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A GENERALIZATION OF A RESULT OF ANDRÉ-JEANNIN CONCERNING SUMMATION OF RECIPROCALS

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1 – Introduction

For p a strictly positive real number define, for all integers n, the sequences

$$
\begin{cases} U_n = p\,U_{n-1} + U_{n-2}\,, & U_0=0,\quad U_1=1\,, \\ V_n = p\,V_{n-1} + V_{n-2}\,, & V_0=2,\quad V_1=p\,. \end{cases} \tag{1.1}
$$

The Binet forms are $U_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$ and $V_n = \alpha^n+\beta^n$ where $\alpha = \frac{p+\sqrt{p^2+4}}{2}$ and $\beta = \frac{p-\sqrt{p^2+4}}{2}$. We see that $\alpha\beta=-1$, $\alpha>1$ and $-1<\beta<0$.

André-Jeannin [1] proved the following theorem.

Theorem 1. *If k is an odd positive integer, then*

$$
\sum_{n=1}^{\infty} \frac{1}{U_{kn}\,U_{k(n+1)}} = \frac{2(\alpha-\beta)}{U_k}\Big[L(\beta^{2k}) - 2L(\beta^{4k}) + 2L(\beta^{8k})\Big] + \frac{\beta^k}{U_k^2}\;; \tag{1.2}
$$

$$
\sum_{n=1}^{\infty} \frac{1}{V_{kn}\,V_{k(n+1)}} = \frac{2}{(\alpha-\beta)\,U_k}\Big[L(\beta^{2k}) - 2L(\beta^{8k})\Big] + \frac{\beta^k}{(\alpha-\beta)\,U_k\,V_k}\;. \tag{1.3}
$$

Here $L(x)$ is the Lambert series defined by $L(x)=\sum_{n=1}^{\infty}\frac{x^n}{1-x^n}$, $|x|<1$. ∎

Our aim in this paper is to generalize Theorem 1.

Remark. Originally, in Theorem 1, k was taken to be an odd integer. The problem with this is that, since $p>0$ and $-1<\beta<0$, a negative k would mean $\beta^{2k}>1$, so that $L(\beta^{2k})$ is undefined. We have stated the theorem with what we believe are the correct constraints on k. □

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2 – Generalization of Theorem 1

We require two lemmas, the first of which appears as Lemma 3′ in [1].

Lemma 1. *If k is an odd positive integer then*

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^{kn}\, U_{kn}} = (\alpha-\beta)\Big[L(\beta^{2k}) - 2L(\beta^{4k}) + 2L(\beta^{8k})\Big]\ ; \tag{2.1}$$

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^{kn}\, V_{kn}} = L(\beta^{2k}) - 2L(\beta^{8k})\ . \ \blacksquare \tag{2.2}$$

Lemma 2. *If k and m are odd integers then*

$$\alpha^{km}\, U_{k(n+m)} + U_{kn} = \alpha^{k(n+m)}\, U_{km}\ ; \tag{2.3}$$

$$\alpha^{km}\, V_{k(n+m)} + V_{kn} = (\alpha-\beta)\, \alpha^{k(n+m)}\, U_{km}\ . \tag{2.4}$$

Proof: We prove only (2.4) since the proof of (2.3) is similar.

$$\begin{aligned}
\alpha^{km}\, V_{k(n+m)} + V_{kn} &= \alpha^{km}(\alpha^{kn+km} + \beta^{kn+km}) + \alpha^{kn} + \beta^{kn} \\
&= \alpha^{kn+2km} - \beta^{kn} + \alpha^{kn} + \beta^{kn} \\
&= \alpha^{kn+2km} + \alpha^{kn} \\
&= \alpha^{kn+km}(\alpha^{km} - \beta^{km}) \\
&= (\alpha-\beta)\, \alpha^{k(n+m)}\, U_{km}\ . \ \blacksquare
\end{aligned}$$

We can now state our generalization of Theorem 1.

Theorem 2. *If k and m are odd positive integers then*

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{U_{kn}\, U_{k(n+m)}} &= \frac{2\,(\alpha-\beta)}{U_{km}} \Big[L(\beta^{2k}) - 2L(\beta^{4k}) + 2L(\beta^{8k})\Big] \\
&\quad - \frac{1}{U_{km}} \sum_{n=1}^{m} \frac{1}{\alpha^{kn}\, U_{kn}}\ ;
\end{aligned} \tag{2.5}$$

$$\sum_{n=1}^{\infty} \frac{1}{V_{kn}\, V_{k(n+m)}} = \frac{1}{(\alpha-\beta)\, U_{km}} \left[2L(\beta^{2k}) - 4L(\beta^{8k}) - \sum_{n=1}^{m} \frac{1}{\alpha^{kn}\, V_{kn}}\right]\ . \tag{2.6}$$

Proof: We first prove (2.5).

$$\frac{1}{\alpha^{kn} U_{kn}} + \frac{1}{\alpha^{k(n+m)} U_{k(n+m)}} = \frac{\alpha^{km} U_{k(n+m)} + U_{kn}}{\alpha^{k(n+m)} U_{kn} U_{k(n+m)}}$$
$$= \frac{\alpha^{k(n+m)} U_{km}}{\alpha^{k(n+m)} U_{kn} U_{k(n+m)}} \quad \text{(by (2.3))}$$
$$= \frac{U_{km}}{U_{kn} U_{k(n+m)}} \ .$$

From this we see that

$$\text{(2.7)} \qquad 2 \sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} U_{kn}} = U_{km} \sum_{n=1}^{\infty} \frac{1}{U_{kn} U_{k(n+m)}} + \sum_{n=1}^{m} \frac{1}{\alpha^{kn} U_{kn}} \ ,$$

and (2.5) follows from (2.1).

Proceeding in the same manner, we first use (2.4) to show that

$$\frac{1}{\alpha^{kn} V_{kn}} + \frac{1}{\alpha^{k(n+m)} V_{k(n+m)}} = \frac{(\alpha-\beta) U_{km}}{V_{kn} V_{k(n+m)}} \ .$$

As before, we sum both sides to obtain

$$\text{(2.8)} \qquad 2 \sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} V_{kn}} = (\alpha-\beta) U_{km} \sum_{n=1}^{\infty} \frac{1}{V_{kn} V_{k(n+m)}} + \sum_{n=1}^{m} \frac{1}{\alpha^{kn} V_{kn}} \ ,$$

and (2.6) follows from (2.2). ∎

When $m=1$ (2.5) and (2.6) reduce to (1.2) and (1.3) respectively.

REFERENCES

[1] André-Jeannin – Lambert series and the summation of reciprocals in certain Fibonacci–Lucas-type sequences, *The Fibonacci Quarterly,* 28(3) (1990), 223–226.

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