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ON NONSYMMETRIC TWO-DIMENSIONAL VISCOUS FLOW THROUGH AN APERTURE

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Abstract: We consider a stationary free boundary problem for the Navier–Stokes equations governing effluence of a viscous incompressible liquid out of unbounded non-expanding at infinity, in general, non-symmetric strip-like domain Ω_- outside which the liquid forms a sector-like jet with free (unknown) boundary and with the limiting opening angle $\theta \in (0, \pi/2)$. Conditions at the free boundary take account of the capillary forces but external forces are absent. The total flux of the liquid through arbitrary cross-section of Ω_- is prescribed and assumed to be small. Under this condition, we prove the existence of an isolated solution of the problem which is found in a certain weighted Hölder space of functions.

1 – Introduction

In the paper [1] there was considered a symmetric viscous flow through an aperture in an infinite straight line. In the present paper we remove the condition of symmetry. We assume that the domain $\Omega \subset R^2$ filled with the liquid consists of two parts, Ω_- and Ω_+, where Ω_- is an infinite domain bounded by two semi-infinite curves, Σ_+ and Σ_-, with endpoints $x_\pm = (\pm d_0, 0)$, and by an aperture $S = \{|x_1| < d_0,\ x_2 = 0\}$, whereas $\Omega_+ \subset R^2_+ = \{x_2 > 0\}$ is a domain bounded by S and by a free surface Γ which consists of two infinite curves, Γ_- and Γ_+, given by the equations

$$x_1 = h_\pm(x_2)\,, \qquad x_2 > 0\ , \tag{1.1}$$

with the functions $h_\pm(x_2)$ satisfying the condition $h_\pm(0) = \pm d_0$, $d_0 > 0$. The problem consists in the determination of Γ_- and Γ_+, i.e., of the functions $h_\pm(x_2)$,

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and of the velocity vector field $\vec{v}(x) = (v_1, v_2)$ and the pressure $p(x)$ satisfying in Ω the Navier–Stokes equations

$$(1.2)\qquad -\nu\,\nabla^2\vec{v} + (\vec{v}\cdot\nabla)\,\vec{v} + \nabla p \;=\; 0\,, \qquad \nabla\cdot\vec{v} = 0\ ,$$

no-slip conditions on Σ and kinematic and dynamic boundary conditions on Γ with the account of capillary forces:

$$(1.3)\qquad \vec{v}|_\Sigma = 0\,, \qquad \vec{v}\cdot\vec{n}|_\Gamma = 0\,, \qquad \vec{\tau}\cdot S(\vec{v})\,\vec{n}|_\Gamma = 0\ ,$$

$$(1.4)\qquad \vec{n}\cdot T(\vec{v},p)\,\vec{n} - \sigma H|_\Gamma = 0\ .$$

In addition, we prescribe conditions at infinity

$$(1.5)\qquad \begin{aligned} &\vec{v}(x)\to 0,\ \ p(x)\to 0\,, \qquad (|x|\to\infty,\ x\in\Omega_+)\ ,\\ &|\vec{v}(x)| + |\nabla p(x)| < \infty\,, \qquad (|x|\to\infty,\ x\in\Omega_-)\ , \end{aligned}$$

the total flux through the aperture S and conditions at the contact points

$$(1.6)\qquad \int_{-d_0}^{d_0} v_2(x_1,0)\,dx_1 = F\ ,$$

$$(1.7)\qquad h'_-(0) = k_-\,, \qquad h'_+(0) = k_+\ .$$

Here ν and σ are positive constants: coefficient of viscosity and of the surface tension, respectively, $\vec{n}$ is a unit exterior normal to Γ, $\vec{\tau}$ is a unit tangential vector to Γ such that $\tau_1 = -n_2$, $\tau_2 = n_1$, $(x\in\Gamma_+)$ and $\tau_1 = n_2$, $\tau_2 = -n_1$, $(x\in\Gamma_-)$, $S(\vec{v}) = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)_{i,j=1,2}$ is the doubled rate-of-strain tensor, $T(p,\vec{v}) = -p\,I + \nu\,S(\vec{v})$ is the stress tensor, I is the unit matrix, and H is the curvature of Γ which is negative at the points of convexity of Γ towards the exterior of Ω.

We assume that $k_+ > k_-$ and that k_- and k_+ coincide with the values of the derivatives of the functions $\hat{h}_\pm(x_2)$ which determine the curves $\Sigma_\pm$ near contact points $x_\pm$ by equations $x_1 = \hat{h}_\pm(x_2)$: $\hat{h}'_\pm(0) = k_\pm$. Hence, the lines $\Gamma_\pm$ are tangential to $\Sigma_\pm$ at the points $x_\pm$, and the contact angle, i.e. the angle between $\Sigma_\pm$ and $\Gamma_\pm$ at $x_\pm$, is equal to π. This assumption seems to be more natural than the prescription of arbitrary contact angles at x_+ and x_-, although this is also possible and this leads only to some inessential technical complications. In similar problems for inviscid fluid (see [2]) this assumption guarantees the continuity of the velocity vector field at the contact points, but in the case of viscous fluid the velocity vanishes at $x_\pm$ and it is continuous for arbitrary value of the contact angle.

We assume also that the curves Σ_+ and Σ_- are smooth and that the function $\mathrm{dist}(x, \Sigma_-)$, $x \in \Sigma_+$, takes values between two positive constants. Further, we assume that an infinite domain Ω_- can be cut into an infinite number of bounded subdomains ω_k by smooth curves S_k, $k = 0, 1, .., S_0 = S$, joining Σ_- and Σ_+ in such a way that

(**i**) the boundary of ω_k consists of S_k, S_{k+1} and of the finite curves $\Sigma_k^- \subset \Sigma_-$ and $\Sigma_k^+ \subset \Sigma_+$, so every ω_k, $k > 0$, has a common boundary only with ω_{k-1} and ω_{k+1}, and ω_0 has a common boundary with Ω_+ and ω_1; $\Omega_m^{(-)} = \bigcup_{j=0}^m \omega_j$ are bounded domains having common boundary with Ω_+ and ω_{m+1} and exhausting Ω_- as $m \to \infty$.

(**ii**) every ω_j can be mapped onto a square $0 \le y_j \le 1$, $i = 1, 2$, by C^{l+2}-smooth mapping $y = Y_j(x)$; C^{l+1}-norms of the Jacobi matrices $J = \left(\frac{\partial y_i}{\partial x_k}\right)_{i,k=1,2}$ are uniformly bounded (l is an arbitrary fixed positive non-integral number).

It is easy to show that there exist smooth functions $\zeta_m(x)$, $x \in \Omega$, such that $\zeta_m(x) = 1$ for $x \in \Omega_m = \Omega_+ \cup \Omega_m^{(-)}$, $\zeta_m = 0$ for $x \in \Omega \backslash \Omega_{m+1}$, $\zeta_m(x) \in (0, 1)$ and $|\nabla \zeta_m| \le c$, c is independent of m.

The domain Ω_- is "strip-like", and it is natural to require that the velocity and the gradient of the pressure should be bounded at infinity (but the pressure can grow without limits), as it is the case for the Poiseuille flow in an infinite strip.

Problem (1.2)–(1.7) will be considered in weighted Hölder spaces whose elements have a specified behavior at infinity (both in Ω_+ and in Ω_-) and near contact points. Let l be a positive non-integral number and $s \in [0, l]$. By $C_s^l(\Omega, b)$ we mean a Banach space of functions given in Ω with a finite norm

$$
\begin{aligned}
|u|_{C_s^l(\Omega,b)} &= \sum_{0 \le |j| < l} \sup_{x \in \Omega} \varrho\Big(x,\, b+|j|,\, |j|-s\Big)\, |D^j u(x)| \\
&\quad + \sup_{x \in \Omega} \varrho\Big(x,\, b+l,\, l-s\Big) [u]^{(l)}_{K(x)} + [u]^{(s)}_{B_{d_0/2}(x_+) \cup B_{d_0/2}(x_-)} \,.
\end{aligned}
\tag{1.8}
$$

Here $D^j = \frac{\partial^{|j|}}{\partial x_1^{j_1} \partial x_2^{j_2}}$, $|j| = j_1 + j_2$, $B_r(x_\pm) = \{x \in \Omega\colon\ |x - x_\pm| < r\}$, $K(x) = \{y \in \Omega\colon\ |y - x| < \frac{1}{2} \varrho(x, 1, 1)\}$, $\varrho(x, b, m)$ is the weight function given by the equation

$$
\varrho(x, b, m) = \begin{cases} |x|^b & , \text{ if } |x| > 4\, d_0,\ x \in \Omega_+\,, \\ |x - x_\pm|^{\max(0,m)} & , \text{ if } |x - x_\pm| < d_0/2\,, \\ 1 & , \text{ if } x_2 < -d_0\ ; \end{cases}
$$

at all the other points $\varrho(x,b,m)$ is a strictly positive function; finally,

$$[u]_G^{(\lambda)} = \sum_{i,j=[\lambda]} \sup_{x,y\in G} |x-y|^{[\lambda]-\lambda} \, |D^j u(x) - D^j u(y)| \,, \qquad \lambda > 0, \text{ non-integral },$$

$$[u]_G^{(\lambda)} = \sum_{i,j=\lambda} \sup_{x,y\in G} |D^j u(x)| \,, \qquad \lambda \text{ a non-negative integer }.$$

It is easily seen that the elements of the space $C_s^l(\Omega,b)$ belong to $C^s(B_{d_0/2}(x_\pm))$ and have a higher regularity, i.e. belong to $C^l(\Omega')$, in every subdomain $\Omega'\subset\Omega$ bounded away from $x_\pm$. As $|x|\to\infty$, $x\in\Omega_+$, they decay like $|x|^{-b}$; on the contrary, in Ω_- they are only bounded and need not decay, as $|x|\to\infty$. The spaces $C_s^l(\Omega,b)$ will also be used in the case when $s<0$; then the norm is given by the same equation (1.8) without the last term.

For the description of properties of the free boundary we use the spaces $C_s^l(R_+,b)$ of functions given on the line $R_+=\{z>0\}$. The norm in $C_s^l(R_+,b)$, $s>0$, is given by

$$|u|_{C_s^l(R_+,b)} = \sum_{j=0}^{[l]} \sup_{R_+} \varrho_1\Big(z,\, b+j,\, j-s\Big) \left|\frac{d^j u(z)}{dz^j}\right| + \sup_{R_+} \varrho_1\Big(z,\, b+l,\, l-s\Big) [u]_{K_1(z)}^{(l)} + [u]_I^{(s)} \tag{1.9}$$

where $I=(0,1)$, $K_1(z)=\{y\in R_+ : |z-y|\le \varrho_1(z,1,1)/2\}$,

$$\varrho_1(z,b,m) = \begin{cases} |z|^b & , \text{ if } z>1, \\ z^{\max(0,m)} & , \text{ if } z\in(0,1)\ . \end{cases}$$

In the case $s<0$ the last term in (1.9) should be omitted.

The main result of the paper is as follows.

Theorem 1. *Assume that $\Sigma_\pm$ belong to the class C^{l+2} and that*

$$0 < \arctan k_+ - \arctan k_- < \pi/2\ . \tag{1.10}$$

If F is sufficiently small, then problem (1.2)–(1.7) has an isolated solution $(h_-,h_+,\vec v,p)$ with the following properties:

$$h_-(x_2) = -d_0 + k_-\,x_2 + \int_0^{x_2} (x_2-t)\, h_-''(t)\,dt\ ,$$

$$h_+(x_2) = +d_0 + k_+\,x_2 + \int_0^{x_2} (x_2-t)\, h_+''(t)\,dt\ ,$$

$$h_\pm'' \in C_{s-1}^{l+1}(R_+,2)\,, \qquad \vec v \in C_s^{l+2}(\Omega,1)\,, \qquad \nabla p \in C_{s-2}^{l}(\Omega,3)\ ,$$

and

$$(1.11)\qquad |\vec{v}|_{C_s^{l+2}(\Omega,1)} + |\nabla p|_{C_{s-2}^{l}(\Omega,3)} + |h''_-|_{C_{s-1}^{l+1}(R_+,2)} + |h''_+|_{C_{s-1}^{l+1}(R_+,2)} \ \le\ c\,|F|$$

Here l is a positive non-integer, $s \in (0,1/2)$.

The proof of Theorem 1.1 is based on the theory of the Stokes and Navier–Stokes equations in domains with noncompact and irregular boundaries. As mentioned above, one could prescribe arbitrary k_+ and k_- (i.e. arbitrary contact angles $\theta_\pm$ at $x_\pm$) satisfying (1.10). In this case one had to work in the Hölder spaces $C^{l+2}_{s_-,s_+}(\Omega,b)$ with different types of singularities at x_+ and x_-, where $s_- < \Re\lambda_-$, $s_+ < \Re\lambda_+$, $\lambda_\pm$ are roots of the equations $\sin 2\theta_\pm\,\lambda_\pm = \lambda_\pm \sin 2\theta_\pm$ with a minimal positive real part different from 1. In the case $\theta_+ = \theta_- = \pi$ we have $\lambda_+ = \lambda_- = 1/2$.

Since $h''_\pm(x_2) = O(|x_2|^{-2})$ for large x_2, the domain Ω_+ has a limiting opening angle $\theta = \arctan h'_+(+\infty) - \arctan h'_-(+\infty)$, although $\Gamma_\pm$ may deviate from the straight lines like $c\log x_2$. For sufficiently small F, $\theta \in (0,\pi/2)$. This condition is important for the analysis of the behavior of the solution for large $|x|$, $x \in \Omega_+$ (see [1], §2). We show that

$$\vec{v}(x) = \frac{F}{\theta}\,\frac{\vec{x}}{|x|^2} + \vec{u}(x), \quad p(x) = -\frac{F^2}{2\,\theta^2\,|x|^2} + q(x)\,, \qquad x \in \Omega_+\ ,$$

and that for large $|x|$ $\vec{u}(x) = O(|x|^{-1-\beta})$, $q(x) = O(|x|^{-2-\beta})$ with a certain $\beta \in (0,1)$. If Ω_- is a semi-infinite strip $\{|x_1| < d_0,\ x_2 < 0\}$ (this case was briefly discussed in [1], §6), then it is possible to clarify the asymptotical behavior of the solution also for $x_2 \to -\infty$, namely, to prove that it tends exponentially to the Poiseuille flow in the strip $\{|x_1| < d_0,\ -\infty < x_2 < +\infty\}$ with the flux F. Under our more general hypotheses concerning Ω_-, this is hardly possible, and we are constrained to seek the solution in a class of vector fields which are only bounded in Ω_- and have there an infinite Dirichlet integral.

The paper is organized as follows. In §2 we consider a linear problem

$$(1.12)\qquad -\nu\,\nabla^2\vec{v} + \nabla p = \vec{f}(x)\ ,$$

$$(1.13)\qquad \nabla\cdot\vec{v} = g(x)\,, \qquad x \in \Omega\ ,$$

$$(1.14)\qquad \vec{v}|_\Sigma = 0\,, \qquad \vec{v}\cdot\vec{n}|_\Gamma = 0\ ,$$

$$(1.15)\qquad \vec{\tau}\cdot S(\vec{v})\,\vec{n}|_\Gamma = 0\ ,$$

$$(1.16)\qquad \vec{v}(x) \to \infty, \quad p(x) \to 0\,, \qquad (|x|\to\infty,\ x \in \Omega_+)\ ,$$

$$(1.17)\qquad |\vec{v}(x)| + |\nabla p(x)| < \infty\,, \qquad (|x|\to\infty,\ x \in \Omega_-)\ ,$$

$$(1.18)\qquad \int_{S_+(r)} \vec{v}\cdot\vec{n}\,dS \,\to\, 0 \qquad (r\to\infty)$$

in a given domain Ω of the type described above; $S_+(r) = \{x \in \Omega_+ :\ |x| = r\}$. We assume that $\operatorname{supp} g \subset \Omega_1$, $\vec{f}(x) = O(|x|^{-3-\beta})$, $g(x) = O(|x|^{-2-\beta})$ for large $|x|$, $x\in\Omega_+$, and that $\vec{f}(x)$ is bounded in Ω_-, and we prove that problem (1.12)–(1.18) possesses a unique generalized solution $(\vec{v}, p)$ such that $D\vec{v}, p \in L_2(\Omega_k)$, for arbitrary $k=0,1,...$, that the Dirichlet integrals of $\vec{v}$ in Ω_k do not exceed ck, and that they are uniformly bounded in every ω_k. In §3 we show that a generalized solution is classical and that $\vec{v} \in C_s^{l+2}(\Omega, 1+\beta)$, $\nabla p \in C_{s-2}^{l}(\Omega, 3+\beta)$; in addition, we consider linear problem with non-homogeneous boundary conditions on Γ:

$$\vec{v}\cdot\vec{n}|_\Gamma = b\,, \qquad \vec{\tau}\cdot S(\vec{v})\,\vec{n}|_\Gamma = d\ .$$

In §4 we study a nonlinear problem

$$-\nu\,\nabla^2\vec{v} + (\vec{v}\cdot\nabla)\,\vec{v} + \nabla p = 0\,, \qquad \nabla\cdot\vec{v} = 0\ ,$$

$$\vec{v}|_\Sigma = 0\,, \qquad \vec{v}\cdot\vec{n}|_\Gamma = 0\,, \qquad \vec{\tau}\cdot S(\vec{v})\,\vec{n}|_\Gamma = 0\ ,$$

$$(1.19)\qquad \int_S v_2\,dx_1 = F\ ,$$

$$\vec{v}(x) \to 0, \quad p(x) \to 0\,, \qquad (|x|\to\infty,\ x \in \Omega_+)\ ,$$

$$|\vec{v}(x)| + |\nabla p(x)| < \infty\,, \qquad (|x|\to\infty,\ x \in \Omega_-)\ ,$$

in the same given Ω and we prove that in the case of small F it has a unique small solution $(\vec{v}, p)$ satisfying the inequality

$$(1.20)\qquad |\vec{v}(x)|_{C_s^{l+2}(\Omega,1)} + |\nabla p(x)|_{C_{s-2}^{l}(\Omega,3)} \,\le\, c\,|F|\ ;$$

moreover, we control the variation of the solution under the variation of Γ. Finally, in §5 we study equation (1.4) for the free boundary, i.e., for the functions $h_\pm(x_2)$, and we prove the solvability of the free boundary problem (1.2)–(1.8) by using the contraction mapping principle.

2 – Auxiliary linear problem (a weak solution)

In this section we consider a linear problem (1.12)–(1.18) in a given domain Ω of the type described in §1 and we prove that this problem possesses a unique weak (generalized) solution. By a weak solution we mean a couple $(\vec{v}(x), p(x))$, $x \in \Omega$, with the following properties:

- **(i)** $\vec{v}$ possesses the first generalized (in the sense of S.L. Sobolev) derivatives $\frac{\partial \vec{v}}{\partial x_i}$, and $\frac{\partial \vec{v}}{\partial x_i}, p \in L_2(\Omega_k)$, in every Ω_k, $k = 0, 1, \ldots$.
- **(ii)** $\vec{v}$ satisfies equation (1.13), conditions (1.14) and (1.18),
- **(iii)** $\vec{v}, p$ satisfy the integral identity

$$\frac{\nu}{2} \int_\Omega S(\vec{v}) : S(\vec{\varphi})\, dx - \int_\Omega p\, \nabla \cdot \vec{\varphi}\, dx = \int_\Omega (\vec{f} \cdot \vec{\varphi} + \nu\, g\, \nabla \cdot \vec{\varphi})\, dx \tag{2.1}$$

for arbitrary $\vec{\varphi}$, with $\operatorname{supp} \vec{\varphi} \subset \Omega_k$, $k = 0, 1, \ldots$ also satisfying boundary conditions (1.14).

Clearly, this identity substitutes the equations (1.12) and boundary condition (1.15).

We prove the following theorem.

Theorem 2. *Assume that $h_\pm(x_2)$ have continuous first derivatives which are sufficiently close to the constants $k_\pm$, that $\vec{f}, g$ have finite norms*

$$\|\vec{f}\|_2 = \sup_{x \in \Omega} \varrho(x, 3+\beta, 2-s)\, |\vec{f}(x)| ,$$

$$\|g\|_1 = \sup_{x \in \Omega} \rho(x, 2+\beta, 1-s)\, |g(x)| , \qquad \beta \in (0, 1) ,$$

(ϱ is the weight function defined in §1), and that $g(x) = 0$ in $\Omega \backslash \Omega_1$. Then problem (1.12)–(1.18) has a unique weak solution and this solution satisfies the estimates

$$\int_{\Omega_k} |D\vec{v}|^2\, dx \le c_1\, k \Big(\|\vec{f}\|_2 + \|g\|_1\Big)^2 , \qquad k = 0, 1, \ldots , \tag{2.2}$$

$$\int_{\Omega_k} |p(x)|^2\, dx \le c_2(k) \Big(\|\vec{f}\|_2 + \|g\|_1\Big)^2 , \qquad k = 0, 1, \ldots , \tag{2.3}$$

$$\int_{\omega_k} |D\vec{v}|^2\, dx \le c_3 \Big(\|\vec{f}\|_2 + \|g\|_1\Big)^2 , \qquad k = 1, 2, \ldots , \tag{2.4}$$

where c_1 and c_3 are independent of k.

Remark. The theorem holds under weaker assumptions concerning $\Gamma_{\pm}$, namely, it is enough to assume that for a certain $M > 0$ the domain $\Omega^{(M)} = \{x \in \Omega_+ : x_2 > M\}$ is a special Lipschitz domain, i.e., it can be defined by the equation $z_2 > F(z_1)$, $z_1 \in R$ in a certain Cartesian coordinate system (z_1, z_2), and the function F satisfies the Lipschitz condition

$$|F(z_1) - F(z_1')| \leq c\,|z_1 - z_1'| \,.$$

It is easily seen that under the hypotheses of the theorem Ω_+ is a special Lipschitz domain in the coordinate system with z_2-axis directed along the line $x_1 = (k_+ + k_-)x_2/2,\ x_2 > 0.$ □

In the proof of Theorem 2, we make use of the following auxiliary propositions.

Proposition 1. *Consider the following problem: find a vector field $\vec{u}(x)$, $x \in \omega_m$, with a bounded Dirichlet integral in ω_m, satisfying the relations*

$$\nabla \cdot \vec{u}(x) = f(x)\,, \qquad x \in \omega_m, \quad \vec{u}(x)|_{\partial\omega_m} = 0\,, \tag{2.5}$$

where f is a given function. For arbitrary $f \in L_2(\omega_m)$ satisfying the condition

$$\int_{\omega_m} f(x)\,dx \;=\; 0 \tag{2.6}$$

there exits a vector field satisfying equations (2.5) and the inequality

$$\|D\vec{u}\|_{L_2(\omega_m)} \;\leq\; c\,\|f\|_{L_2(\omega_m)} \tag{2.7}$$

with the constant independent of m.

The same problem is solvable in Ω_k, $k = 0, 1, ...$, for arbitrary $f \in L_2(\Omega_k)$, without additional condition of the type (2.6) (but the constant in the estimate (2.7) tends to infinity as $k \to \infty$). ■

The first statement of the proposition follows from Lemma 2.4 in [3]; the constants in (2.7) are independent of m because arbitrary ω_m may be mapped onto the unit square $0 \leq y_i \leq 1$, $i = 1, 2$, and the Jacobi matrices of the corresponding transformations are uniformly bounded (in this connection see, for instance, [4], §2, in particular, Lemma 2.2). The second statement follows from the fact that Ω_k is a sum of a finite number of domains ω_j for which problem (2.5),(2.6) is solvable, and of the Lipschitz domain Ω_+ for which this problem is solvable without condition (2.6).

Proposition 2. *For arbitrary vector field $\vec{v}(x)$, $x \in \Omega_m$, which has a bounded Dirichlet integral in Ω_j and satisfies the boundary conditions (1.14) there holds the Korn inequality*

$$\|D\vec{v}\|_{L_2(\Omega_m)} \le c\,\|S(\vec{v})\|_{L_2(\Omega_m)} \tag{2.8}$$

with the constant independent of m. ∎

This result follows from the fact that the Korn inequality holds in every ω_m (which can be proved exactly as in [5], Proposition 2.3) and in the Lipschitz domain Ω_+.

Proof of Theorem 2: First of all, we reduce problem (1.12)–(1.18) to an analogous problem with $g(x) = 0$ by construction of an auxiliary vector field $\vec{V}(x)$ satisfying the conditions

$$\nabla\cdot\vec{V} = g\,, \qquad \lim_{r\to\infty}\int_{S_+(r)} \vec{V}\cdot\vec{n}\,dS = 0$$

and vanishing at $\partial\Omega = \Sigma\cup\Gamma$. We define $\vec{V}(x)$ as the sum $\vec{V}(x) = \vec{V}_1(x) + \vec{V}_2(x)$ where

$$\vec{V}_1(x) = -G\,\xi_-(x)\,\vec{A}(x)\ ,$$

$G = \int_{\Omega_1} g(x)\,dx$, $\xi_-(x)$ is a smooth function given in Ω which is equal to 1 in $\Omega\backslash\Omega_1$ and to zero in Ω_+; $\vec{A}(x)$, $x\in\Omega_-$, is a smooth solenoidal vector field such that $\int_{S'} \vec{A}\cdot\vec{n}\,dS = 1$ for arbitrary cross-section S' of Ω_-. We set

$$\vec{A}(x) = \Big(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\Big)$$

where $\psi(x)$ is a smooth bounded function defined in Ω_- and equal to 1 in the neighbourhood of Σ_+ and to zero in the neighbourhood of Σ_-. Clearly,

$$g_1(x) \equiv \nabla\cdot\vec{V}_1(x) = -G\,\nabla\xi_-(x)\cdot\vec{A}(x)$$

is a smooth function with $\operatorname{supp} g_1 \subset \omega_0$ and with

$$\int_{\omega_0} g_1(x)\,dx = G\int_{\omega_0}\nabla(1-\xi_-(x))\cdot\vec{A}(x)\,dx = G\int_S A_2\,dx_1 = G\ .$$

We extend $g_1(x)$ by zero into Ω_+ and set

$$g_2(x) = g(x) - g_1(x)\ .$$

We have

$$\|g_2\|_1 \le \|g\|_1 + c\,|G| \le c\,\|g\|_1 \ ,$$

and, as a consequence,

$$\|g_2\|_{L_2(\Omega)} \le c\,\|g\|_1 \ ;$$

in addition,

$$\int_{\Omega_1} g_2\,dx = \int_{\Omega_1} g\,dx - G = 0 \ .$$

By Proposition 1, there exists the vector field $\vec{V}_2(x)$ such that

$$\nabla\cdot \vec{V}_2(x) = g_2(x)\,, \qquad x \in \Omega_1, \quad \vec{V}_2|_{\partial\Omega_1} = 0 \ ,$$

and

$$\|D\vec{V}_2\|_{L_2(\Omega_1)} \le c\,\|g_1\|_{L_2(\Omega)} \le c\,\|g\|_1 \ .$$

We extend it by zero into $\Omega\backslash\Omega_1$ and set $\vec{V} = \vec{V}_1 + \vec{V}_2$. Clearly, $D\vec{V} \in L_2(\Omega)$,

$$\|D\vec{V}\|_{L_2(\Omega)} \le c\,\|g\|_2 \tag{2.9}$$

and

$$\int_{S_+(r)} \vec{V}\cdot\vec{n}\,dS = \int_{S_+(r)} \vec{V}_2\cdot\vec{n}\,dS = -\int_{\Omega^{(r)}} g_2\,dx \longrightarrow 0 \qquad (r\to\infty)$$

where $\Omega^{(r)} = \{x\in\Omega_+ :\ |x| > r\}$. A new unknown vector field $\vec{u} = \vec{v} - \vec{V}$ should be divergence free: $\nabla\cdot\vec{u} = 0$, satisfy boundary conditions (1.14), condition (1.18), have a finite Dirichlet integral in every Ω_k and satisfy the integral identity

$$\frac{\nu}{2}\int_\Omega S(\vec{u}):S(\vec{\eta})\,dx = \int_\Omega \vec{f}\cdot\vec{\eta}\,dx - \frac{\nu}{2}\int_\Omega S(\vec{V}):S(\vec{\eta})\,dx \tag{2.10}$$

with an arbitrary solenoidal $\vec{\eta}$ having a bounded Dirichlet integral, vanishing outside a certain Ω_j and also satisfying (1.14).

To prove the existence of such $\vec{u}(x)$, we fix arbitrary $m > 0$, introduce the space $J(\Omega_m)$ of solenoidal vector fields with $\|D\vec{\eta}\|_{L_2(\Omega_m)} < \infty$ satisfying the boundary conditions

$$\vec{\eta}\cdot\vec{n}|_\Gamma = 0\,, \qquad \vec{\eta}|_{\partial\Omega_m\backslash\Gamma} = 0$$

and consider the auxiliary problem of the determination of $\vec{u}_m \in J(\Omega_m)$ such that

$$\frac{\nu}{2}\int_{\Omega_m} S(\vec{u}_m):S(\vec{\eta})\,dx = \int_{\Omega_m} \vec{f}\cdot\vec{\eta}\,dx - \frac{\nu}{2}\int_{\Omega_m} S(\vec{V}):S(\vec{\eta})\,dx\,, \quad \forall\,\vec{\eta}\in J(\Omega_m)\,. \tag{2.11}$$

The existence of $\vec{u}_m$ can be proved in a standard way. Due to the Korn inequality (2.8) which holds for arbitrary $\vec{\eta} \in J(\Omega_m)$, the bilinear form in the left-hand side of (2.11) can be considered as a new scalar product in $J(\Omega_m)$, whereas the right-hand side is a linear continuous functional in $J(\Omega_m)$. Indeed, for arbitrary $\vec{\eta} \in J(\Omega_m)$ we have

$$\begin{aligned}\left|\int_{\Omega_m} \vec{f}\cdot\vec{\eta}\,dx\right| &\le \|\vec{f}\|_1 \int_{\Omega_m} \varrho^{-1}(x,\,3+\beta,\,2-s)\,|\vec{\eta}(x)|\,dx \\ &\le \|\vec{f}\|_1 \Bigg(\int_{|x-x_+|<d_0/2} |x-x_+|^{s-2}\,|\vec{\eta}(x)|\,dx \\ &\quad + \int_{|x-x_-|<d_0/2} |x-x_-|^{s-2}\,|\vec{\eta}(x)|\,dx \\ &\quad + \int_{|x|>2d_0,x_2>0} |x|^{-(3+\beta)}\,|\vec{\eta}(x)|\,dx \,+\, c\int_{\Omega'_m} |\vec{\eta}(x)|\,dx\Bigg)\end{aligned}$$

where $\Omega'_m = \Omega_m^{(-)} \cup \{x \in \Omega_+ \colon |x| < 2d_0\}$, and $c = \max(1, \sup_{\Omega'_m} \varrho)$. Clearly,

$$\int_{\Omega'_m} |\vec{\eta}(x)|\,dx \le |\Omega'_m|^{1/2}\left(\int_{\Omega'_m} |\vec{\eta}(x)\,dx\right)^{1/2} \le c\sqrt{m}\,\|D\vec{\eta}\|_{L_2(\Omega'_+)}\ ,$$

by virtue of the Friedrichs inequality.

Evaluating other integrals in the right-hand side with the aid of the Hardy inequality we prove that it is less than

$$c\sqrt{m}\,\|\vec{f}\|_2\,\|D\vec{\eta}\|_{L_2(\Omega_m)} \le c\sqrt{m}\,\|\vec{f}\|_2\,\|S(\vec{\eta})\|_{L_2(\Omega_m)}\ .$$

In addition, we have, by (2.9),

$$\left|\int_{\Omega_m} S(\vec{V}) : S(\vec{\eta})\,dx\right| \le \|S(\vec{V})\|_{L_2(\Omega_m)}\,\|S(\vec{\eta})\|_{L_2(\Omega_m)} \le c\,\|g\|_1\,\|S(\vec{\eta})\|_{L_2(\Omega_m)}\ ,$$

hence,

$$\left|\int_{\Omega_m} \vec{f}\,\vec{\eta}\,dx - \frac{\nu}{2}\int_{\Omega_m} S(\vec{V}) : S(\vec{\eta})\,dx\right| \le c\sqrt{m}\left(\|\vec{f}\|_2 + \|g\|_1\right)\|S(\vec{\eta})\|_{L_2(\Omega_m)}\ , \tag{2.12}$$

and the existence of $\vec{u}_m$ follows from the Riesz representation theorem. Setting $\vec{\eta} = \vec{u}_m$ in (2.11) we get

$$\|D\vec{u}_m\|_{L_2(\Omega_m)} \le \|S(\vec{u}_m)\|_{L_2(\Omega_m)} \le c\sqrt{m}\left(\|\vec{f}\|_2 + \|g\|_1\right)\ . \tag{2.13}$$

It is also easily seen that

$$\int_{S_+(r)} \vec{u}_m \cdot \vec{n}\, dS \;=\; \int_{S_{m+1}} \vec{u}_m \cdot \vec{n}\, dS \;=\; 0$$

for arbitrary large $r > 0$.

Our next step is the prove of the estimate

$$\|D\vec{u}_m\|_{L_2(\Omega_k)} \;\le\; c\sqrt{k}\left(\|\vec{f}\|_2 + \|g\|_1\right) . \tag{2.14}$$

for arbitrary $k < m$. We reproduce the proof from [4], Theorem 3.1. We choose the test function $\vec{\eta}(x)$ in (2.11) in a special way, namely, we require that $\vec{\eta}(x)$ should be equal to $\vec{u}_m(x)$ in Ω_k, to zero in $\Omega_m\backslash\Omega_{k+1}$ and that it should be solenoidal everywhere in Ω_m. We set

$$\vec{\eta}(x) \;=\; \zeta_k(x)\,\vec{u}_m(x) + \vec{U}_{km}(x)\,, \qquad x \in \omega_{k+1}\ ,$$

where $\zeta_k(x)$ is a smooth function which is equal to 1 in Ω_k, to zero in $\Omega\backslash\Omega_{k+1}$, $|\zeta_k(x)| + |\nabla\zeta_k(x)| \le c$ (see §1), and $\vec{U}_{km}(x)$ is a solution of the problem of type (2.5) in ω_{k+1}, namely

$$\nabla\cdot\vec{U}_{km}(x) = -\nabla\zeta_k(x)\cdot\vec{u}_m(x)\,, \qquad x \in \omega_{k+1}, \quad \vec{U}_{km}|_{\partial\omega_{k+1}} = 0\ ,$$

$$\|D\vec{U}_{km}\| \;\le\; c\,\|\nabla\zeta_k(x)\cdot\vec{u}_m(x)\| \;\le\; c\,\|D\vec{u}_m\|_{L_2(\omega_{k+1})}\ . \tag{2.15}$$

Since

$$\int_{\omega_{k+1}} \nabla\zeta_k(x)\cdot\vec{u}_m(x)\,dx \;=\; \int_{S_k} \vec{u}_m(x)\cdot\vec{n}\,dS \;=\; 0\ ,$$

such a vector field $\vec{U}_{km}(x)$ exists, and it is easy to verify that $\vec{\eta}(x)$ defined above is an element of $J(\Omega_m)$. We denote it by $\vec{u}_{km}(x)$.

The identity (2.11) with $\vec{\eta} = \vec{u}_{km}$ takes the form

$$\begin{aligned}\frac{\nu}{2}\int_{\Omega_{k+1}} S(\vec{u}_{km}):S(\vec{u}_{km})\,dx \;=\; &\frac{\nu}{2}\int_{\Omega_{k+1}} S(\vec{u}_{km}-\vec{u}_m):S(\vec{u}_{km})\,dx \\ &+\int_{\Omega_{k+1}} \vec{f}\cdot\vec{u}_{km}\,dx \\ &-\frac{\nu}{2}\int_{\Omega_{k+1}} S(\vec{V}):S(\vec{u}_{km})\,dx\ .\end{aligned} \tag{2.16}$$

When we evaluate the left-hand side from below by the Korn inequality, and the right-hand side from above by (2.12), we obtain

$$\|D\vec{u}_{km}\|^2_{L_2(\Omega_{k+1})} \;\le$$

$$\le\; c\Big(\|D(\vec{u}_{km}-\vec{u}_m)\|_{L_2(\omega_{k+1})}\,\|D\vec{u}_{km}\|_{L_2(\omega_{k+1})} + \sqrt{k}\Big(\|\vec{f}\|_2+\|g\|_1\Big)\,\|D\vec{u}_{km}\|_{L_2(\Omega_{k+1})}\Big),$$

and in virtue of (2.15) and of the Cauchy inequality,

$$\|D\vec{u}_{km}\|^2_{L_2(\Omega_{k+1})} \le$$
$$\le c\bigg(\|D\vec{u}_m\|^2_{L_2(\omega_{k+1})} + \frac{\varepsilon}{2}\,\|D\vec{u}_m\|^2_{L_2(\Omega_{k+1})} + \frac{k}{2\varepsilon}\Big(\|\vec{f}\|_1 + \|g\|_2\Big)^2\bigg), \qquad \forall\,\varepsilon > 0$$

Choosing now ε small enough we easily arrive at

$$y_k \le c(y_{k+1} - y_k) + Mk$$

or

$$y_k \le b\,y_{k+1} + \frac{M}{1+c}\,k\,, \tag{2.17}$$

where $y_k = \|D_x\vec{u}_m\|^2_{L_2(\Omega_{k+1})}$, $M = c(\|\vec{f}\|_1 + \|g\|_2)^2$, $b = \frac{c}{1+c} \in (0,1)$. Since (2.17) holds for all $k = 1, ..., m-1$, we have

$$\begin{aligned} y_k &\le \frac{Mk}{1+c} + \frac{M(k+1)}{1+c} + b^2\,y_{k+2} \\ &\le ... \\ &\le \frac{M}{1+c}\Big(k + b(k+1) + \cdots + b^{m-k-2}(m-2)\Big) + b^{m-k-1}\,y_{m-1}\,. \end{aligned}$$

Making use of the estimate (2.13), we finally obtain

$$y_k \le c_1\,M\bigg(k\sum_{j=0}^{\infty} b^j + \sum_{j=0}^{\infty} j\,b^j\bigg) \le c_2\Big(\|\vec{f}\|_1 + \|g\|_2\Big)^2\,k\,,$$

and (2.14) is proved.

The final estimate is that of $\int_{\omega_k} |D\vec{u}_m|^2\,dx$, $k < \frac{m}{2}$. To obtain it, we set in (2.11)

$$\vec{\eta} = \vec{u}_{k+j,m}(x) - \vec{u}_{k-j-2,m}(x) = \begin{cases} \vec{u}_{k+j,m}(x)\,, & x \in \omega_{k+j+1}, \\ \vec{u}_m(x)\,, & x \in \omega_{k-j} \cup \cdots \cup \omega_{k+j}, \\ \vec{u}_m(x) - \vec{u}_{k-j-2,m}\,, & x \in \omega_{k-j-1}, \\ 0, & x \in \Omega_{k-j-2}\,, \end{cases}$$

where $j \le k-2$.

Instead of (2.16), we now have

$$\frac{\nu}{2}\int_{\Omega_{k+j+1}\setminus\Omega_{k-j-2}}\Big|S(\vec{u}_{k+j,m}-\vec{u}_{k-j-2,m})\Big|^2\,dx =$$

$$= \frac{\nu}{2}\int_{\omega_{k+j+1}} S\Big(\vec{u}_{k+j,m}(x)-\vec{u}_m(x)\Big):S(\vec{u}_{k+j,m})\,dx$$
$$-\ \frac{\nu}{2}\int_{\omega_{k-j-1}} S(\vec{u}_{k-j-2,m}):S(\vec{u}_m-\vec{u}_{k-j-2,m})\,dx$$
$$+\int_{\Omega_{k+j+1}\setminus\Omega_{k-j-2}} \vec{f}\cdot\Big(\vec{u}_{k+j,m}(x)-\vec{u}_{k-j-2,m}(x)\Big)\,dx$$
$$-\ \frac{\nu}{2}\int_{\Omega_{k+j+1}\setminus\Omega_{k-j-2}} S(\vec{V}):S(\vec{u}_{k+j,m}-\vec{u}_{k-j-2,m})\,dx\ .$$

Using the Korn inequality and (2.15), we arrive at

$$z_j\ \le\ c(z_{j+1}-z_j)+M(j+1)\ ,$$

or

$$z_j\ \le\ b\,z_{j+1}+\frac{M}{1+c}\,(j+1)\ , \tag{2.18}$$

where

$$z_j\ =\int_{\Omega_{k+j+1}\setminus\Omega_{k-j-2}}|D\vec{u}_m|^2\,dx\,,\qquad j=0,1,...,k-2\ .$$

We have already seen that (2.18) implies

$$z_0\ \le\ \frac{M}{1+c}\Big(1+2\,b+\cdots+(k-2)\,b^{k-3}\Big)+b^{k-2}\,z_{k-2}\ ,$$

and since $z_{k-2}\le c\,M(k-1)$, (by virtue of (2.14)), we obtain

$$\int_{\omega_k}|D\vec{u}_m|^2\,dx\ \le\ z_0\ \le\ c\Big(\|\vec{f}\|_2+\|g\|_1\Big)^2\ , \tag{2.19}$$

q.e.d..

Now, we prove the existence of the vector field $\vec{u}$ satisfying the identity (2.10) by passing to a limit in (2.11). By virtue of (2.14), there exists a subsequence $\vec{u}_{m_s}$ such that $D\vec{u}_{m_s}$ are weakly convergent in $L_2(\Omega_k)$ for arbitrary $k>0$, as $s\to\infty$, to $D\vec{u}(x)$ where $\vec{u}\in J(\Omega_k)$, $k=1,\dots$. Clearly, the derivatives $D\vec{u}$ satisfy inequalities (2.14) and (2.19), i.e.,

$$\|D\vec{u}\|_{L_2(\Omega_k)}\ \le\ c\sqrt{k}\,\Big(\|\vec{f}\|_2+\|g\|_1\Big)\ ,$$

$$\|D\vec{u}\|_{L_2(\omega_k)}\ \le\ c\,\Big(\|\vec{f}\|_2+\|g\|_1\Big)\ .$$

Further, if we fix arbitrary solenoidal $\vec{\eta}$ with $\|D\vec{\eta}\|_{L_2(\Omega)} < \infty$ satisfying (1.14) whose support is contained in a certain Ω_j and pass in (2.11) to the limit as $m = m_s \to \infty$, we arrive at (2.10). Hence, $\vec{v} = \vec{u} + \vec{V}$ satisfies inequalities (2.2),(2.4), conditions (1.14),(1.18) and the integral identity

$$\frac{\nu}{2}\int_\Omega S(\vec{v}):S(\vec{\eta})\,dx \;=\; \int_\Omega \vec{f}\cdot\vec{\eta}\;dx \;=\; \int_\Omega (\vec{f}\cdot\vec{\eta} + \nu\, g\,\nabla\cdot\vec{\eta})\,dx \;,$$

with the same $\vec{\eta}$ as in (2.11). It is well known (see, for instance, [3]) that there exists a function $p(x)$ square integrable in every Ω_k and satisfying (2.3) and the integral identity (2.1). So, $(\vec{v}, p)$ is a weak solution of problem (1.12)–(1.18). The uniqueness of the solution follows from the fact that the difference $\vec{u} = \vec{v} - \vec{v}'$ of two weak solutions satisfies the integral identity

$$\frac{\nu}{2}\int_\Omega S(\vec{u}):S(\vec{\eta})\,dx \;=\; 0 \;.$$

Then, as we saw, $y_k = \int_{\Omega_{k+1}} |D\vec{u}|^2\,dx$ satisfy the inequalities $y_k \le b\,y_{k+1}$ which implies $y_k = 0$ and $\vec{u} = 0$. The theorem is proved. ∎

3 – Auxiliary linear problem (a classical solution)

In this section we consider the nonhomogeneous linear problem:

$$\tag{3.1}\begin{gathered}
-\nu\,\nabla^2\vec{v} + \nabla p = \vec{f}(x), \quad \nabla\cdot\vec{v} = g(x)\,, \qquad x\in\Omega\;, \\
\vec{v}|_\Sigma = 0\;, \\
\vec{v}\cdot\vec{n}|_\Gamma = b(x)\,, \qquad \vec{\tau}\cdot S(\vec{v})\,\vec{n}|_\Gamma = d(x)\;, \\
\vec{v}(x)\to 0, \quad p(x)\to 0\,, \qquad (|x|\to\infty, \quad x\in\Omega_+)\;, \\
|\vec{v}(x)| + |\nabla p(x)| < \infty\,, \qquad (|x|\to\infty, \quad x\in\Omega_-)\;, \\
\int_{S_+(r)} \vec{v}\cdot\vec{n}\,dS \to 0 \qquad (r\to\infty)\;,
\end{gathered}$$

and prove the following theorem.

Theorem 3. *Assume that $\Sigma_\pm \in C^{l+2}$, $\Gamma\pm$ are defined by equations (1.1) with*

$$h_\pm(x_2) \;=\; \pm d_0 + k_\pm\,x_2 + \int_0^{x_2}(x_2-t)\,h''_\pm(t)\,dt\,, \qquad h''_\pm \in C^{l+1}_{s-1}(R_+, 2)\;,$$

and the limits $h'_\pm(\infty) = k_\pm + \int_0^\infty h''_\pm(t)\,dt$ *satisfy the condition*

$$0 \;<\; \arctan h'_+(\infty) - \arctan h'_-(\infty) \;<\; \pi/2 \;. \tag{3.2}$$

For arbitrary $\vec f \in C^l_{s-2}(\Omega, 3+\beta)$, $g \in C^{l+1}_{s-1}(\Omega, 2+\beta)$, $b \in C^{l+2}_s(\Gamma, 1+\beta)$, $d \in C^{l+1}_{s-1}(\Gamma, 2+\beta)$ *such that*

$$\operatorname{supp} g \subset \Omega_1\,, \qquad b(x_\pm) = 0\;,$$

problem (3.1) has a unique solution $\vec v \in C^{l+1}_s(\Omega, 1+\beta)$, $\nabla p \in C^l_{s-2}(\Omega, 3+\beta)$ *and*

$$\begin{aligned} &|\vec v|_{C^{l+2}_s(\Omega,1+\beta)} + |\nabla p|_{C^l_{s-2}(\Omega,3+\beta)} \;\le \\ &\le\; c\left(|\vec f|_{C^l_{s-2}(\Omega,3+\beta)} + |g|_{C^{l+1}_{s-1}(\Omega,2+\beta)} + |b|_{C^{l+2}_s(\Gamma,1+\beta)} + |d|_{C^{l+1}_{s-1}(\Gamma,2+\beta)}\right)\,. \end{aligned} \tag{3.3}$$

The condition $b \in C^{l+2}_s(\Gamma, 1+\beta)$ *means that*

$$b_\pm(x_2) \;=\; b|_{x_1=h_\pm(x_2)} \;\in\; C^{l+2}_s(R_+, 1{+}\beta)$$

and

$$|b|_{C^{l+2}_s(\Gamma,1+\beta)} \;=\; |b_+|_{C^{l+2}_s(R_+,1+\beta)} + |b_-|_{C^{l+2}_s(R_+,1+\beta)}\;.$$

The condition $d \in C^{l+1}_{s-1}(\Gamma, 2{+}\beta)$ *has an analogous meaning. The number* l *is arbitrary positive and non-integral,* $s \in (0, 1/2)$, $\beta \in (0, \min(1, \pi/\theta - 2))$.

Proof: We prove the theorem in two steps. First of all, we consider the particular case $b = 0$, $d = 0$. In this case, we have a weak solution of problem (3.1), and we show that it possesses regularity properties indicated in the statement of the theorem and satisfies inequality (3.3), i.e., we prove Theorem 3 for $b = 0$, $d = 0$. Then we reduce problem (3.1) to the problem with homogeneous boundary conditions by construction of a special auxiliary vector field.

Thus, we assume that $b = 0$, $d = 0$. It follows from the regularity theorem proved in [6] that a weak solution belongs to $C^{l+2}(\omega) \times C^{l+1}(\omega)$ in arbitrary bounded domain ω such that $\operatorname{dist}(\omega, x_\pm) > 0$. Moreover, for the solution the following "local estimate" holds. Let $x_0 \in \bar\Omega$, $B_\rho(x_0) = \{x \in \Omega\colon\, |x - x_0| < \rho\}$. If the domain $B_\rho(x_0)$ is bounded away from $x_\pm$: $\operatorname{dist}(B_\rho(x_0), x_\pm) \ge \rho_1 > 0$ and $\vec f \in C^l(B_\rho(x_0))$, $g \in C^{l+1}(B_\rho(x_0))$, then for arbitrary $\rho' < \rho$

$$\begin{aligned} &|\vec v|_{C^{l+2}(B_{\rho'}(x_0))} + |\nabla p|_{C^l(B'_\rho(x_0))} \;\le \\ &\le\; c(\rho, \rho')\left(|\vec f|_{C^l(B_\rho(x_0))} + |g|_{C^{l+1}(B_\rho(x_0))} + \|\vec v\|_{L_2(B_\rho(x_0))} + \|p\|_{L_2(B_\rho(x_0))}\right)\,. \end{aligned}$$

The pressure is defined up to an additive constant, and we can normalize it by the condition $\int_{B_\rho(x_0)} p\,dx = 0$. It is then well known (see [3]) that

$$\|p\|_{L_2(B_\rho(x_0))} \le c\left(\|D\vec{v}\|_{L_2(B_\rho(x_0))} + \|f\|_{L_2(B_\rho(x_0))}\right),$$

hence,

$$|\vec{v}|_{C^{l+2}(B_{\rho'}(x_0))} + |\nabla p|_{C^l(B_{\rho'}(x_0))} \le$$
$$\le c(\rho,\rho')\left(|\vec{f}|_{C^l(B_\rho(x_0))} + |g|_{C^{l+1}(B_\rho(x_0))} + \|D\vec{v}\|_{L_2(B_\rho(x_0))} + \|\vec{v}\|_{L_2(B_\rho(x_0))}\right).$$

It follows from this inequality that

$$|\vec{v}|_{C^{l+2}(\omega_k)} + |\nabla p|_{C^l(\omega_k)} \le c\left(|\vec{f}|_{C^l(\omega_k^*)} + |g|_{C^{l+1}(\omega_k^*)} + \|D\vec{v}\|_{L_2(\omega_k^*)} + \|\vec{v}\|_{L_2(\omega_k^*)}\right),$$

where $\omega_k^* = \omega_k \cup \omega_{k-1} \cup \omega_{k+1}$, $k > 1$. As $\vec{v}|_\Sigma = 0$, the Friedrichs inequality implies

$$\|\vec{v}\|_{L_2(\omega_k^*)} \le c\,\|D\vec{v}\|_{L_2(\omega_k^*)},$$

so, taking into account estimate (2.4), we obtain

$$|\vec{v}|_{C^{l+2}(\omega_k)} + |\nabla p|_{C^l(\omega_k)} \le c\left(|\vec{f}|_{C^l_{s-1}(\Omega,3+\beta)} + |g|_{C^l_{s-1}(\Omega,2+\beta)}\right), \qquad k = 2, \dots \tag{3.4}$$

with the constant independent of k.

The properties of the solution near contact points $x_\pm$ can be easily investigated with the help of results of [7]. Local analysis of a weak solution near contact points was carried out in Theorem 4.3 in [7] from which it follows that

$$|\vec{v}|_{\overset{\circ}{C}{}^{l+2}_s(B_\rho(x_\pm))} + |\nabla p|_{\overset{\circ}{C}{}^{l}_{s-2}(B_\rho(x_\pm))} \le c\left(|\vec{f}|_{C^l_{s-2}(\Omega,3+\beta)} + |g|_{C^{l+1}_{s-1}(\Omega,2+\beta)}\right) \tag{3.5}$$

where ρ is a certain sufficiently small (but fixed) number and the norms in the left-hand side are defined by

$$|\vec{u}|_{\overset{\circ}{C}{}^{l+2}_s(B_\rho(x_\pm))} = \sum_{0\le|j|<l} \sup_{B_\rho} |x-x_\pm|^{|j|-s}\,|D^j u(x)| + \sup_{B_\rho(x_\pm)} |x-x_\pm|^{l-s}\,[u]^l_{K_\pm(x)},$$

$K_\pm(x) = \{y \in B_\rho(x_\pm)\colon\, |x-y| < |x-x_\pm|/2\}$.

It remains to analyze $\vec{v}$, p for large $|x|$, $x \in \Omega_+$, in particular, to establish necessary decay properties. We shall work in the domain $\Omega^{(R)} = \{x\in\Omega\colon\, x_2 > R\}$. Let D_θ be an infinite sector

$$\kappa^{(-)} y_2 - d_0 < y_1 < \kappa^{(+)} y_2 + d_0, \qquad y_2 > -\frac{2\,d_0}{\kappa^{(+)} - \kappa^{(-)}}$$

where

$$\kappa^{(\pm)} = h'_\pm(\infty) = k_{(\pm)} + \int_0^\infty h''_\pm(t)\,dt \ .$$

The opening of this sector is equal to

$$\theta = \arctan\kappa^{(+)} - \arctan\kappa^{(-)} \in (0,\pi/2) \ .$$

Further, let $\zeta \in C^\infty(R)$, $\zeta(t) = 1$ for $t > 2/3$, $\zeta(t) = 0$ for $t < 1/3$, and consider the mapping $x = \mathcal{Z}(y)$, $y \in D_\theta$, defined by the formulas

$$x_1 = y_1 + \Phi(y), \quad x_2 = y_2\,, \qquad y \in D_\theta \ ,$$

(3.6)

$$\Phi(y) = \Big[\lambda_+(y)\Big(h_+(y_2) - \kappa^{(+)}y_2 - d_0\Big) + \lambda_-(y)\Big(h_-(y_2) - \kappa^{(-)}y_2 + d_0\Big)\Big]\,\zeta_R(y_2) \ ,$$

where $\zeta_R(y_2) = \zeta(y_2/R)$, $\lambda_\pm(y) = \zeta\Big(\pm\dfrac{2\,y_1 - (\kappa^{(+)}+\kappa^{(-)})\,y_2}{2\,d_0 + (\kappa^{(+)}-\kappa^{(-)})\,y_2}\Big)$. It is easy to verify that this mapping possesses the following properties:

(i) if $y_2 < R/3$, then $\Phi = 0$ and $\mathcal{Z}$ is an identical transformation;

(ii) if $y_1 = \kappa^{(+)}y_2 + d_0$, then $\Phi(y) = (h_+(y_2) - \kappa^{(+)}y_2 - d_0)\,\zeta_R(y_2)$ and

$$x_1 = h_+(x_2)\,\zeta_R(x_2) + (\kappa^{(+)}x_2 + d_0)\,(1 - \zeta_R(x_2)) \equiv h_+^{(R)}(x_2) \ ;$$

if $y_1 = \kappa^{(-)}y_2 - d_0$, then $\Phi(y) = (h_-(y_2) - \kappa^{(-)}y_2 + d_0)\,\zeta_R(y_2)$ and

$$x_1 = h_-(x_2)\,\zeta_R(x_2) + (\kappa^{(-)}x_2 - d_0)\,(1 - \zeta_R(x_2)) \equiv h_-^{(R)}(x_2) \ ;$$

clearly, $h_\pm^{(R)}(x_2) = h_\pm(x_2)$ for $x_2 > 2R/3$ and $h_\pm^{(R)}(x_2) = \kappa^{(\pm)}x_2 \pm d_0$ for $x_2 < R/3$.

Since

$$h_\pm(y_2) - \kappa^{(\pm)}y_2 \mp d_0 = -\int_0^\infty \min(y_2,t)\,h''_\pm(t)\,dt \ ,$$

we have for $y_2 > R/3$

$$\begin{aligned}\Big|h_\pm(y_2) - \kappa^{(\pm)}y_2 \mp d_0\Big| &\le \sup_{z>0}\rho(z,2,1-s)\,|h''_\pm(z)| \int_0^\infty \min(y_2,t)\,\frac{dt}{\rho(t,2,1-s)}\\ &\le c\,(1+\log y_2)\,\sup_{z>0}\rho(z,2,1-s)\,|h''_\pm(z)| \ ,\end{aligned}$$

$$\Big|\frac{d}{dy_2}\Big(h_\pm(y_2) - \kappa^{(\pm)}y_2 \mp d_0\Big)\Big| \le \Big|\int_{y_2}^\infty h''(t)\,dt\Big| \le c\,y_2^{-1}\,\sup_{z>0}\rho(z,2,1-s)\,|h''_\pm(z)|$$

and, as a consequence,

$$|\nabla\Phi(y)| \le c\,R^{-\gamma}\Big(|h''_+|_{C^{l+1}_{s-1}(R_+,2)} + |h''_-|_{C^{l+1}_{s-1}(R_+,2)}\Big)\,, \qquad \forall\,\gamma\in(0,1), \;\; y\in D_\theta\;,$$

which shows that the mapping $\mathcal{Z}$ is invertible, if R is large enough. Moreover, $\nabla\Phi$ belongs to the space $\overset{\circ}{C}{}^{l+2}_{-\gamma}(D_\theta)$ and satisfies the inequality

$$|\nabla\Phi(y)|_{\overset{\circ}{C}{}^{l+2}_{-\gamma}(D_\theta)} \le c\,\Big(|h''_+|_{C^{l+1}_{s-1}(R_+,2)} + |h''_-|_{C^{l+1}_{s-1}(R_+,2)}\Big)\;. \tag{3.7}$$

The norm in this space is defined by

$$|u|_{\overset{\circ}{C}{}^{l}_{s}(D_\theta)} = \sum_{0\le|j|<l}\sup_{D_\theta}|y-y_0|^{|j|-s}\,|D^j u(y)| + \sup|y-y_0|^{l-s}\,[u]^{(l)}_{K(y)}$$

where $y_0 = \left(-d_0\,\dfrac{\kappa^{(+)}+\kappa^{(-)}}{\kappa^{(+)}-\kappa^{(-)}},\, -\dfrac{2\,d_0}{\kappa^{(+)}-\kappa^{(-)}}\right)$ is the vertex of the sector D_θ and $K(y) = \{z\in D_\theta\colon\, |z-y|\le\frac{|y-y_0|}{2}\}$.

Thus, $\mathcal{Z}$ is an invertible mapping of D_θ onto the domain

$$D^{(R)}_\theta = \left\{h^{(R)}_-(x_2) < x_1 < h^{(R)}_+(x_2),\;\; x_2 > -\frac{2\,d_0}{\kappa^{(+)}-\kappa^{(-)}}\right\}\;;$$

whose intersection with the half-plane $x_2 > R$ coincides with $\Omega^{(R)}$. We denote by $\Gamma^{(R)}_\pm$ the curves $x_1 = h^{(R)}_\pm(x_2),\; x_2 > -\frac{2d_0}{\kappa^{(+)}-\kappa^{(-)}}$.

Let us make the change of variables $x = \mathcal{Z}(y)$ in (3.1) assuming that $x_2 > R$. It is easy to see that equations and boundary conditions in (3.1) (with $b = 0$, $d = 0$) take the form

$$\begin{gathered} -\nu\,\hat\nabla^2\vec{\hat v} + \hat\nabla\hat p = \vec{\hat f}(y)\,, \qquad \hat\nabla\cdot\vec{\hat v} = \hat g(y)\;\; y\in D_\theta\;\;(y_2>R)\,,\\ \vec{\hat v}\cdot\vec n|_{\Gamma(\infty)} = 0\,, \qquad \vec\tau\cdot\hat S(\vec{\hat v})\,\vec n|_{\Gamma(\infty)} = 0\;, \end{gathered} \tag{3.8}$$

where $\vec{\hat v}(y) = \vec v(\mathcal{Z}(y))$, $\hat p(y) = p(\mathcal{Z}(y))$, $\hat\nabla = J^T\nabla = \left(\displaystyle\sum_{m=1}^{2} J_{mk}\,\frac{\partial}{\partial y_m}\right)_{k=1,2}$, $J_{mk} = \left(\dfrac{\partial y_m}{\partial x_k}\right)$ are elements of the Jacobi matrix

$$J = \begin{pmatrix} \dfrac{1}{1+\Phi_{y_1}} & -\dfrac{\Phi_{y_2}}{1+\Phi_{y_1}} \\[2ex] 0 & 1 \end{pmatrix}\;,$$

of the transformation $\mathcal{Z}^{-1}$,

$$\hat{S}(\vec{w}) = \left(\sum_{m=1}^{2} \left(J_{mi} \frac{\partial w_j}{\partial y_m} + J_{mj} \frac{\partial w_i}{\partial y_m} \right) \right)_{i,j=1,2}$$

and $\Gamma_\pm^{(\infty)}$ are straight lines $y_1 = \kappa^{(\pm)} y_2 \pm d_0$. By $\vec{n}$ and $\vec{\tau}$ in the boundary conditions we mean normal and tangential vectors to $\Gamma_\pm$ given by standard formulas:

$$\vec{n}|_{\Gamma_\pm^{(\infty)}} = \pm \left(\frac{1}{\sqrt{1+h_\pm'^2(y_2)}}, -\frac{h_\pm'(y_2)}{\sqrt{1+h_\pm'^2(y_2)}} \right) \equiv \vec{n}_\pm \, , \tag{3.9}$$

$$\vec{\tau}|_{\Gamma_\pm^{(\infty)}} = \left(\frac{h_\pm'(y_2)}{\sqrt{1+h_\pm'^2(y_2)}}, \frac{1}{\sqrt{1+h_\pm'^2(y_2)}} \right) \equiv \vec{\tau}_\pm \, .$$

Now, we multiply (3.8) by $\zeta_{2R}(y_2)$ and introduce new functions $\vec{u} = \vec{\hat{v}}\, \zeta_{2R}$, $q = \hat{p}\, \zeta_{2R}$. It is easy to see that these functions satisfy the equations

$$-\nu\, \nabla^2 \vec{u} + \nabla q = \vec{\hat{f}}\, \zeta_{2R} + \vec{f}_1 + \vec{l}_1(\vec{u}, q), \quad \nabla\cdot \vec{u} = \hat{g}\, \zeta_{2R} + g_1 + l_2(\vec{u})\, , \qquad y \in D_\theta \, ,$$

$$\vec{u}\cdot \vec{n}_\pm^{(\infty)}|_{\Gamma_\pm^{(\infty)}} = l_3^{(\pm)}(\vec{u})\, , \qquad \vec{\tau}_\pm^{(\infty)} \cdot S(\vec{u})\, \vec{n}_\pm^{(\infty)}|_{\Gamma_\pm^{(\infty)}} = d_1^{(\pm)} + l_4^{(\pm)}(\vec{u})\, , \tag{3.10}$$

where $\vec{n}_\pm^{(\infty)}$, $\vec{\tau}_\pm^{(\infty)}$ are normal and tangential vectors to $\Gamma_\pm^{(\infty)}$, i.e.,

$$\vec{n}_\pm^{(\infty)} = \pm \left(\frac{1}{\sqrt{1+\kappa^{(\pm)2}}}, -\frac{\kappa^{(\pm)}}{\sqrt{1+\kappa^{(\pm)2}}} \right), \quad \vec{\tau}_\pm^{(\infty)} = \left(\frac{\kappa^{(\pm)}}{\sqrt{1+\kappa^{(\pm)2}}}, \frac{1}{\sqrt{1+\kappa^{(\pm)2}}} \right),$$

and

$$\vec{l}_1(\vec{u}, q) = \nu(\hat{\nabla}^2 - \nabla^2)\, \vec{u} + (\nabla - \hat{\nabla})\, q \, ,$$

$$l_2(\vec{u}) = (\nabla - \hat{\nabla}) \cdot \vec{u} \, ,$$

$$l_3^{(\pm)}(\vec{u}) = \vec{u} \cdot (\vec{n}_\pm^{(\infty)} - \vec{n}_\pm) \, ,$$

$$l_4^{(\pm)}(\vec{u}) = \vec{\tau}_\pm^{(\infty)} \cdot S(\vec{u}) \vec{n}_\pm^{(\infty)} - \vec{\tau}_\pm \cdot S(\vec{u})\, \vec{n}_\pm + \vec{\tau}_\pm \Big(S(\vec{u}) - \hat{S}(\vec{u}) \Big)\, \vec{n}_\pm \, ,$$

$$\vec{f}_1 = -2\,\nu\, \hat{\nabla} \vec{\hat{v}}\, \hat{\nabla}\zeta_{2R} - \nu\, \vec{\hat{v}}\, \hat{\nabla}^2 \zeta_{2R} + p\, \hat{\nabla}\zeta_{2R} \, ,$$

$$g_1 = \vec{\hat{v}}\, \hat{\nabla}\zeta_{2R} \, ,$$

$$d_1^{(\pm)} = (\vec{\hat{v}} \cdot \vec{\tau}^{(\pm)})\, (\vec{n}^{(\pm)} \cdot \hat{\nabla}\zeta_{2R}) + (\vec{\hat{v}} \cdot \vec{n}^{(\pm)})\, (\vec{\tau}^{(\pm)} \cdot \hat{\nabla}\zeta_{2R}) \, .$$

Clearly, $\vec{f}_2 \equiv \hat{\vec{f}}\zeta_{2R} + \vec{f}_1 \in \overset{\circ}{C}{}^{l}_{b-2}(D_\theta)$, $g_2 \equiv \hat{g}\,\zeta_{2R} + g_1 \in \overset{\circ}{C}{}^{l+1}_{b-1}(D_\theta)$, $d_1^{\pm} \in \overset{\circ}{C}{}^{l+1}_{b-1}(\Gamma^{(\infty)}_{\pm})$ with arbitrary $b \geq -1-\beta$. Further, since $\hat{\nabla} - \nabla = (J^T - I)\,\nabla$, and the elements of the matrix

$$J^T - I = \begin{pmatrix} -\dfrac{\Phi_{y_1}}{1+\Phi_{y_1}} & 0 \\ -\dfrac{\Phi_{y_2}}{1+\Phi_{y_2}} & 0 \end{pmatrix}$$

are proportional to the derivatives Φ_{y_j} for which estimate (3.7) holds, we have

$$\begin{aligned} &|\vec{l}_1(\vec{u},q)|_{\overset{\circ}{C}{}^{l+1}_{b-\alpha-2}(D_\theta)} + |l_2(\vec{u})|_{\overset{\circ}{C}{}^{l+1}_{b-\alpha-1}(D_\theta)} \leq \\ &\leq c\,R^{-(\gamma-\alpha)}\Big(|h''_+|_{C^{l+1}_{s-1}(R_+,2)} + |h''_-|_{C^{l+1}_{s-1}(R_+,2)}\Big)\Big(|\vec{u}|_{\overset{\circ}{C}{}^{l+2}_{b}(D_\theta)} + |q|_{\overset{\circ}{C}{}^{l+1}_{b-1}(D_\theta)}\Big) \end{aligned} \tag{3.11}$$

for arbitrary $0<\alpha<\gamma<1$, $\vec{u} \in \overset{\circ}{C}{}^{l+1}_{b-1}(D_\theta)$, $q \in \overset{\circ}{C}{}^{l+2}_{b}(D_\theta)$. Expressions $l_3^{(\pm)}(\vec{u})$ and $l_4^{(\pm)}(\vec{u})$ contain also terms with $\vec{u}$ or $D_j\vec{u}$ multiplied by $(\vec{n}^{(\infty)}_{\pm} - \vec{n}_{\pm})_i$ or $(\vec{\tau}^{(\infty)}_{\pm} - \vec{\tau}_{\pm})_i$ where $\vec{n}_{\pm}$ and $\vec{\tau}_{\pm}$ are normal and tangential vector to $\Gamma^{(R)}_{\pm}$, respectively, given by formulas (3.9) with $h^{(R)}_{\pm}$ instead of $h_{\pm}$. Since

$$\begin{aligned} h^{(R)\prime}_{\pm}(y_2) - \kappa^{(\pm)} &= \Big(h'_{\pm}(y_2) - \kappa^{(\pm)}\Big)\,\zeta_R + \zeta'_R(y_2)\,\Big(h_{\pm}(y_2) - \kappa^{(\pm)}y_2 \mp d_0\Big) \\ &= -\zeta_R\int_{y_2}^{\infty} h''_{\pm}(t)\,dt \;-\; \zeta'_R(y_2)\int_0^{\infty}\min(y_2,t)\,h''_{\pm}(t)\,dt\ , \end{aligned}$$

we have

$$|\vec{n}^{(\infty)}_{\pm} - \vec{n}_{\pm}|_{C^{l+2}_{-\gamma}(\Gamma^{(\infty)}_{\pm})} \leq c\,|h''_{\pm}|_{C^{l+1}_{s-1}(R_+,2)}$$

so $l_3(\vec{u})$ and $l_4(\vec{u})$ also satisfy inequality similar to (3.11):

$$\begin{aligned} &|l_3^{(\pm)}(\vec{u})|_{\overset{\circ}{C}{}^{l+2}_{b-\alpha}(\Gamma^{(\infty)}_{\pm})} + |l_4^{(\pm)}(\vec{u})|_{\overset{\circ}{C}{}^{l+1}_{b-\alpha-1}(\Gamma^{(\infty)}_{\pm})} \leq \\ &\leq c\,R^{-\gamma+\alpha}\,|h''_{\pm}|_{C^{l+1}_{s-1}(R_+,2)}\,|\vec{u}|_{\overset{\circ}{C}{}^{l+2}_{b}(D_\theta)}\,, \qquad \forall\,\vec{u} \in \overset{\circ}{C}{}^{l+2}_{b}(D_\theta)\ . \end{aligned} \tag{3.12}$$

We consider (3.10) as a boundary value problem for $\vec{u}$, q in D_θ with given $\vec{f}_2 \equiv \hat{\vec{f}}\,\zeta_{2R} + \vec{f}_1$, $g_2 = \hat{g}\,\zeta_{2R} + g_1$, $d_1^{\pm}$. By virtue of (3.11), (3.12), we can prove the solvability of this problem in the case of large R, using the contraction mapping principle in the space $\overset{\circ}{C}{}^{l+2}_{b}(D_\theta) \times \overset{\circ}{C}{}^{l+1}_{b-1}(D_\theta)$ with small positive b. From (3.11), (3.12) we may conclude that $\vec{f}_2 + \vec{l}_1(\vec{u},q) \in \overset{\circ}{C}{}^{l}_{b_1-2}(D_\theta)$, $g_2 + l_2(\vec{u}) \in \overset{\circ}{C}{}^{l+1}_{b_1-1}(D_\theta)$, $l_3(\vec{u}) \in \overset{\circ}{C}{}^{l+2}_{b_1}(\Gamma^{(\infty)}_{\pm})$, $d_1^{(\pm)} + l_4(\vec{u}) \in \overset{\circ}{C}{}^{l+1}_{b_1}(\Gamma^{(\infty)}_{\pm})$ with $b_1 = b - \alpha < 0$; hence, in virtue of general results of V.G. Maz'ya and B.A. Plamenevskii [8,9] (see also [7], §2), we have

$$\vec u\in \overset{\circ}{C}{}^{l+2}_{b}(D_\theta)\cap \overset{\circ}{C}{}^{l+2}_{b_1}(D_\theta)\,,\qquad q\in \overset{\circ}{C}{}^{l+1}_{b-1}(D_\theta)\cap \overset{\circ}{C}{}^{l+1}_{b_1-1}(D_\theta)\ .$$

Since $b>0$ and $b_1<0$, $D\vec u$ and q belong to $L_2(D_\theta)$.

In the original coordinates (x_1,x_2) problem (3.10) takes the form

$$-\nu\,\nabla^2\check{\vec u}+\nabla\check q=\check{\vec f}_2,\quad \nabla\cdot\check{\vec u}=\check g_2\,,\qquad x\in D^{(R)}_\theta\ ,$$

$$\check{\vec u}\cdot\vec n|_{\Gamma^{(R)}_\pm}=0\,,\qquad \vec\tau\cdot S(\check{\vec u})\,\vec n|_{\Gamma^{(R)}_\pm}=d_1\ , \tag{3.13}$$

where $\check f = f\circ\mathcal{Z}^{-1}$, and we have shown that it has two solutions with a finite Dirichlet integral: $\vec v\,\zeta_{2R},\ p\,\zeta_{2R}$ where $\vec v,p$ is a weak solution of problem (3.1) and $\check{\vec u},\check q$ obtained as a solution of problem (3.10). The difference of these functions is a weak solution of a homogeneous problem (3.13), hence, they coincide. Thus, we have shown that $\vec u=\hat{\vec v}\,\zeta_{2R}\in \overset{\circ}{C}{}^{l+2}_{b_1}(D_\theta),\ q=\hat p\,\zeta_{2R}\in\overset{\circ}{C}{}^{l+1}_{b_1-1}(D_\theta)$ with $b_1<0$. We can apply again estimates (3.11), (3.12) with $b=b_1$ and the same result of V.G. Maz'ya and B.A. Plamenevskii to prove that $\vec u\in\overset{\circ}{C}{}^{l+2}_{b_1-\alpha}(D_\theta),\ q\in\overset{\circ}{C}{}^{l+1}_{b_1-\alpha-1}(D_\theta)$ and repeat these arguments until we show that $\vec f_2+l_1(\vec u,q)\in\overset{\circ}{C}{}^{\,l}_{-3-\beta}(D_\theta)$, $g_2+l_2(\vec u)\in\overset{\circ}{C}{}^{l+1}_{-2-\beta}(D_\theta),\ l_3^{(\pm)}\in\overset{\circ}{C}{}^{l+2}_{-1-\beta}(\Gamma^{(\infty)}_\pm),\ d_1+l_4^{(\pm)}\in\overset{\circ}{C}{}^{l+1}_{-2-\beta}(\Gamma^{(\infty)}_\pm)$. Then we can make use of Propositions 2.1 and 2.2 in [1] to conclude that $\vec u\in\overset{\circ}{C}{}^{l+2}_{-1-\beta}(D_\theta)$, $q\in\overset{\circ}{C}{}^{l+1}_{-2-\beta}(D_\theta)$, due to conditions (3.2) and $\lim_{r\to\infty}\int_{S_+(r)}\vec v\cdot\vec n\,dS=0$, and that

$$\begin{aligned}
&|\vec u|_{\overset{\circ}{C}{}^{l+2}_{-1-\beta}(D_\theta)}+|q|_{\overset{\circ}{C}{}^{l+1}_{-2-\beta}(D_\theta)}\ \le\\
&\le\ c\left(|\vec f_2|_{\overset{\circ}{C}{}^{l}_{-3-\beta}(D_\theta)}+|g_2|_{\overset{\circ}{C}{}^{l+1}_{-2-\beta}(D_\theta)}+|d_1^{(+)}|_{\overset{\circ}{C}{}^{l+1}_{-2-\beta}(\Gamma^{(\infty)}_+)}+|d_2^{(-)}|_{\overset{\circ}{C}{}^{l+1}_{-2-\beta}(\Gamma^{(\infty)}_-)}\right)\\
&\le\ c\left(|\vec f|_{C^l_{s-2}(\Omega,3+\beta)}+|g|_{C^{l+1}_{s-1}(\Omega,2+\beta)}\right)\ .
\end{aligned}$$

This estimate in combination with (3.5) and (3.4) yields a desired inequality

$$|\vec v|_{C^{l+2}_s(\Omega,1+\beta)}+|\nabla p|_{C^l_{s-2}(\Omega,3+\beta)}\ \le c\left(|\vec f|_{C^l_{s-2}(\Omega,3+\beta)}+|g|_{C^{l+1}_{s-1}(\Omega,2+\beta)}\right)\,, \tag{3.14}$$

so, in the case $b=0$, $d=0$ the theorem is proved.

The general case reduces to the case of homogeneous boundary conditions by construction of a vector field $\vec V\in C^{l+2}_s(\Omega,1+\beta)$ such that

$$\vec V|_\Sigma=0\,,\qquad \vec V|_G=b\,\vec n\,,\qquad \frac{\partial\vec V}{\partial n}\bigg|_G=\left(d-\vec n\cdot\frac{\partial b\vec n}{\partial\tau}\right)\vec\tau\ \equiv\ \vec b_1 \tag{3.15}$$

and

$$
\begin{aligned}
(3.16)\qquad |\vec{V}|_{C^{l+2}_s(\Omega,1+\beta)} &\leq c\left(|b\,\vec{n}|_{C^{l+2}_s(\Gamma,1+\beta)} + |\vec{b}_1|_{C^{l+1}_{s-1}(\Gamma,2+\beta)}\right)\\
&\leq c\left(|b|_{C^{l+2}_s(\Gamma,1+\beta)} + |d|_{C^{l+1}_{s-1}(\Gamma,2+\beta)}\right) .
\end{aligned}
$$

The construction of such a vector field is possible, because $s < 1/2$ and $b(x)$ satisfies the compatibility conditions $b(x_\pm) = 0$. It is a standard problem of the theory of functions; for weighted spaces the methods of construction of functions satisfying boundary conditions of the type (3.15) are given, for instance, in [10], Theorem 4.1 and in [7], Lemmas 2.2, 4.3. It is easy to see that (3.15) implies

$$
\vec{V}\cdot\vec{n}|_\Gamma = c\,, \qquad \vec{\tau}\cdot S(\vec{V})\,\vec{n}|_\Gamma \;=\; \vec{\tau}\cdot\frac{\partial\vec{V}}{\partial n} + \vec{n}\cdot\frac{\partial\vec{V}}{\partial\tau}\bigg|_\Gamma = d\ .
$$

For $\vec{u} = \vec{v} - \vec{V}$, p we obtain problem (3.1) with homogeneous boundary conditions and with $\vec{f}$ and g replaced by

$$
\vec{f} + \nu\,\nabla^2\vec{V} \in C^l_{s-2}(\Omega, 3+\beta) \quad\text{ and }\quad g - \nabla\cdot\vec{V} \in C^{l+1}_{s-1}(\Omega, 2+\beta)\ ,
$$

respectively. We have already shown that this problem in uniquely solvable. Estimate (3.3) follows from (3.14) and (3.16). This proves Theorem 3 in the general case. ∎

4 – Auxiliary nonlinear problem

This section is devoted to a nonlinear problem (1.19) in a given domain Ω of the same type as in §3. We prove the following theorem.

Theorem 4.

1. *Let $\Sigma_\pm$ belong to C^{l+2}, let $\Gamma_\pm$ be given by equations (1.1) with*

$$
(4.1)\qquad h_\pm(x_2) \;=\; \pm d_0 + k_\pm x_2 + \int_0^{x_2} (x_2 - t)\, h''_\pm(t)\,dt\,, \qquad h''_\pm \in C^{l+1}_{s-1}(R_+, 2)\ ,
$$

and assume that the norms of $h''_\pm$, as well as the number F, are sufficiently small. Then problem (1.19) has a solution $(\vec{v}, p)$, $\vec{v}\in C^{l+2}_s(\Omega, 1)$, $\nabla p\in C^l_{s-2}(\Omega, 3)$, $s<1/2$, satisfying the inequality

$$
(4.2)\qquad |\vec{v}|_{C^{l+2}_s(\Omega,1)} + |\nabla p|_{C^l_{s-2}(\Omega,3)} \;\leq\; c\,|F|\ ,
$$

and the solution is unique.

2. *Let Ω_1 and Ω_2 be two domains corresponding to the functions $h_\pm^{(1)}$ and $h_\pm^{(2)}$ satisfying the above conditions, and let*

$$t_\pm^{(i)}(x_2) \;=\; \vec{n}_i \cdot T(\vec{v}^{(i)}, p^{(i)})\, \vec{n}_i|_{x_1=h_\pm^{(i)}(x_2)}$$

where $\vec{n}_i$ is a normal to $\Gamma_\pm^{(i)}$ and $(\vec{v}^{(i)}, p^{(i)})$ is a solution of problem (1.19) in Ω_i. There holds the inequality

$$(4.3)\quad |t_+^{(1)} - t_+^{(2)}|_{C_{s-1}^{l+1}(R_+,2)} \le c\Big(|h_-^{(1)\prime\prime} - h_-^{(2)\prime\prime}|_{C_{s-1}^{l+1}(R_+,2)} + |h_+^{(1)\prime\prime} - h_+^{(2)\prime\prime}|_{C_{s-1}^{l+1}(R_+,2)}\Big).$$

Proof: We reduce problem (1.19) to a problem of the type (3.1) with additional small linear and nonlinear terms in the equations which can be solved in the case of small F with the aid of the contraction mapping principle. To this end, we introduce an auxiliary vector field

$$(4.4)\qquad \vec{w}(x) \;=\; \xi_+(x)\, \vec{v}^{(+)}(x) + \xi_-(x)\, \vec{v}^{(-)}(x)$$

where $\xi_+(x)$ and $\xi_-(x)$ are smooth cut-off functions such that

$$\xi_-(x) = 1 \;\text{ for } x \in \Omega_-\backslash\omega_0 \quad \text{and} \quad \xi_-(x) = 0 \;\text{ for } x \in \Omega_+\;,$$

$$\xi_+(x) = 1 \;\text{ for } |x| \ge 4\,d_0,\; x\in\Omega_+, \quad \text{and}$$
$$\xi_+(x) = 0 \;\text{ for } x \in \Omega_- \;\text{ and for }\; |x| \le 2\,d_0,\; x\in\Omega_+\;;$$

further, $\vec{v}^{(+)} \;=\; \dfrac{F}{\theta}\,\dfrac{\vec{x}}{|x|^2}$ where θ is a limiting opening angle of the domain Ω_+ (see §3), and $\vec{v}^{(-)}(x) \;=\; F\Big(-\dfrac{\partial\psi(x)}{\partial x_2}, \dfrac{\partial\psi(x)}{\partial x_1}\Big)$ where $\psi(x)$ is a bounded smooth function given in Ω_- and equal to 1 in the neighbourhood of Σ_+ and to 0 in the neighbourhood of Σ_-. It is easily seen that

$$\int_{S_+(r)} \vec{v}^{(+)} \cdot \vec{n}\; dS \;\to\; F \quad (r \to \infty)\,, \qquad \int_{S'} \vec{v}^{(-)} \cdot \vec{n}\; dS \;=\; F$$

for arbitrary cross-section S' of Ω_-. The vector field $\vec{w}(x)$ is as smooth as ξ_+, ξ_-, ψ are (i.e., $\vec{w}(x) \in C^{l+2}(\Omega)$), moreover, it behaves like $|x|^{-1}$ for large $|x|$, $x \in \Omega_+$, and it is bounded together with its derivatives in Ω_-, hence, $\vec{w} \in C_s^{l+2}(\Omega, 1)$ and

$$(4.5)\qquad |\vec{w}|_{C_s^{l+2}(\Omega,1)} \;\le\; c\,|F|\;.$$

In addition, $\vec{w}$ possesses the following properties:

1. $\displaystyle\int_{S'} \vec{w}\cdot\vec{n}\,dS = F$ for arbitrary cross-section S' of $\Omega_-\backslash\omega_0$ and

$\displaystyle\int_{S_+(r)} \vec{w}\cdot\vec{n}\,dS \underset{r\to\infty}{\longrightarrow} F;$

2. $\nabla\cdot\vec{w} \equiv -g(x)$ is a smooth function with a compact support and

$$|g|_{C^{l+2}_{s-1}(\Omega,2+\beta)} \le c\,|F|\,, \qquad \beta\in(0,1)\ ;$$

3. $\displaystyle \vec{w}\cdot\vec{n}|_{\Gamma_\pm} = \frac{F}{\theta}\,\xi_+\frac{\vec{x}\cdot\vec{n}}{|x|^2}\Big|_{\Gamma_\pm} \equiv -b_\pm,$ and, as

$$\vec{n}|_{\Gamma_\pm} = \pm\left(\frac{1}{\sqrt{1+h'^2_\pm}},\,-\frac{h'_\pm}{\sqrt{1+h'^2_\pm}}\right)\ ,$$

we have:

$$\frac{\vec{x}\cdot\vec{n}}{|x|^2}\Big|_{\Gamma_\pm} = \frac{x_2n_2+x_1n_1}{|x|^2}\Big|_{\Gamma_\pm} = \pm\frac{h_\pm - x_2\,h'_\pm(x_2)}{|x|^2\sqrt{1+h'^2_\pm}}$$
$$= \pm\frac{\pm d_0 - \displaystyle\int_0^{x_2} t\,h''_\pm(t)\,dt}{|x|^2\sqrt{1+h'^2_\pm}}\ ,$$

from which it is clear that $b_\pm\in C^{2+l}_s(R_+,1{+}\beta)$ and

$$|b_\pm|_{C^{2+l}_s(R_+,1+\beta)} \le c\,|F|\ ;$$

4.

$$\vec{\tau}\cdot S(\vec{w})\,\vec{n}|_{\Gamma_\pm} = \frac{F}{\theta}\Big(\frac{\vec{\tau}\cdot\vec{x}}{|x|^2}\,\frac{\partial\xi_+}{\partial n} + \frac{\vec{n}\cdot\vec{x}}{|x|^2}\,\frac{\partial\xi_+}{\partial\tau} - 4\,\xi_+\,\frac{(\vec{x}\cdot\vec{\tau})\,(\vec{x}\cdot\vec{n})}{|x|^4}\Big)\Big|_{\Gamma_\pm}\ ,$$

hence, $d_\pm = -\vec{\tau}\cdot S(\vec{w})\cdot\vec{n}|_{x_1=h_\pm(x_2)}\in C^{l+1}_{s-1}(R_+,2{+}\beta),$ and

$$|d_\pm|_{C^{l+1}_{s-1}(R_+,2+\beta)} \le c\,|F|\ ;$$

5. By the Gauss formula,

$$\int_\Omega g\,dx - \int_{\Gamma_+} b_+\,dS - \int_{\Gamma_-} b\,dS = -\lim_{r\to\infty}\int_{S_+(r)}\vec{w}\cdot\vec{n}\,dS + \int_{S_1}\vec{w}\cdot\vec{n}\,dS$$
$$= -\lim_{r\to\infty}\int_{S_+(r)}\vec{w}\cdot\vec{n}\,dS + F = 0\ ;$$

6. Let $s(x) = \xi_+ p^+(x)$, $p^{(+)}(x) = -F^2/(2\,\theta^2\,|x|^2)$. Since $(\vec{v}^{(+)}, p^{(+)}(x))$ satisfy the Navier–Stokes equations, we have

$$\begin{aligned}\vec{f}(x) &\equiv \nu\,\nabla^2\vec{w} - (\vec{w}\cdot\nabla)\,\vec{w} - \nabla s \\ &= \xi_+(x)\,(1-\xi_+(x))\,(\vec{v}^{(+)}\cdot\nabla)\,\vec{v}^{(+)} - \vec{v}^{(+)}\,\xi_+(\vec{v}^{(+)}\cdot\nabla)\,\xi_+ \\ &\quad - (\vec{v}^{(+)}\xi_+\cdot\nabla)\,(\xi_-\,\vec{v}^{(-)}) - (\vec{v}^{(-)}\xi_-\cdot\nabla)\,(\xi_+\,\vec{v}^{(+)}) - (\vec{v}^{(-)}\xi_-\cdot\nabla)\,(\xi_-\,\vec{v}^{(-)}) \\ &\quad + 2\,\nu\,\nabla\vec{v}^{(+)}\,\nabla\xi_+ + \nu\Big(\vec{v}^{(+)}\,\nabla^2\xi_+ + \nabla^2(\xi_-\,\vec{v}^{(-)})\Big) - p^{(+)}\,\nabla\xi_+\ .\end{aligned}$$

It is easily seen that $\vec{f}(x)$ is a smooth vector field whose support is contained in $\Omega_-\cup\operatorname{supp}\nabla\xi_+$, and

$$|\vec{f}|_{C^l_{s-2}(\Omega,3+\beta)} \le c\,|F|\ .$$

For $\vec{u} = \vec{u} - \vec{w}$, $q = p - s$ we obtain the problem

$$(4.6)\qquad \begin{aligned} &-\nu\,\nabla^2\vec{u} + \nabla q = \vec{F}[\vec{u}] + \vec{f},\quad \nabla\cdot\vec{u} = g\,, \qquad (x\in\Omega)\ ,\\ &\vec{u}|_\Sigma = 0\,,\qquad \vec{u}\cdot\vec{n}|_{\Gamma_\pm} = b_\pm\,,\qquad \vec{\tau}\cdot S(\vec{u})\,\vec{n}|_{\Gamma_\pm} = d_\pm\ ,\\ &\vec{u}(x) = 0,\quad q(x)\to 0\,,\qquad (|x|\to\infty,\ x\in\Omega_+)\ ,\\ &|\vec{u}(x)| + |\nabla q(x)| < \infty\,,\qquad (|x|\to\infty,\ x\in\Omega_-)\ ,\end{aligned}$$

with the function

$$F[\vec{u}] = -(\vec{w}\cdot\nabla)\,\vec{u} - (\vec{u}\cdot\nabla)\,\vec{w} - (\vec{u}\cdot\nabla)\,\vec{u}$$

satisfying the inequalities

$$\begin{aligned}|F[\vec{u}]|_{C^l_{s-2}(\Omega,3+\beta)} &\le c\Big(|\vec{w}|_{C^{l+2}_s(\Omega,1)} + |\vec{u}|_{C^{l+2}_s(\Omega,1)}\Big)\,|\vec{u}|_{C^{l+2}_s(\Omega,1+\beta)}\\ &\le c\,|\vec{u}|_{C^{l+2}_s(\Omega,1+\beta)}\Big(|F| + |\vec{u}|_{C^{l+2}_s(\Omega,1+\beta)}\Big)\ ,\end{aligned}$$

$$\Big|F[\vec{u}_1] - F[\vec{u}_2]\Big|_{C^l_{s-2}(\Omega,3+\beta)} \le$$

$$\le c\,|\vec{u}_1 - \vec{u}_2|_{C^{l+2}_s(\Omega,1+\beta)}\Big(|F| + |\vec{u}_1|_{C^{l+2}_s(\Omega,1+\beta)} + |\vec{u}_2|_{C^{l+2}_s(\Omega,1+\beta)}\Big)\ .$$

Therefore, for small $|F|$, the solvability of problem (4.6) follows from Theorem 3 and the contraction mapping principle, and for the solution there can be obtained the estimate

$$(4.7)\qquad |\vec{u}|_{C^{l+2}_s(\Omega,1+\beta)} + |\nabla s|_{C^l_{s-2}(\Omega,3+\beta)} \le c\,|F|\ .$$

Such a solution is certainly unique. Clearly, $\vec{v} = \vec{w} + \vec{u}$, $p = q + s$ is a solution of problem (1.19). Estimate (4.2) follows from (4.5) and (4.7). The first statement of the theorem is proved.

To prove the second statement, we should construct a mapping $\mathcal{X}$ of Ω_1 onto Ω_2, in order to evaluate the differences $\vec{u}_1 - \vec{u}_2 \circ \mathcal{X}^{-1}$, $q_1 - q_2 \circ \mathcal{X}^{-1}$. We define it as follows:

$$x_1 = y_1 + \Phi(y), \quad x_2 = y_2\,, \qquad y \in \Omega_1\ . \tag{4.8}$$

The mapping $\mathcal{X}$ should transform $\Gamma_\pm^{(1)}$ into $\Gamma_\pm^{(2)}$ and leave Σ invariant, therefore $\Phi(y)$ should satisfy the conditions

$$\Phi|_\Sigma = 0\,, \qquad \Phi(y)\big|_{y_1 = h_\pm^{(1)}(y_2)} = h_\pm^{(2)}(y_2) - h_\pm^{(1)}(y_2)\ . \tag{4.9}$$

We set

$$\begin{aligned}\Phi(y) \;&=\; \Big(h_+^{(2)}(y_2) - h_+^{(1)}(y_2)\Big)\,\chi_+\left(\frac{y - x_+}{|y - x_+|}\right)\tilde{\lambda}_+(y)\\ &\quad+\;\Big(h_-^{(2)}(y_2) - h_-^{(1)}(y_2)\Big)\,\chi_-\left(\frac{y - x_-}{|y - x_+|}\right)\tilde{\lambda}_-(y)\,, \qquad y \in \Omega_+\ ,\\ \Phi(y) \;&=\; 0\,, \qquad y \in \Omega_-\ ,\end{aligned}$$

where

$$\tilde{\lambda}_\pm(y) \;=\; \zeta\left(\pm\,\frac{2\,y_1 - (k_+ + k_-)\,y_2}{2\,d_0 + (k_+ - k_-)\,y_2}\right)\,,$$

$\zeta(t)$ is the same function as in (3.6) and $\chi_\pm(\xi)$ are smooth functions defined on the unit circle $|\xi| = 1$ and chosen in such a way that $\chi_\pm\left(\dfrac{y - x_\pm}{|y - x_\pm|}\right) = 1$ when $y \in \Gamma_\pm$ and $\chi_\pm\left(\dfrac{y - x_\pm}{|y - x_\pm|}\right) = 0$ when $y_2 \le 0$ (this is the case if $\chi_\pm(\xi) = 1$ in a large enough neighbourhood of the point $\xi_1 = k_\pm\,\xi_2$, $\xi_2 > 0$, and $\chi_\pm(\xi) = 0$ in the neighbourhood of the points $(\mp 1, 0)$). Since $\lambda_+(y) = 1$, $\lambda_-(y) = 0$, when $y_1 > 2d_0/3 + (5\,k_+ + k_-)y_2/6$, and $\lambda_+(y) = 0$, $\lambda_-(y) = 1$, when $y_1 < -2d_0/3 + (5\,k_- + k_+)y_2/6$, our function $\Phi(y)$ satisfies (4.9). In addition, as

$$h_\pm^{(2)}(y_2) - h_\pm^{(1)}(y_2) \;=\; \int_0^{y_2} (y_2 - t)\,\Big(h_\pm^{(2)\prime\prime}(t) - h_\pm^{(1)\prime\prime}(t)\Big)\,dt\ ,$$

we have

$$\begin{aligned}|\nabla\Phi(y)| \;&\le\; c\left(\int_0^{y_2}\Big|h_+^{(2)\prime\prime}(t) - h_+^{(1)\prime\prime}(t)\Big|\,dt + \int_0^{y_2}\Big|h_-^{(2)\prime\prime}(t) - h_-^{(1)\prime\prime}(t)\Big|\,dt\right)\\ &\le\; c\left(\Big|h_+^{(2)\prime\prime} - h_+^{(1)\prime\prime}\Big|_{C^{l+1}_{s-1}(R_+,2)} + \Big|h_-^{(2)\prime\prime} - h_-^{(1)\prime\prime}\Big|_{C^{l+1}_{s-1}(R_+,2)}\right)\end{aligned}$$

(this guarantees the invertibility of $\mathcal{X}$, if $h_\pm^{(i)}$ are close to each other) and, moreover,

$$(4.10)\quad |\nabla\Phi(y)|_{C_s^{l+2}(\Omega,0)} \le c\left(\left|h_+^{(2)\prime\prime} - h_+^{(1)\prime\prime}\right|_{C_{s-1}^{l+1}(R_+,2)} + \left|h_-^{(2)\prime\prime} - h_-^{(1)\prime\prime}\right|_{C_{s-1}^{l+1}(R_+,2)}\right).$$

The proof of this inequality relies on the estimates

$$|D^j\chi_\pm| \le \frac{c}{|y-x_\pm|^{|j|}}, \qquad |D^j\lambda_\pm| \le \frac{c}{(1+|y_2|)^{|j|}}.$$

From (4.10) it follows that the elements J_{km} of the Jacobi matrix

$$J = \begin{pmatrix} \dfrac{1}{1+\Phi_{y_1}} & -\dfrac{\Phi_{y_2}}{1+\Phi_{y_2}} \\ 0 & 1 \end{pmatrix}$$

of the transformation $\mathcal{X}^{-1}$ satisfy the inequality

$$(4.11)\quad \begin{aligned} &|J_{km} - \delta_{km}|_{C_s^{l+2}(\Omega,0)} \le \\ &\le c\left(\left|h_+^{(2)\prime\prime} - h_+^{(1)\prime\prime}\right|_{C_{s-1}^{l+1}(R_+,2)} + \left|h_-^{(2)\prime\prime} - h_-^{(1)\prime\prime}\right|_{C_{s-1}^{l+1}(R_+,2)}\right). \end{aligned}$$

Consider problems (4.6) for $\vec{u} = \vec{u}_i$, $q = q_i$, $i = 1, 2$:

$$(4.12)\quad \begin{aligned} &-\nu\nabla^2\vec{u}_i + \nabla q_i = \vec{F}_i[\vec{u}_i] + \vec{f}_i, \quad \nabla\cdot\vec{u}_i = g_i(x) \quad (x\in\Omega_i), \\ &\vec{u}_i|_\Sigma = 0, \quad \vec{u}_i\cdot\vec{n}_i|_{\Gamma^{(i)}} = b_\pm^{(i)}, \quad \vec{\tau}_i\cdot S(\vec{u}_i)\,\vec{n}_i|_{\Gamma^{(i)}} = d_\pm^{(i)}, \\ &\vec{u}_i(x)\to 0, \quad q_i(x)\to 0, \quad (|x|\to\infty,\ x\in\Omega_{i+}), \\ &|\vec{u}_i(x)| + |\nabla q_i(x)| < \infty, \quad (|x|\to\infty,\ x\in\Omega_{i-}). \end{aligned}$$

Here $\vec{u}_i$, $\vec{\tau}_i$ are normal and tangential vectors to $\Gamma_\pm^i$, and $\vec{f}_i$, g_i, $b_\pm^{(i)}$, $d_\pm^{(i)}$ are expressed, as indicated above, in terms of $\vec{v}^{(+)}(x) = \vec{v}_i^{(+)}(x) = F\vec{x}/\theta_i|x|^2$ where

$$\theta_i = \arctan\left(k_+ + \int_0^\infty h_+^{(i)\prime\prime}(t)\,dt\right) - \arctan\left(k_- + \int_0^\infty h_+^{(i)\prime\prime}(t)\,dt\right).$$

It is clear that

$$(4.13)\quad |\theta_1 - \theta_2| \le c\left(\int_0^\infty |h_+^{(1)\prime\prime} - h_+^{(2)\prime\prime}|\,dt + \int_0^\infty |h_-^{(1)\prime\prime} - h_-^{(2)\prime\prime}|\,dt\right).$$

We make the change of variables (4.8) in the problem (4.12) for $\vec{u}_2$, q_2, denote the transformed functions by $\hat{\vec{u}}_2$, $\hat{q}$ etc. (in general, $\hat{f} = f\circ\mathcal{X}^{-1}$) and observe that the gradient $\nabla = \left(\dfrac{\partial}{\partial x_k}\right)_{k=1,2}$ is transformed into $\hat{\nabla} = \left(\sum_{m=1}^{2} J_{km}\dfrac{\partial}{\partial y_m}\right)_{k=1,2}$. Therefore after this change of variables we obtain

$$\begin{aligned}
&-\nu\,\hat{\nabla}^2\vec{\hat{u}}_2 + \hat{\nabla}\hat{q}_2 = \vec{\hat{F}}_2[\vec{\hat{u}}_2] + \vec{\hat{f}}_2\,, \qquad \hat{\nabla}\cdot\vec{\hat{u}}_2 = \hat{g}_2(x) \quad (x\in\Omega_1)\,,\\
&\vec{\hat{u}}_2|_\Sigma = 0\,, \qquad \vec{\hat{u}}_2\cdot\vec{n}_2|_{\Gamma_\pm^{(1)}} = b_\pm^{(2)}\,, \qquad \vec{\tau}_2\cdot\hat{S}(\vec{\hat{u}}_2)\,\vec{n}_2|_{\Gamma_\pm^{(1)}} = d_\pm^{(2)}\,,\\
&\qquad \vec{\hat{u}}_2(y)\to 0, \quad \hat{q}_2(y)\to 0\,, \qquad (|y|\to\infty,\ y\in\Omega_{1+})\,,\\
&\qquad |\vec{\hat{u}}_2(y)| + |\nabla\hat{q}_2(y)| < \infty\,, \qquad (|y|\to\infty,\ y\in\Omega_{1-})\,,
\end{aligned}$$

with $\vec{\hat{F}}_2[\vec{\hat{u}}_2] = -(\vec{\hat{w}}_2\cdot\hat{\nabla})\,\vec{\hat{u}}_2 - (\vec{\hat{u}}_2\cdot\hat{\nabla})\,\vec{\hat{w}}_2 - (\vec{\hat{u}}_2\cdot\hat{\nabla})\,\vec{\hat{u}}_2$ and

$$\hat{S}(\vec{w}) \;=\; \left(\sum_{m=1}^{2}\left(J_{mj}\,\frac{\partial \vec{w}_i}{\partial y_m} + J_{mi}\,\frac{\partial \vec{w}_j}{\partial y_m}\right)\right)_{i,j=1,2}.$$

We rewrite this problem in the form

$$\begin{aligned}
&-\nu\,\nabla^2\vec{\hat{u}}_2 + \nabla\hat{q}_2 = \vec{\hat{F}}_2[\vec{\hat{u}}_2] + \vec{\hat{f}}_2 + L_1(\vec{\hat{u}}_2,\hat{q}_2)\,, \qquad \nabla\cdot\vec{\hat{u}}_2 = \hat{g}_2(x) + L_2(\vec{\hat{u}}_2) \quad (x\in\Omega_1)\,,\\
&\vec{\hat{u}}_2|_\Sigma = 0\,, \qquad \vec{\hat{u}}_2\cdot\vec{n}_1|_{\Gamma_\pm^{(1)}} = b_\pm^{(2)} + L_3(\vec{\hat{u}}_2)\,, \qquad \vec{\tau}_1\cdot S(\vec{\hat{u}}_2)\,\vec{n}_1|_{\Gamma_\pm^{(1)}} = d_\pm^{(2)} + L_4(\vec{\hat{u}}_2)\,,\\
&\qquad \vec{\hat{u}}_2(y) = 0, \quad \hat{q}_2(y)\to 0\,, \qquad (|y|\to\infty,\ y\in\Omega_{1+})\,,\\
&\qquad |\vec{\hat{u}}_2(y)| + |\nabla\hat{q}_2(y)| < \infty\,, \qquad (|y|\to\infty,\ y\in\Omega_{1-})\,,
\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}
L_1(\vec{\hat{u}}_2,\hat{q}_2) &= \nu\,(\hat{\nabla}^2 - \nabla^2)\,\vec{\hat{u}}_2 + (\nabla - \hat{\nabla})\,\hat{q}_2\,,\\
L_2(\vec{\hat{u}}_2) &= (\nabla - \hat{\nabla})\cdot\vec{\hat{u}}_2\,,\\
L_3(\vec{\hat{u}}_2) &= \vec{\hat{u}}_2\cdot(n_1 - n_2)|_{\Gamma_\pm^{(1)}}\,,\\
L_4(\vec{\hat{u}}_2) &= \vec{\tau}_1\cdot\Big(S(\vec{\hat{u}}_2) - \hat{S}(\vec{\hat{u}}_2)\Big)\,\vec{n}_1 + \vec{\tau}_1\cdot S(\vec{\hat{u}}_2)\,\vec{n}_1 - \vec{\tau}_2\cdot S(\vec{\hat{u}}_2)\,\vec{n}_2|_{\Gamma_\pm^{(1)}}\,,
\end{aligned}$$

and we subtract relations (4.14) from the corresponding relations (4.12) for $(\vec{u}_1, q_1)$. This leads to the following problem for the differences $\vec{U} = \vec{u}_1 - \vec{\hat{u}}_2$, $Q = q_1 - \hat{q}_2$:

$$-\nu\,\nabla^2\vec{U}+\nabla Q \;=\; \vec{f}_1-\vec{\hat{f}}_2+F_1[\vec{u}_1]-\vec{\hat{F}}_2[\vec{u}_2]+L_1(\vec{\hat{u}}_2,\hat{q}_2)\;,$$
$$\nabla\cdot\vec{U}\;=\;g_1-\vec{\hat{g}}_2+L_2(\vec{\hat{u}}_2)\,,\qquad \vec{U}\cdot\vec{n}_1|_{\Gamma^{(1)}_\pm}\;=\;b^{(1)}_\pm-b^{(2)}_\pm+L_3(\vec{\hat{u}}_2)\;,$$
$$\vec{\tau}_1\cdot S(\vec{U})\,\vec{n}_1|_{\Gamma^{(1)}_\pm}\;=\;d^{(1)}_\pm-d^{(2)}_\pm+L_4(\vec{\hat{u}}_2)\;,$$
$$\vec{U}(y)\to 0,\quad Q(y)\to 0\qquad (y\to\infty,\; y\in\Omega_{1+})\;,$$
$$|\vec{U}(y)|+|\nabla Q(y)|<\infty\qquad (y\to\infty,\; y\in\Omega_{1-})\;.$$

Now, we make use of the estimate (3.3):

$$(4.15)\qquad |\vec{U}|_{C^{l+2}_s(\Omega_1,1+\beta)}+|\nabla Q|_{C^{l+1}_{s-1}(\Omega_1,2+\beta)}\;\le$$
$$\le\; c\Big(|\vec{f}_1-\vec{\hat{f}}_2|_{C^{l}_{s-2}(\Omega_1,3+\beta)}+|g_1-\hat{g}_2|_{C^{l+1}_{s-1}(\Omega_1,2+\beta)}$$
$$+\;|b_1-b_2|_{C^{l+2}_s(\Gamma,1+\beta)}+|d_1-d_2|_{C^{l+1}_{s-1}(\Gamma,2+\beta)}\Big)$$
$$+\;c\Big(\vec{F}_1[\vec{u}_1]-\vec{\hat{F}}_2[\vec{u}_2]|_{C^{l}_{s-2}(\Omega_1,3+\beta)}+|L_1(\vec{\hat{u}}_2,q_2)|_{C^{l}_{s-2}(\Omega_1,3+\beta)}$$
$$+\;|L_2(\vec{\hat{u}}_2,q_2)|_{C^{l+1}_{s-1}(\Omega_1,2+\beta)}+|L_3(\vec{\hat{u}}_2,q_2)|_{C^{l+2}_s(\Gamma,1+\beta)}+|L_4(\vec{\hat{u}}_2,q_2)|_{C^{l+1}_{s-1}(\Gamma,2+\beta)}\Big)\,.$$

We evaluate the differences $\vec{f}_1-\vec{\hat{f}}_2$, $g_1-\hat{g}_2$, b_1-b_2, d_1-d_2 using the above explicit formulas and estimate (4.13). After straightforward calculations we find that the sum of the norms of these differences does not exceed

$$c\,|F|\,\Big(|h^{(1)\prime\prime}_+-h^{(2)\prime\prime}_+|_{C^{l+1}_{s-1}(R_+,2)}+|h^{(1)\prime\prime}_--h^{(2)\prime\prime}_-|_{C^{l+1}_{s-1}(R_+,2)}\Big)\,.$$

The expressions L_k contain $\vec{\hat{u}}_2$, their derivatives and derivatives of $\hat{q}_2$ multiplied by the coefficients $J_{km}-\delta_{km}$ which satisfy the inequality (4.11) or by

$$\vec{n}_1(y_2)-\vec{n}_2(y_2)\;=\;\int_0^{y_2}\frac{d}{d\xi}\Big(\vec{n}_1(\xi_2)-\vec{n}_2(\xi_2)\Big)\,d\xi$$

satisfying inequality of the same kind:

$$(4.16)\qquad |\vec{n}_1-\vec{n}_2|_{C^{l+2}_s(R_+,0)}\;\le$$
$$\le\; c\Big(|h^{(1)\prime\prime}_+-h^{(2)\prime\prime}_+|_{C^{l+1}_{s-1}(R_+,2)}+|h^{(1)\prime\prime}_--h^{(2)\prime\prime}_-|_{C^{l+1}_{s-1}(R_+,2)}\Big)\,.$$

Using these inequalities and the estimate (4.7) for $\vec{u}_2$, we find that the sum of all the other terms in (4.15) can be evaluated by

$$c\,|F|\left(|\vec{U}|_{C_s^{l+2}(\Omega_1,1+\beta)} + |h_+^{(1)\prime\prime} - h_+^{(2)\prime\prime}|_{C_{s-2}^{l}(R_+,2)} + |h_-^{(1)\prime\prime} - h_-^{(2)\prime\prime}|_{C_{s-2}^{l}(R_+,2)}\right) .$$

Hence, if F is small, we have

$$\begin{aligned} &|\vec{U}|_{C_{s-2}^{l+2}(\Omega_1,1+\beta)} + |\nabla Q|_{C_{s-2}^{l}(\Omega_1,3+\beta)} \;\le \\ &\le\; c\,|F|\left(|h_+^{(1)\prime\prime} - h_+^{(2)\prime\prime}|_{C_{s-1}^{l+1}(R_+,2)} + |h_-^{(1)\prime\prime} - h_-^{(2)\prime\prime}|_{C_{s-1}^{l+1}(R_+,2)}\right) \end{aligned} \tag{4.17}$$

and, as a consequence,

$$|Q|_{C_{s-1}^{l+1}(\Gamma,2+\beta)} \le c\,|F|\left(|h_+^{(1)\prime\prime} - h_+^{(2)\prime\prime}|_{C_{s-1}^{l+1}(R_+,2)} + |h_-^{(1)\prime\prime} - h_-^{(2)\prime\prime}|_{C_{s-1}^{l+1}(R_+,2)}\right). \tag{4.18}$$

The last two estimates imply (4.3). Indeed,

$$\begin{aligned} t_\pm^{(i)}(y_2) \;&=\; n_i\cdot T(\vec{u}_i,q_i)\,\vec{n}_i\big|_{y_1=h_\pm^{(i)}(y_2)} \\ &+\;\left[\left(-p_i^{(+)} + 2\,\nu\,\vec{n}_i\cdot\frac{\partial\vec{v}_i^{(+)}}{\partial n_i}\right)\xi_+ + \,2\,\nu\,(\vec{n}_i\cdot\vec{v}_i^{(+)})\,\frac{\partial\xi_+}{\partial n_i}\right]\Bigg|_{y_1=h_\pm^{(i)}(y_2)} \end{aligned}$$

and

$$-p_i^{(+)} + 2\,\nu\,\vec{n}_i\cdot\frac{\partial\vec{v}_i^{(+)}}{\partial n_i} \;=\; -\frac{F^2}{2\,\theta_i^2\,|x|^2} + 2\,\nu\,\frac{F(1-n_{ir}^2)}{\theta_i\,|x|^2}\;;$$

so the differences $t_\pm^{(1)} - t_\pm^{(2)}$ are easily estimated with the help of (4.13),(4.16)–(4.18). The theorem is proved. ∎

5 – Proof of Theorem 1

According to a standard scheme, the proof of Theorem 1 reduces to the proof of the solvability of equation (1.4) for the free boundary where $(\vec{v},p)$ is a solution of auxiliary problem (4.1). Since

$$H|_{\Gamma_\pm} = \pm\,\frac{d}{dx_2}\,\frac{h'_\pm(x_2)}{\sqrt{1+h'^2_+(x_2)}}$$

this equation is equivalent to the system

$$\frac{d}{dx_2}\frac{h'_+(x_2)}{\sqrt{1+h'^2_+(x_2)}} = \frac{1}{\sigma}\, t_+(x_2)\ ,$$
$$-\frac{d}{dx_2}\frac{h'_-(x_2)}{\sqrt{1+h'^2_-(x_2)}} = \frac{1}{\sigma}\, t_-(x_2)\,, \qquad x \in R_+\ ,$$

completed with the conditions

$$h_\pm(0) = \pm d_0\,, \qquad h'_\pm(0) = k_\pm\ .$$

It is easy to show (see [1]) that $h_+(x_2)$ is expressed in terms of t_+ by the formula

$$h_+(x_2) = d_0 + \int_0^{x_2} \frac{\kappa_+ - I_+(y)}{\sqrt{1-(\kappa_+ - I_+(y))^2}}\, dy$$

where

$$\kappa_+ = \frac{k_+}{\sqrt{1+k_+^2}} + \frac{1}{\sigma}\int_0^\infty t_+(y)\,dy\,, \qquad I_+(y) = \frac{1}{\sigma}\int_y^\infty t_+(\xi)\,d\xi\ .$$

Similar formula holds for h_-:

$$h_-(x_2) = -d_0 + \int_0^{x_2} \frac{\kappa_- + I_-(y)}{\sqrt{1-(\kappa_- + I_-(y))^2}}\ ,$$
$$\kappa_- = \frac{k_-}{\sqrt{1+k_-^2}} - \frac{1}{\sigma}\int_0^\infty t_-(y)\,dy\,, \qquad I_-(y) = \frac{1}{\sigma}\int_y^\infty t_-(\xi)\,d\xi\ .$$

These formulas imply

$$h''_+(x_2) = \frac{1}{\sigma}\frac{t_+(x_2)}{\Big(1-(\kappa_+ - I_+(x_2))^2\Big)^{3/2}}\ ,$$
$$h''_-(x_2) = -\frac{1}{\sigma}\frac{t_-(x_2)}{\Big(1-(\kappa_- + I_-(x_2))^2\Big)^{3/2}}\ .$$

We consider this system as an equation for $h'' = (h''_+, h''_-)$ in the space $(C^{l+1}_{s-1}(R_+,2))^2$ which we write in the form

(5.1) $$h'' = \mathcal{Y}(h'')\ .$$

The right-hand side $\mathcal{Y}(h'') = (Y_1(h''), Y_2(h''))$ where

$$Y_1(h'') = \frac{t_+(x_2)}{\sigma\Big(1-(\kappa_+ - I_+(x_2))^2\Big)^{3/2}}\,, \qquad Y_2(h'') = -\frac{t_-(x_2)}{\sigma\Big(1-(\kappa_- + I_-(x_2))^2\Big)^{3/2}}$$

can be indeed considered as an quantity depending on h'', because $h_\pm(x_2)$ are expressed in terms of $h''_\pm$ by the formula (4.1), and $h_\pm(x_2)$ determine $\vec{v}$, p and $t_\pm(x_2)$.

Let us show that equation (5.1) is uniquely solvable, if F is small. By (4.2), we have

$$|\mathcal{Y}(h'')|_{(C^{l+1}_{s-1}(R_+,2))^2} = |Y_1(h'')|_{C^{l+1}_{s-1}(R_+,2)} + |Y_2(h'')|_{C^{l+1}_{s-1}(R_+,2)} \leq c_1\,|F|$$

and, in virtue of (4.3),

$$\begin{aligned} |\mathcal{Y}(h^{(1)''}) - \mathcal{Y}(h^{(2)''})|_{(C^{l+1}_{s-1}(R_+,2))^2} &\leq \\ &\leq c\,|F|\,\Big(|h^{(1)''}_+ - h^{(2)''}_+|_{C^{l+1}_{s-1}(\Re_+,2)} + |h^{(1)''}_- - h^{(2)''}_-|_{C^{l+1}_{s-1}(\Re_+,2)}\Big) \\ &\leq c_2\,|F|\,|h^{(1)''} - h^{(2)''}|_{(C^{l+1}_{s-1}(\Re_+,2))^2}\,. \end{aligned}$$

Hence, the operator $\mathcal{Y}$ maps the ball $|h''|_{(C^{l+1}_{s-1}(\Re_+,2))^2} \leq c_1\,|F|$ of the space $(C^{l+1}_{s-1}(R_+,2))^2$ into itself and it is a contraction operator in this ball, if $|F|$ so small that $c_2\,|F| < 1$.

It remains to apply the contraction mapping principle. Once the free boundary is bound, $(\vec{v}, p)$ are defined from the problem (1.19). The proof is completed. ∎

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