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A COUNTEREXAMPLE TO A RESULT CONCERNING CLOSURE OPERATORS

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Abstract: In 1960, José Morgado gave a necessary and sufficient condition on a poset P in order that closure operators on P, ordered pointwise, form a complete lattice. This result was based on a notion of relative quasi-infimum in posets. This note shows that Morgado's result is flawed.

1 – Introduction

One important issue of the research on closure operators (closures for short) has been the lattice-theoretic structure of the poset $uco(P)$ of all closures on a given poset P, partially ordered by standard pointwise ordering, here denoted by $\sqsubseteq$. In particular, the focus has been on results stating when $\langle uco(P), \sqsubseteq\rangle$ turns out to be a complete lattice. In fact, it is easy to observe that, in general, it is not true that $uco(P)$ is a complete lattice for any poset P (e.g., see [3, Example 1, p. 106]). The first basic result is a well-known and easy theorem by Ward [5, Theorem 4.2] and Monteiro and Ribeiro [2, Theorem 8.2]: if P is a complete lattice then $uco(P)$ is a complete lattice as well. On the other hand, the converse of this theorem does not hold, i.e., a poset P need not be a complete lattice in order that $uco(P)$ be a complete lattice: e.g., see [3, Example 6, p. 124] for an example involving an infinite poset, otherwise observe that the finite poset Q of Figure 1 is not a lattice while $uco(P)$ consists of only one closure, namely the identity operator, and therefore it is trivially a complete lattice. Baer [1, Theorem 5.3] improved the above theorem as follows: if P is a complete join semilattice (i.e.,

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least upper bounds of all nonempty subsets of P exist) then $uco(P)$ is a complete lattice; moreover, if P is a lattice then that sufficient condition becomes also necessary for $uco(P)$ to be a complete lattice. Recently, Ranzato [4, Corollary 4.6] gave a considerably improved result: if P is a directed-complete poset (i.e., least upper bounds of all directed subsets of P exist) then $uco(P)$ is a complete lattice — this result plays a useful role in theoretical computer science, see [4]. Yet, the converse of this latter result does not hold (see [4, Example 4.7]).

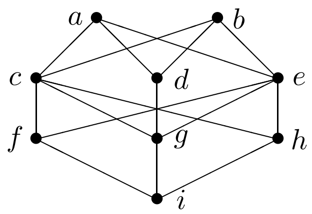

Fig. 1 – The poset Q.

In 1960, José Morgado [3, Theorem 28] gave a necessary and sufficient condition on a poset P in order that $\langle uco(P), \sqsubseteq\rangle$ is a complete lattice. Morgado's approach was based on a notion of relative quasi-infimum in posets [3, Definition 4]. This note gives a finite and fairly simple counterexample showing that Morgado's characterization is flawed, and points out where Morgado's proof fails.

2 – On the notion of relative quasi-infimum

Let us first introduce some notation. If X and Y are sets then $X \subseteq_{\varnothing} Y$ denotes that X is a nonempty subset of Y. Let $\langle P, \leq\rangle$ be a poset. If $x \in P$ then $\uparrow x \overset{\text{def}}{=} \{y \in P \mid x \leq y\}$. If $x \in P$ and $Y \subseteq P$ then we write $x \leq Y$ when, for any $y \in Y$, $x \leq y$. For any $x \in P$, we define $\mathbb{P}_x(P) \overset{\text{def}}{=} \{Y \subseteq P \mid Y \neq \varnothing,\ x \leq Y\}$. Note that if $x \leq y$ then $\mathbb{P}_y(P) \subseteq \mathbb{P}_x(P)$.

Let us recall from [3] the following definition of relative quasi-infimum in posets, which is the main new notion introduced in Morgado's paper.

Definition 2.1 ([3, Definitions 4 and 5, pp. 118, 120])**.** Let P be a poset, $x \in P$ and $Y \in \mathbb{P}_x(P)$. An element $\overset{x}{\triangle} Y \in P$ is the *quasi-infimum of Y relative to x* if the following three conditions hold:

(**i**) $x \leq \overset{x}{\triangle} Y \leq Y$;

(ii) For any $S \subseteq_{\varnothing} Y$ and $s \in P$ such that $x \leq s \leq S$, there exists some $w \in P$ such that $s \leq w \leq S$ and $\overset{x}{\vartriangle} Y \leq w$;

(iii) For any $z \in P$ satisfying the conditions (i) and (ii), $z \leq \overset{x}{\vartriangle} Y$.

If $\overset{x}{\vartriangle} Y$ exists for any $x \in P$ and $Y \in \mathbb{P}_x(P)$, then P is called *relatively quasi-inf-complete*. ▫

Thus, $\overset{x}{\vartriangle} Y$ is the greatest element in P satisfying conditions (i) and (ii) for x and Y. Note that if a relative quasi-infimum exists then this is necessarily unique.

Many proofs in Morgado's paper [3] from Section 5 forward, including the proof of the main theorem [3, Theorem 27, p. 138], make crucial use of the following property of relative quasi-infima [3, property (a), p. 118]:

$$\text{If } x \leq y,\ Z \in \mathbb{P}_y(P) \text{ (and hence } Z \in \mathbb{P}_x(P)), \text{ and both } \overset{x}{\vartriangle} Z \text{ and } \overset{y}{\vartriangle} Z \text{ exist, then } \overset{x}{\vartriangle} Z \leq \overset{y}{\vartriangle} Z. \qquad (*)$$

However, the following example shows that this property $(*)$ does not hold.

Example 2.2. Consider the finite poset Q diagrammed in Figure 1. Then, it turns out that $\overset{i}{\vartriangle}\{a,b\} = g$ while $\overset{h}{\vartriangle}\{a,b\} = h$, and therefore these equalities show that the above property $(*)$ actually is not true. Let us show how to get these relative quasi-infima.

– $\overset{i}{\vartriangle}\{a,b\} = g$. First, notice that $g \in Q$ satisfies the conditions (i) and (ii) of Definition 2.1: the only non-obvious case is condition (ii) for $S = \{a,b\}$ and either $s = f$ or $s = h$; in this case, by choosing, respectively, $w = c$ and $w = e$, condition (ii) results to be satisfied. Moreover, notice that g is the only element of Q satisfying both conditions (i) and (ii): just the elements f and h are not trivial to check; both elements do not satisfy the condition (ii) by choosing $S = \{a,b\}$ and $s = d$. Hence, g actually is the quasi-infimum of $\{a,b\}$ relative to i.

– $\overset{h}{\vartriangle}\{a,b\} = h$. This case is simpler than the previous one, as one can easily check that h is the only element of Q satisfying both conditions (i) and (ii), and thus h itself is the quasi-infimum of $\{a,b\}$ relative to h. ▫

This property $(*)$, or some consequence of it, is deeply used throughout Morgado's paper in many key proofs. Indeed, we found that Morgado's proofs are correct except for the fact of using such false property $(*)$. Thus, it is exactly the lack of property $(*)$ for relative quasi-infima that invalidates Morgado's results.

3 – A counterexample to Morgado's characterization

Let us first recall some basic definitions about closure operators. An (upper) closure operator on a poset $\langle P, \leq\rangle$ is an operator $\rho\colon P \to P$ which is isotone (i.e., $\forall\, x, y \in P$, $(x \leq y) \Rightarrow (\rho(x) \leq \rho(y))$), idempotent and increasing (i.e., $\forall\, x \in P$, $x \leq \rho(x)$). Fixed points of a closure are also called closed elements. We denote by $uco(P)$ the set of all closures on the poset P. Closures on posets are partially ordered by standard pointwise ordering, i.e., if $\rho_1, \rho_2 \in uco(P)$ then $\rho_1 \sqsubseteq \rho_2$ iff $\forall\, x \in P$, $\rho_1(x) \leq \rho_2(x)$, and therefore $\langle uco(P), \sqsubseteq\rangle$ is a poset. It turns out that any closure $\rho \in uco(P)$ is uniquely determined by its image $\rho(P)$, which coincides with the set of closed elements of ρ: if $x \in P$ then the set $\{y \in \rho(P) \mid x \leq y\}$ is nonempty and contains its greatest lower bound, let us say x_ρ, and it turns out that $\rho(x) = x_\rho$. Following Morgado's terminology [3, Definition 3, p. 115], an element $e \in P$ is called *essentially closed* when it is closed for any closure on P, i.e., for any $\rho \in uco(P)$, $\rho(e) = e$. We will denote by $ec(P)$ the set of essentially closed elements of P, i.e. $ec(P) \overset{\text{def}}{=} \bigcap_{\rho \in uco(P)} \rho(P)$.

Morgado's characterization is based on the following condition involving relative quasi-infima and closure operators. Let P be a poset.

For any $x \in P$ and for any (nonempty) family $\{\rho_i\}_{i \in I} \subseteq uco(P)$, there exists the relative quasi-infimum $\overset{x}{\triangle}((\bigcup_{i \in I} \rho_i(P)) \cap \uparrow x)$. (C)

Then, the following is the main result in Morgado's paper [3].

[3, Theorem 27, p. 138]**.** *Let P be a poset. Then, $uco(P)$ is a complete lattice if and only if P satisfies the condition* (C) *and for any $x \in P$, $\uparrow x \cap ec(P) \neq \varnothing$.* ∎

The following counterexample shows that Morgado's statement is flawed.

Example 3.1. Consider the poset R depicted in Figure 2. Since R is a finite poset, $\langle uco(R), \sqsubseteq\rangle$ is a complete lattice (e.g., see [1, Theorem 5.4]). However, it turns out that R does not satisfy the condition (C). Let us consider the closures $\rho_1, \rho_2 \in uco(R)$ defined by the following sets of closed elements: $\rho_1(P) = \{a, b\}$ and $\rho_2(P) = \{a, c\}$. Then, the quasi-infimum of $(\rho_1(P) \cup \rho_2(P)) \cap \uparrow i = \{a, b, c\}$ relative to i does not exist. In fact, for $x = i$ and $Y = \{a, b, c\}$, note that $\{g, h, i\}$ is the set of elements of R satisfying both conditions (i) and (ii) of Definition 2.1. Thus, the greatest element satisfying both conditions (i) and (ii) of Definition 2.1 does not exist, i.e. the quasi-infimum of $(\rho_1(P) \cup \rho_2(P)) \cap \uparrow i$ relative to i does not exist. This therefore contradicts the above equivalence stated in Morgado's result [3, Theorem 27, p. 138]. □

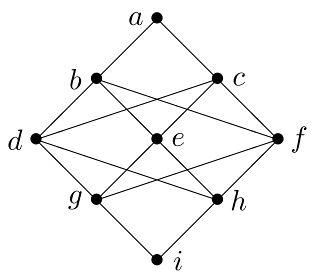

Fig. 2 – The poset R.

It is worth noting that Example 3.1 also shows that the class of relatively quasi-inf-complete posets does not include even the class of finite posets. This fact contradicts other results in Morgado's paper, like [3, Theorem 13 and successive Corollary, p. 120].

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