x\in S_1} = \widehat{\varphi}_1 \ , \\ & w|_{t=t_0} = 0 \ . \end{aligned} \tag{29}$$

Note that by construction a regularizer one can prove the existence of the solution of problem (29) on the small time interval $[t_0, t_0 + \tau)$ in the same way as for problem (19). Because of uniform parabolicity of the operator L and condition (20) for the operator B, τ can be chosen one and the same for any $t_0 \in [0, T)$.

Thus, we find $w \in \widetilde{W}_p^{2,1}(\Omega \times [t_0, t_0 + \tau))$ such that $w_t|_{x\in S_1\times[t_0,t_0+\tau)} \in \widetilde{W}_p^{1-1/p,\,1/2-1/2p}(S_1 \times [t_0, t_0 + \tau))$, $w|_{t=t_0} = 0$, $w_t|_{\substack{x\in S_1 \\ t=t_0}} = 0$. We make the zero extension $\widehat{w}$ of the function w to Ω_{t_0}. Obviously,

$$\widehat{w} \in \underset{\circ}{\widetilde{W}}{}_p^{2,1}(\Omega_{t_0+\tau}) \,, \qquad \widehat{w}_t|_{x\in S_{1,t_0+\tau}} \in \underset{\circ}{\widetilde{W}}{}_p^{1-1/p,\,1/2-1/2p}(S_{1,t_0+\tau}) \ .$$

For $t \in [0, 2t_0)$ we put $u = \widehat{u} + \widehat{w}$, it is a solution of problem (19) with the all required properties.

In this way we can construct the solution of problem (19) in a finite number of steps. ∎

4 – Boundary from $W_p^{2-1/p}(\mathbb{R}^{n-1})$

In this section we reduce the assumptions on the smoothness of the boundary and consider the case when the boundary belongs to the space $W_p^{2-1/p}(\mathbb{R}^{n-1})$. First we consider the periodic case, as in section 3.

Assume that the surface S_1 can be given by the equation $x_n = \xi(x')$, where ξ is a periodic function which belongs to the space $\widetilde{W}_p^{2-1/p}(\mathbb{R}^{n-1})$, $p > n+2$. In this case Theorem 5 is also valid. To prove it we transform Ω_T to the domain which lies between two smooth surfaces and then apply the corresponding result (Theorem 5) to the obtained problem.

First we construct $\sigma \in \widetilde{W}_p^2(\mathbb{R}^n)$ possessing the following properties:

$$\sigma(y', 0) = \xi(y') , \qquad |\sigma_{y_n}| \le \frac{1}{2} ,$$

$$\|\sigma\|_{\widetilde{W}_p^2(\mathbb{R}^n)} \le c \|\xi\|_{\widetilde{W}_p^{2-1/p}(\mathbb{R}^{n-1})} .$$

We put for example

$$\sigma(y', y_n) = \int_{\mathbb{R}^{n-1}} K(z') \, \xi(y' + z' \lambda \, y_n) \, dz' ,$$

where the kernel $K \in C_0^\infty(\mathbb{R}^{n-1})$, $\int_{\mathbb{R}^{n-1}} K(z') \, dz' = 1$, $\lambda > 0$ is a sufficiently small number.

We make the coordinate transformation $y = Y(x)$ the inverse of which $-Y^{-1}(y)$ is given by the formulas

$$x' = y' , \qquad x_n = y_n + \sigma(y', y_n) . \tag{30}$$

The transform $Y(x)$ maps the domain Ω onto the domain $\widetilde{\Omega}$ located between the plane $y_n = 0$ and the surface $\widetilde{S}$ which is the image of S_2. $\sigma(y)$ is a smooth function for $y_n > 0$, so the surface $\widetilde{S}$ is smooth.

Note that if a function $v(x, t)$ is a periodic function with respect to the space variables x_i $(i = 1, ..., n-1)$ with period 1 then the function $\widetilde{v}(y, t) = v(Y^{-1}(y), t)$ is also periodic with respect to the space variables y_i $(i = 1, ..., n-1)$ with the same period.

In order to obtain the problem in the domain $\widetilde{\Omega}$ corresponding to the problem (19) we use the following formulas:

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i} - \frac{\sigma_{y_i}}{1+\sigma_{y_n}} \frac{\partial}{\partial y_n}, \qquad i = 1, ..., n-1 ,$$
$$\frac{\partial}{\partial x_n} = \frac{1}{1+\sigma_{y_n}} \frac{\partial}{\partial y_n} .$$

As a result Lu takes the form

$$\widetilde{L}\widetilde{u} = \widetilde{u}_t - \sum_{i,j=1}^{n} \alpha_{ij}(y,t)\, \widetilde{u}_{y_i y_j} - \sum_{i=1}^{n} \alpha_i(y,t)\, \widetilde{u}_{y_i} - \alpha(y,t)\, \widetilde{u}(y,t) ,$$

where

$$\widetilde{u}(y,t) = u(Y^{-1}(y),t) ,$$

$$\alpha_{ij}(y,t) = \widetilde{a}_{ij}(y,t) \quad \text{for} \quad i,j = 1, ..., n-1 ,$$
$$\alpha_{in}(y,t) = \alpha_{ni}(y,t) = \widetilde{a}_{in}(y,t)\, \frac{1}{1+\sigma_{y_n}} - 2\sum_{j=1}^{n-1} \widetilde{a}_{ij}(y,t)\, \frac{\sigma_{y_j}}{1+\sigma_{y_n}} \quad \text{for } i=1, ..., n-1 ,$$
$$\alpha_{nn}(y,t) = \frac{1}{(1+\sigma_{y_n})^2}\Big(\sum_{i,j=1}^{n-1} \widetilde{a}_{ij}(y,t)\, \sigma_{y_i}\, \sigma_{y_j} - 2\sum_{i=1}^{n} \widetilde{a}_{in}(y,t)\, \sigma_{y_i} + \widetilde{a}_{nn}(y,t)\Big) ,$$

$$\alpha_i(y,t) = \widetilde{a}_i(y,t) \quad \text{for} \quad i = 1, ..., n-1 ,$$
$$\begin{aligned}\alpha_n(y,t) = {} & \sum_{i,j=1}^{n-1} \widetilde{a}_{ij}(y,t)\Big(-\Big(\frac{\sigma_{y_i}}{1+\sigma_{y_n}}\Big)_{y_j} + \frac{\sigma_{y_j}}{1+\sigma_{y_n}}\Big(\frac{\sigma_{y_i}}{1+\sigma_{y_n}}\Big)_{y_n}\Big) \\ & - \frac{2}{1+\sigma_{y_n}} \sum_{i=1}^{n-1} \widetilde{a}_{in}(y,t)\Big(\frac{\sigma_{y_i}}{1+\sigma_{y_n}}\Big)_{y_n} + \widetilde{a}_{nn}(y,t)\, \frac{1}{1+\sigma_{y_n}}\Big(\frac{1}{1+\sigma_{y_n}}\Big)_{y_n} \\ & - \frac{1}{1+\sigma_{y_n}} \sum_{i=1}^{n-1} \widetilde{a}_i(y,t)\, \sigma_{y_i} + \frac{1}{1+\sigma_{y_n}}\, \widetilde{a}_n(y,t) ,\end{aligned}$$

$$\alpha(y,t) = \widetilde{a}(y,t) .$$

As $\sigma \in \widetilde{W}_p^2(\mathbb{R}^n)$, $p > n+2$, by the embedding theorem [18], σ_{y_i}, $i = 1, ..., n$, satisfy the Hölder condition. It has been assumed also that $|\sigma_{y_n}| < \frac{1}{2}$, therefore the function $\frac{1}{1+\sigma_{y_n}}$ is continuous. The functions $\widetilde{a}_{ij}(y,t) = a_{ij}(Y^{-1}(y),t)$ are continuous as compositions of the continuous functions. Hence, the coefficients α_{ij} $(i,j = 1, ..., n)$ at the second order derivatives in the operator $\widetilde{L}$ are continuous.

As $a_i(x,t)$, $a(x,t) \in \widetilde{L}_p(\Omega)$, and the Jacobean of transformation (30) is positive, we conclude that

$$\widetilde{a}_i(y,t) = a_i(Y^{-1}(y),t)\,, \qquad \widetilde{a}(y,t) = a(Y^{-1}(y),t) \in \widetilde{L}_p(\widetilde{\Omega})\;.$$

The coefficient α_n is a sum of products of the type $f_1(y,t)\,f_2(y,t)$, where $f_1(y,t) \in \widetilde{L}_p(\widetilde{\Omega})$, $f_2(y,t) \in C(\widetilde{\Omega})$ and, consequently, also belongs to $\widetilde{L}_p(\widetilde{\Omega})$.

Thus $\alpha_i(y,t)$, $i = 1,...,n$, $\alpha(y,t) \in \widetilde{L}_p(\widetilde{\Omega})$.

The boundary condition on the surface S_1 is transformed into the boundary condition on the plane $y_n = 0$ and takes the form

$$\mu\,\widetilde{u}_t - \sum_{i=1}^{n-1} \widetilde{b}_i(y,t)\,\widetilde{u}_{y_i} + \frac{1}{1+\sigma_{y_n}}\Big(\widetilde{b}_n(y,t) - \sum_{i=1}^{n-1} \widetilde{b}_i(y,t)\,\sigma_{y_i}\Big)\,\widetilde{u}_{y_n} = \widetilde{\varphi}_1\;. \tag{31}$$

We consider the case when the surface S_1 is given by the equation $x_n = \xi(x')$, so the outward normal to the surface S_1 is $\boldsymbol{\nu} = (-\xi_{x_1}, -\xi_{x_2}, ..., -\xi_{x_{n-1}}, 1)^T$ and condition (20) reads

$$b_n(x,t) - \sum_{i=1}^{n-1} b_i(x,t)\,\xi_{x_i} > 0\;.$$

Thus, the coefficient at the derivative $\widetilde{u}_{y_n}$ in the boundary condition (31) is positive and, because of the fact that unit vector $\boldsymbol{e}_n$ is the outward normal vector to the plane $y_n = 0$, we see that condition (20) for (31) holds.

Taking into account that σ_{y_i} satisfies the Hölder condition, we see that

$$\widetilde{b}_i\;,\quad \widetilde{b}_n - \sum_{i=1}^{n-1} \widetilde{b}_i\,\sigma_{y_i} \;\in\; \widetilde{W}_p^{1-1/p,\,1/2-1/2p}(\mathbb{R}^{n-1})\;.$$

So, we obtain in the domain $\widetilde{\Omega}$ the problem for which all the assumptions of Theorem 5 hold. We apply this theorem and find the solution $\widetilde{u} \in \widetilde{W}_p^{2,1}(\widetilde{\Omega}_T)$. Then, we do the inverse transform (30). As the Jacobean of transform (30) is positive and $\sigma \in \widetilde{W}_p^2(\mathbb{R}^n)$, $p > n+2$, we conclude that

$$u(x,t) = \widetilde{u}(Y(x),t) \in \widetilde{W}_p^{2,1}(\Omega_T)\;.$$

$u(x,t)$ is a solution of problem (19). Estimate (21) for $u(x,t)$ follows from the corresponding estimate for $\widetilde{u}(y,t)$.

Now we consider the problem in a bounded domain. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega = S$. We assume that there exists $d > 0$ such that in the ball of radius d with the center at any point $x_0 \in S$ the surface S can be given in local coordinates by the equation $y_n = F(y_1, ..., y_{n-1})$, where

$$F \in W_p^{2-1/p}(\mathbb{R}^{n-1}) \ ,$$

$$(32) \qquad \begin{aligned} &Lu = f\,, \qquad (x,t) \in \Omega_T \ , \\ &Bu|_{x\in S} = \varphi \ , \\ &u|_{t=0} = 0 \ , \end{aligned}$$

where L and B are the operators defined in problem (19).

Theorem 6. *Let $a_{ij} \in C(\Omega_T)$, $a_i, a \in L_p(\Omega_T)$, $b_i \in W_p^{1-1/p,\,1/2-1/2p}(S_T)$ with a certain $p > n+2$, and boundary operator B satisfies condition (20).*

Then, for any $f \in L_p(\Omega_T)$, $\varphi \in \overset{\circ}{W}{}_p^{1-1/p,\,1/2-1/2p}(S_T)$, problem (32) has a unique solution $u \in \overset{\circ}{W}{}_p^{2,1}(\Omega_T)$, this solution has the additional smoothness on the surface S:

$$u_t|_{x\in S} \in \overset{\circ}{W}{}_p^{1-1/p,\,1/2-1/2p}(S_T) \ ,$$

and the estimate

$$\|u\|_{W_p^{2,1}(\Omega_T)} + \|u_t\|_{W_p^{1-1/p,\,1/2-1/2p}(S_T)} \le C\Big(\|f\|_{p,\Omega_T} + \|\varphi\|_{W_p^{1-1/p,\,1/2-1/2p}(S_T)}\Big)$$

holds.

This result can be proved by construction a regularizer. If in consideration of problems with frosen coefficients in subdomains closed to the boundary S we use the coordinate transform similar to (30), we reduce these problems to the problems of the same type in a half-space.

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REFERENCES

[1] Bazalii, B.V. – The Stefan problem, *Dokl. Akad. Nauk Ukrain. SSR, Ser. A,* 11 (1986), 3–7.

[2] Radkevich, C.V. – *On the solvability of general nonstationary free boundary problems*, in “Some Application of Functional Analysis to the Problems of Mathematical Physics”, Novosibirsk, 1986, p. 85–111.

[3] BIZHANOVA, G.I. and SOLONNIKOV, V.A. – On a solvability of the initial boundary value problem with time derivative in the boundary condition for a second order parabolic equation in a weighted Hölder function space, *Algebra i Analiz,* 5(1) (1993), 109–142; English transl. in *St. Petersburg Math. J.*, 5(1) (1994).

[4] ANISUTIN, B.M. – Energy estimates of a heat conduction problem with time derivative in the boundary condition, *Dinamika Sploshnou Steady,* 95 (1990), 24–39.

[5] GRIGORIEVA, N.A. and MOGILEVSKII, I.SH. – On a boundary value problem for the heat equation, *Geom. Ques. Funct. Theory,* (1985), 27–42.

[6] TEMIRBULATOR, S.I. – Incorrect missed problems of heat conduction, *Dokl. Akad. Nauk SSSR,* 246(1) (1982), 45–47.

[7] LIZORKIN, P.L. – Generalized Liouville differentiation and functional spaces $\mathbb{L}_p^{(r)}(\mathbb{R}^n)$, *Mat. Sb.,* 60(3) (1963), 325–353.

[8] VOLEVICH, L.P. – Solvability of the boundary value problems for general elliptic systems, *Mat. Sb.,* 68(3) (1965), 373–416.

[9] LADYZHENSKAYA, O.A., SOLONNIKOV, V.A. and URAL'TSEVA, N.N. – *Linear and quasilinear equations of parabolic type*, M. Nauka, 1967; English transl., Transl. Math. Monographs, Vol. 23, Amer. Math. Soc., Providence, RI, 1968.

[10] SOLONNIKOV, V. and FROLOVA, E.V. – L_p-theory for the Stefan problem, *Zap. Nauchn. Semin. POMI,* 243 (1997), 299–323.

[11] FROLOVA, E.V. – L_p-estimates for solutions of model problems with time derivative in the boundary condition or in the conjugation condition, *Izv. St. Petersburg Electrotechnical Univ.,* 501 (1996), 69–79.

[12] KOCH, H. – Classical solutions to phase transition problems are smooth, preprint *(SFB 359),* (1996), 96–46, Univ. of Heidelberg.

[13] BAZALII, B.V. and DEGTYAREV, S.P. – Classical solvability of the multidimensional Stefan problem under convective movement of viscous incompressible fluid, *Mat. Sb.,* 132(1) (1987), 3–19.

[14] BIZHANOVA, G.I. – Solvability of the initial boundary value problem with time derivative in the conjugation condition for a second order parabolic equation in a weighted Hölder function space, *Algebra i Analiz,* 6(1) (1994), 64–94.

[15] FROLOVA, E.V. – An initial boundary value problem with a noncoercive boundary condition in domains with edges, *Zap. Nauchn. Semin. POMI,* 213 (1994), 206–223.

[16] MOGILEVSKII, I.SH. – Estimates of a solution of a general initial boundary value problem for the linear nonstationary Navier–Stokes system in a half-space, *Zap. Nauchn. Semin. POMI,* 84 (1979), 147–173.

[17] BIZHANOVA, G.I. – Investigation of the solvability of the multidimensional two-phase Stefan problems and the Florin problem on nonstationary filtration for second order parabolic equations in a weighted Hölder function space (Cauchy–Stefan and Cauchy–Florin problems), *Zap. Nauchn. Sem. POMI,* 213 (1994), 14–47.

[18] SOLONNIKOV, V.A. – A priory estimates for second order parabolic equations, *Trydy Mat. Inst. Steklov,* 70 (1964), 133–212.

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