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K_W DOES NOT IMPLY K_W^*

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Abstract: We prove that the cyclic monotonically normal space T of M.E. Rudin is a K_W-space which is not a K_W^*-space. This answers a question in [3]. In order to do this, we first prove that if a space X has $D^*(\mathbb{R};\leq)$ then X is a K_W-space (it is well known that X is also a K_1-space; this does not necessarily mean that X is a K_{1W}-space.).

Theorem 1. *If a space (X,τ) has the property $D^*(\mathbb{R};\leq)$ then X is a K_W-space.*

Proof: By Theorem 10 of [4] (i.e. $D^*(\mathbb{R};\leq)$ if and only if $D^*(\mathbb{R};\leq;cch)$), let $\gamma\colon C^*(F)\to C^*(X)$ be a monotone extender such that $\gamma(a_F)=a_X$ for each $a\in\mathbb{R}$. Then for each $U\in\tau|F$, let

$$\mu(U) = \bigcup\Big\{\gamma(f)^{-1}(]-\infty,1[) \mid f\in C(F,[-2,2]),\ \ f(F-U)\subset\{2\}\Big\} ,$$

$$v(U) = \bigcup\Big\{\gamma(f)^{-1}(]-1,\infty[) \mid f\in C(F,[-2,2]),\ \ f(F-U)\subset\{-2\}\Big\} ,$$

$$k(U) = \mu(U)\cup v(U) .$$

If $U\in\tau|F$ and $z\in U$, then there exists $f\in C(F,[-2,2])$ such that $f(z)=-2$ and $f(F-U)\subset\{2\}$. Since γ is an extender, we get that $F\cap\mu(U)=U$; similarly, $F\cap v(U)=U$. Hence, $F\cap k(U)=U$, for each $U\in\tau|F$. Clearly, $k(F)=X$ and $k(\emptyset)=\emptyset$, because $\gamma(\pm 2_F)=\pm 2_X$.

It is obvious that $k(U)\subset k(V)$ whenever $U\subset V$.

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Next, we prove that if $U \cup V = F$ then $k(U) \cup k(V) = X$ ($\underline{W \log}$, let us assume that $U \neq F \neq V$). Let $x \in X$ and suppose that $x \notin \mu(U)$. Then, for each $f \in C(F,[-2,2])$ such that $f(F-U) = 2$, we get that $\gamma(f)(x) \geq 1$. Pick $h \in C(F,[-2,2])$ such that $h(F-V) = -2$ and $h(F-U) = 2$ (recall that F is normal). It follows that $\gamma(h)(x) \geq 1$, which implies that $x \in v(V)$. Similarly, if $x \notin v(V)$ then $x \in \mu(U)$. Consequently, we get that $x \in k(U) \cup k(V)$, as required.

Finally, we prove that, for each $U \in \tau|F$, $\overline{k(U)} \cap F = \overline{U}$: Suppose there is $p \in F$ such that $p \in \overline{k(U)}$ and $p \notin \overline{U}$. Pick $h\colon F \to [-2,2]$ such that $h(\overline{U}) = -2$ and $h(p) = 2$. Then $h \leq f$ for all $f : F \to [-2,2]$ such that $f(F-U) = 2$, which implies that $\mu(U) \subset \gamma(h)^{-1}(]-\infty,1[)$. Since $\gamma(h)^{-1}(]-\infty,1[) \cap \gamma(h)^{-1}(]1,\infty[) = \emptyset$ and $p \in \gamma(h)^{-1}(]1,\infty[)$, we get that $p \notin \overline{\mu(U)}$. Similarly, $p \notin \overline{v(U)}$, and this proves that $p \notin \overline{k(U)}$. Therefore, $\overline{k\,U} \cap F = \overline{U}$. This completes the proof. ∎

Theorem 2. *If a space (X,τ) is a K_W^*-space then, for each closed subspace F of X there exists a function $k\colon \tau|F \to \tau$ such that*

- (**i**) $F \cap k(U) = U$, *for each* $U \in \tau|F$, $k(F) = (X)$, $k(\emptyset) = \emptyset$;
- (**ii**) $k(U) \subset k(V)$ *whenever* $U \subset V$;
- (**iv**) $U, V \in \tau|F$, $\overline{U \cap V} = \overline{U} \cap \overline{V}$ *implies* $k(U \cap V) = k(U) \cap k(V)$;
- (**iv**) $\overline{k(U)} \cap F = \overline{U}$.

Proof: Let $v\colon \tau|F \to \tau$ be a K_W^*-function and define $k\colon \tau|F \to \tau$ by

$$k(U) \;=\; U \cup \Big(X - \Big[F \cup \overline{v(F-\overline{U})}\Big]\Big)\,.$$

From the proof of Theorem 4 of [3], we immediately get that k satisfies (i), (ii) and (iv). To verify (iii), note that

$$\begin{aligned}
k(U) \cap k(V) \;&=\; \Big[U \cup \Big(X - \Big[F \cup \overline{v(F-\overline{U})}\Big]\Big)\Big] \cap \Big[V \cup \Big(X - \Big[F \cup \overline{v(F-\overline{V})}\Big]\Big)\Big] \\
&=\; (U \cap V) \cup \Big[\Big(X - \Big[F \cup \overline{v(F-\overline{U})}\Big]\Big) \cap \Big(X - \Big[F \cup \overline{v(F-\overline{V})}\Big]\Big)\Big] \\
&\qquad \Big(\text{because } U \cap \Big(X - \Big[F \cup \overline{v(F-\overline{V})}\Big]\Big) = \emptyset = \\
&\qquad\qquad = \Big(X - \Big[F \cup \overline{v(F-\overline{U})}\Big]\Big) \cap V, \text{ since } U \subset F \text{ and } V \subset F\Big) \\
&=\; (U \cap V) \cup \Big(X - \Big[F \cup \overline{v(F-\overline{U})} \cup \overline{v(F-\overline{V})}\Big]\Big) \;=
\end{aligned}$$

$$= (U \cap V) \cup \left(X - \left[F \cup \overline{v(F - \overline{U}) \cup v(F - \overline{V})}\,\right]\right)$$

$$= (U \cap V) \cup \left(X - \left[F \cup \overline{v\Big((F - \overline{U}) \cup (F - \overline{V})\Big)}\,\right]\right)$$

(because v is a K_W^*-function)

$$= (U \cap V) \cup \left(X - \left[F \cup \overline{v(F - \overline{U} \cap \overline{V})}\,\right]\right)$$

$$= (U \cap V) \cup \left(X - \left[F \cup \overline{v(F - \overline{U \cap V})}\,\right]\right)$$

$$= k(U \cap V)\,, \qquad \text{because } \overline{U \cap V} = \overline{U} \cap \overline{V}\ ,$$

which completes the proof. ∎

We conjecture that the converse of Theorem 2 is false and we have not been able to find a characterization of K_W^*-spaces analogous to the characterization of K_W-spaces which appears in Theorem 4 of [3].

Theorem 3. *There is a K_W-space T which is not a K_W^*-space.*

Proof: The space T is the space described by M.E. Rudin in [6]. We already know from Theorem 1 (recall that monotonically normal spaces have $D^*(\mathbb{R}; \leq)$) that T is a K_W-space.

Assuming that T is a K_W^*-space, let $k : \tau|F \to \tau$ satisfy the conditions of Theorem 3(b). Since the sets U_{xi} and U_{rxi} defined on p. 305 of [6] are easily seen to be clopen, then we get that

$$\overline{\bigcap_{i<3} U_{xi}} = \bigcap_{i<3} \overline{U_{xi}} \quad \text{and} \quad k\Big(\bigcap_{i<3} U_{xi} \cap Y\Big) = \bigcap_{i<3} k(U_{xi} \cap Y)$$

and, similarly,

$$k\Big(U_{rxij} \cap U_{rxi(j-1)} \cap Y\Big) \;=\; k(U_{rxij} \cap Y) \cap k(U_{rxi(j-1)} \cap Y)\ .$$

Consequently, M.E. Rudin's argument, verbatim, also proves that the above k cannot exist, a contradiction. This completes the proof. ∎

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