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A NOTE ON DUAL DISCRIMINATOR VARIETIES

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Introduction

Discriminator varieties were introduced by A. Pixley in 1971. Classical examples of discriminator varieties, amongst which is the variety $\mathcal{B}$ of Boolean algebras, may be found in the first survey on discriminator varieties, a monograph by H. Werner [W], published in 1978. Since then discriminator varieties have deserved special interest. For instance, in the book [B&S]; in a paper by S. Burris [4], attention is focused on discriminator varieties as a tool for computational research and on the reduction of first-order logic to equational logic using discriminator varieties, given by McKenzie in [13]; in a paper by B. Jónsson [10].

Dual discriminator varieties were investigated by E. Fried and A. Pixley in [7] as a generalization of discriminator varieties. The best-known example of a dual discriminator variety is the variety $\mathcal{D}$ of distributive lattices. The variety of median algebras, independently studied by many authors under different names and different approaches, is a dual discriminator variety, as pointed out by H.-J. Bandelt and J. Hedlíkova in [2]. Other important examples were studied in [7] and [8].

These varieties are congruence distributive, semisimple and have equationally definable principal congruences. A dual discriminator variety is a discriminator variety if and only if it is congruence permutable [7]. This fact suggested to Pixley the observation that certain properties of discriminator varieties might be obtained as corollaries of more general properties on dual discriminator varieties by adding the extra hypothesis of congruence permutability. Some examples of results presented in parallel for the two theories were given in [15].

In this note some results on (dual) discriminator varieties are revisited. As a consequence of analyzing the characterizations of (dual) discriminator varieties

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by equational conditions and by congruence properties, given in [13], [7] and [15], some different proofs and new characterizations came to light. In section 1, the basic definitions and results are recalled. In section 2, it is shown that in the equational characterization of dual discriminator varieties, given in [7; 3.2], one of the conditions can be omitted. It is also shown that in the equational characterization of discriminator varieties, given in [13; Th. 1.3], two conditions can be omitted. Other characterizations of (dual) discriminator varieties by congruence properties, related to those in [15; Th. DD] and [6; Th. 4.16], [15; Th. D] are established. From these characterizations we can conclude that, if a variety has the congruence extension property, being a (dual) discriminator variety may be tested by congruence properties of its free algebra on three generators. In section 3, some propositions regarding congruences of algebras of varieties more general than dual discriminator varieties are established. They provide alternate arguments for proving results of [7].

For basic facts on Universal Algebra the reader is referred to [ALV], [B&S], and [GR]. For congruence distributive varieties, including discriminator varieties and varieties with equationally definable principal congruences, the survey paper [10], which contains the necessary references, is recommended.

1 – Preliminaries

Let A be any nonvoid set. The ternary *discriminator* (resp. *dual discriminator*) *operation* d (resp. q) on A is defined by

$$d(x,y,z)=\begin{cases} z & \text{if } x=y \\ x & \text{if } x\neq y \end{cases} \qquad \left(q(x,y,z)=\begin{cases} x & \text{if } x=y \\ z & \text{if } x\neq y \end{cases} \right).$$

We first recall some important identities satisfied by these operations.

The discriminator operation d is a Pixley operation [14] ($\frac{2}{3}$-minority operation):

$$\mathrm{P}_1.\ d(x,x,z)=z\,; \qquad \mathrm{P}_2.\ d(x,y,y)=x\,; \qquad \mathrm{P}_3.\ d(x,y,x)=x\ .$$

The dual discriminator operation q is a majority operation:

$$\mathrm{M}_1.\ q(x,x,z)=x\,; \qquad \mathrm{M}_2.\ q(x,y,y)=y\,; \qquad \mathrm{M}_3.\ q(x,y,x)=x\ .$$

For any Małcev operation $p(x,y,z)$ (i.e., an operation satisfying P_1 and P_2):

$$\text{D.}\quad d(x,y,z) = p\Big(x, q(x,y,z), z\Big)\ .$$

$$\text{DD.}\quad q(x,y,z) = p\Big(x, d(x,y,z), z\Big)\ ,$$

particularly,

$$\text{DD}_1.\quad q(x,y,z) = d\Big(x, d(x,y,z), z\Big)\ .$$

$$\text{I.}\quad q\Big(x,y,q(x,y,z)\Big) = q(x,y,z)\ ,$$

$$\text{J}_1.\quad d\Big(x,y,d(x,y,z)\Big) = d(x,y,z)\ .$$

$$\text{J.}\quad d\Big(x, d(x,y,z), y\Big) = y\ .$$

For any n-ary operation f, the following identity also holds for d:

$$\text{D}_f.\quad q\Big(x,y,f(z_1,...,z_n)\Big) \;=\; q\bigg(x,y,f\Big(q(x,y,z_1),...,q(x,y,z_n)\Big)\bigg)$$

$\Big($and whenever f is an idempotent operation,

$$\text{D}'_f.\quad q\Big(x,y,f(z_1,...,z_n)\Big) \;=\; f\Big(q(x,y,z_1),...,q(x,y,z_n)\Big)\ \ \Big)\ .$$

Remarks.

(**1.1**) Identities I and D'_f imply D_f:

$$\begin{aligned} q\Big(x,y,f(z_1,...,z_n)\Big) \;&=\; q\bigg(x,y,q\Big(x,y,f(z_1,...,z_n)\Big)\bigg) \\ &=\; q\bigg(x,y,f\Big(q(x,y,z_1),...,q(x,y,z_n)\Big)\bigg)\ . \end{aligned}$$

(**1.2**) Identities J and D'_f imply D_f as in (1.1), noting that J implies J_1:

$$d\Big(x,y,d(x,y,z)\Big) \;=\; d\bigg(x, d\Big(x,d(x,y,z),y\Big), d(x,z,y)\bigg) \;=\; d(x,y,z)\ .$$

(**1.3**) If $|A| \le 2$, then the DD-operation q is a symmetric operation, i.e.,

$$\text{S}_2.\quad q(x,y,z) = q(z,y,x) \qquad \Big(\text{or}\ \ \text{S}_3.\quad q(x,y,z) = q(z,x,y)\Big)\ .$$

Conversely, if q is a DD-operation on A satisfying S_2, then $|A| \le 2$. In fact, for $a,b,c \in A$, with $a \ne b$, $c = q(a,b,c) = q(c,b,a) = a$ or b.

Analogously, a D-operation d on a set A satisfies

S_2. $d(x, y, z) = d(z, y, x)$

iff $|A| \leq 2$. In fact, for $a, b, c \in A$, with $a \neq b$, we have that $a = d(a, b, c) = d(c, b, a) = x$, where $x = c$ if $c \neq b$, and $x = a$ if $c = b$. ▫

A ternary term d (resp. q) in the language of an algebra $\mathbb{A}$ is called a *discriminator term on* $\mathbb{A}$ (resp. *dual discriminator on* $\mathbb{A}$), (a *D-term* (resp. a *DD-term*) for short) if $d^{\mathbb{A}}$ ($q^{\mathbb{A}}$) is the (dual) discriminator operation on A. An algebra is called a *discriminator algebra* (resp. *dual discriminator algebra*), (a *D-algebra* (resp. a *DD-algebra*) for short) if it has a D-term (resp. a DD-term).

A ternary term d (q) in the language of a class $\mathcal{H}$ of algebras is called a *D-term* (a *DD-term*) *on* $\mathcal{H}$ if q is a D-term (a DD-term) on every algebra in $\mathcal{H}$.

A class $\mathcal{H}$ is a *discriminator* (a *dual discriminator*) *class* (a *D-class* (a *DD-class*) for short) if there is a D-term (a DD-term) on $\mathcal{H}$.

A variety is a *discriminator* (a *dual discriminator*) *variety* (a *D-variety* (a *DD-variety*) for short), if it is generated by a D-class (a DD-class).

An algebra $(A; d)$ (resp. $(A; q)$) of type (3) is called a *pure (dual) discriminator algebra* if d (q) is the dual discriminator operation on A. The pure (dual) discriminator variety $\mathcal{PD}$ ($\mathcal{PDD}$) is the variety of type (3) generated by the class of all pure D-algebras (DD-algebras) ([2], [4]).

The following results are quoted from [7].

- [7, Th. 2.1] *Any subalgebra of a DD-algebra is simple.*
- [7, Th. 2.2] *A DD-variety is CD.*
- [7, Lem. 2.2 (iii)] *A DD-variety is a D-variety iff it is CP.*

The first characterization of DD-varieties may be proved by arguments analogous to those in [10; Th. 4.5] for D-varieties.

Theorem 1.1. *A variety* $\mathcal{V}$ *is a DD-variety if and only if* $\mathrm{Si}\,\mathcal{V}$ *is a DD-class.*

Corollary 1.2. *Let* $\mathcal{V} = \mathcal{V}(\mathcal{H})$ *be a variety generated by a DD-class* $\mathcal{H}$, *with* q *a dual discriminator term. Then, for every non-trivial algebra* $\mathbb{A} \in \mathcal{V}$, *t.f.a.e.:*

(i) $\mathbb{A} \in \mathrm{Si}\,\mathcal{V}$;

(ii) $\mathbb{A}$ *is simple;*

(iii) $\mathbb{A} \in \mathrm{ISPu}\,\mathcal{H}$;

(iv) q *is a DD-term on* $\mathbb{A}$.

By DD_1, any D-algebra is a DD-algebra, and any D-variety is a DD-variety. By P_1, P_2 and P_3, D-varieties are CD and CP [10; Th. 2.4].

A variety $\mathcal{V}$ is semisimple (SS) if every nontrivial subdirectly irreducible algebra in $\mathcal{V}$ is simple, equivalently, every algebra in $\mathcal{V}$ is a subdirect product of simple algebras (said semisimple). By the preceding results, any (D)D-variety is subsemisimple (Sub SS), i.e., is simple and every nontrivial subalgebra of a simple algebra is simple.

2 – On characterizations of (dual) discriminator varieties

In every DD-variety $\mathcal{V}$ there exists a ternary term q for which the identities M_1, M_2, M_3, I and D_f (for every basic $\mathcal{V}$-operation symbol f) hold. In the next theorem the converse is proven. This shows that, in the characterization of DD-varieties by equational conditions given in [7; Th. 3.2], the equation (c) $q(z, q(x, y, z), q(x, y, w)) = q(x, y, z)$ can be omitted. Another characterization of DD-varieties, involving an identity and a property of congruences, is also provided in the same theorem.

Theorem 2.1. *For a variety $\mathcal{V}$ and a ternary term q, t.f.a.e.:*

(**1**) *$\mathcal{V}$ is a DD-variety and q a DD-term on* $\mathrm{Si}\,\mathcal{V}$.

(**2**) *$\mathcal{V}$ satisfies the following identities:*

M_1. $q(x, x, y) = x$;

M_2. $q(x, y, y) = y$;

M_3. $q(x, y, x) = x$;

I. $q(x, y, q(x, y, z)) = q(x, y, z)$;

and, for each n-ary operation symbol f of $\mathcal{V}$,

D_f. $q(x, y, f(z_1, ..., z_n)) = q(x, y, f(q(x, y, z_1), ..., q(x, y, z_n)))$

(or D'_f. $q(x, y, f(z_1, ..., z_n)) = f(q(x, y, z_1), ..., q(x, y, z_n))$ if f induces an idempotent operation).

(**3**) *$\mathcal{V}$ satisfies the identity*

M_1. $q(x, x, y) = x$,

and for any $\mathbb{A} \in \mathcal{V}$, and $a, b \in A$, there exists $\gamma(a, b) \in \mathrm{Con}\,\mathbb{A}$ such that:

(**i**) *if $a \neq b$, $(a, b) \notin \gamma(a, b)$;*

(**ii**) *for any $c \in A$, $(c, q(a, b, c)) \in \gamma(a, b)$.*

Proof: (**1**)⇒(**2**) Obvious.

(**2**)⇒(**3**) Suppose the identities in (2) hold in $\mathcal{V}$. Let $\mathbb{A} \in \mathcal{V}$ and $a, b \in A$. The identities D_f ensure that the equivalence relation defined by $\gamma(a,b) = \{(x,y) \in A{\times}A : \, q(a,b,x) = q(a,b,y)\}$ is a congruence of $\mathbb{A}$. By M_1, $\gamma(a,a) = \nabla$, so condition (ii) holds whenever $a = b$. Assume now that $a \neq b$.

(**i**) $(a,b) \notin \gamma(a,b)$ if $a \neq b$, by M_2 and M_3.

(**ii**) $(c, q(a,b,c)) \in \gamma(a,b)$, for all $c \in A$, by the identity I.

(*****) For any $c \in A$, $(q(a,b,c), q(a,a,c)) \in \theta(a,b)$, i.e., $(q(a,b,c), a) \in \theta(a,b)$, by M_1. This, together with (ii), yields $q(a,b,c) \in c/\gamma(a,b) \cap a/\theta(a,b)$, and for any $c \in A$, $c/\gamma(a,b) \cap a/\theta(a,b) \neq \emptyset$. This yields

$$\begin{aligned} \nabla &= \gamma(a,b) \circ \theta(a,b) \circ \theta(a,b) \circ \gamma(a,b) \\ &= \gamma(a,b) \circ \theta(a,b) \circ \gamma(a,b) \\ &= \gamma(a,b) \vee \theta(a,b) \ . \end{aligned}$$

(**3**)⇒(**1**) Since M_1 holds in $\mathcal{V}$, it remains to show that, for any nontrivial $\mathbb{A} \in \mathrm{Si}\,\mathcal{V}$, and any $a,b,c \in A$, $a \neq b$ implies $q(a,b,c) = c$.

Let $\mathbb{A} \in \mathrm{Si}\,\mathcal{V}$ be nontrivial with monolith $\theta(x,y)$. Let $\gamma(x,y)$ be a congruence of $\mathbb{A}$ satisfying all the conditions in (3). So $\theta(x,y) \not\subseteq \gamma(x,y)$, which implies $\gamma(x,y) = \Delta$, since $\theta(x,y)$ is the monolith. Then, by (*), $\theta(x,y) = \nabla$ and $\mathbb{A}$ is a simple algebra.

Now let $a,b \in A$, with $a \neq b$. Let $\gamma(a,b)$ be a congruence of $\mathbb{A}$ satisfying all the conditions in (3). By the simplicity of $\mathbb{A}$, $\gamma(a,b) = \Delta$, since, by (i), $\gamma(a,b) \neq \nabla$. Hence, by (ii), $q(a,b,c) = c$, for all $c \in A$. ∎

Remarks. The proof of (2)⇒(3) combines arguments from the proofs of Theorems 3.2 and 3.8 in [7]. The proof of (3)⇒(1) could be slightly shortened by invoking a characterization of semisimple algebras from [16]: an algebra $\mathbb{A}$ is semisimple iff for each principal congruence $\theta(a,b)$, with $a \neq b$, there exists a congruence θ' such that $(a,b) \notin \theta'$ and $\theta(a,b) \vee \theta' = \nabla$. □

In the next theorem we show that two conditions in the characterization of D-varieties by equational conditions, due to R. McKenzie [13; Th. 1.3] can be omitted.

Theorem 2.2. *For a variety $\mathcal{V}$ and a ternary term d, t.f.a.e.:*

(**1**) *$\mathcal{V}$ is a D-variety with d a D-term on* $\mathrm{Si}\,\mathcal{V}$.

(**2**) *$\mathcal{V}$ satisfies the following identities:*

P_2. $d(x, y, y) = x$;

J. $d(x, d(x, y, z), y) = y$;

and, for each n-ary operation symbol f of $\mathcal{V}$,

D_f. $d(x, y, f(z_1, ..., z_n)) = d(x, y, f(d(x, y, z_1), ..., d(x, y, z_n)))$.

(**3**) *$\mathcal{V}$ satisfies the identity*

P_1. $d(x, x, y) = y$;

and, for each $\mathbb{A} \in \mathcal{V}$, and any $a, b, c \in A$, with $a \neq b$, $(a, b) \notin \theta(a, d(a, b, c))$.

Proof: (**1**)⇒(**2**) These identities hold in $\mathcal{V}$, since they hold in $\mathrm{Si}\,\mathcal{V}$ (see §1).

(**2**)⇒(**3**) First note that P_2 and J imply P_1: By J, $d(x, d(x, z, z), z) = z$, which implies $d(x, x, z) = z$, by P_2. Analogously, J and P_1 is equivalent to J and P_3.

Now, let $\mathbb{A} \in \mathcal{V}$, and $a, b \in A$, with $a \neq b$. The identities D_f ensure that the equivalence relation $\tau = \{(x, y) \in A \times A : d(a, b, x) = d(a, b, y)\} \in \mathrm{Con}\,\mathbb{A}$. To show that $\tau = \theta(a, b)$, first observe that, by P_2 and P_3, $(a, b) \in \tau$. Let $\theta \in \mathrm{Con}\,\mathbb{A}$ with $(a, b) \in \theta$. Then, for all $x, y \in A$, $(d(a, a, x), d(a, b, x)) \in \theta$, and $(d(a, a, y), d(a, b, y)) \in \theta$. By P_1, we obtain $(x, d(a, b, x)) \in \theta$ and $(y, d(a, b, y)) \in \theta$, for all $x, y \in A$. Let $(x, y) \in \tau$, i.e., $d(a, b, x) = d(a, b, y)$. Then, by the transitivity of θ, $(x, y) \in \theta$. Hence, $\tau \subseteq \theta$.

By P_3, $d(a, d(a, b, c), a) = a$; by J, $d(a, d(a, b, c), b) = b$. So $(a, b) \notin \theta(a, d(a, b, c))$.

(**3**)⇒(**1**) Let $\mathbb{A} \in \mathrm{Si}\,\mathcal{V}$ be nontrivial with monolith $\theta(a, b)$. By the hypothesis, $(a, b) \notin \theta(a, d(a, b, c))$, for all $c \in A$, since $a \neq b$. Hence $a = d(a, b, c)$, for all $c \in A$. This implies, by P_1, $(a, c) = (d(a, b, c), d(a, a, c)) \in \theta(a, b)$, for all $c \in A$. Hence $\theta(a, b) = \nabla$, and $\mathbb{A}$ is a simple algebra. Thus, for any $x, y \in A$, $x \neq y$, we have $\theta(x, y) = \nabla$, and as, by the hypothesis, $(x, y) \notin \theta(x, d(x, y, z))$, it must be $x = d(x, y, z)$, for all $z \in A$. This concludes the proof that d is a D-term on $\mathrm{Si}\,\mathcal{V}$, since $\mathcal{V}$ satisfies P_1. ∎

An algebra $\mathbb{A}$ is said to have the *principal congruence intersection property* (PCI) if whenever a principal congruence $\theta(a, b)$ has a complement θ', then $c/\theta' \cap a/\theta(a, b) \neq \emptyset$, for all $c \in A$ [15]. A variety $\mathcal{V}$ is said to have PCI if every algebra in $\mathcal{V}$ has PCI. A variety $\mathcal{V}$ has *complemented principal congruences* (PCC, for short) if each of its algebras has complemented principal congruences [15], [6].

In [15; Th. DD], DD-varieties were characterized as those CD varieties having PCC and PCI. The characterization of D-varieties as those CD varieties having PCC and CP, due to Fried and Kiss [6; Th. 4.16], was then obtained as a corollary [15; Cor. D], by using [7; Lem. 2.2(iii)].

Some properties of a variety $\mathcal{V}$, equivalent to Małcev conditions, such as being CD or CP or CD and CP, can be tested in its free algebra on three generators $\mathbb{F}_{\mathcal{V}}(3)$ [10; Th. 6.3]. Our aim will be to show that, for any variety $\mathcal{V}$ having CEP, being a (D)D-variety can be tested on the free algebra $\mathbb{F}_{\mathcal{V}}(3)$. This will follow as a corollary from a new characterization of DD-varieties close to that in [15; Th. DD]. The following propositions will be needed.

Proposition 2.3. *If a CD algebra $\mathbb{A}$ has PCC, then so has $H(\mathbb{A})$.*

Proof: Let $h\colon \mathbb{A} \to \mathbb{B}$ be an onto homomorphism. There exists an order isomorphism $\eta\colon \operatorname{Con}\mathbb{B} \to [\ker h, \nabla] \subseteq \operatorname{Con}\mathbb{A}$ [B&S; 6.20]. To $\theta(h(a), h(b)) \in \operatorname{Con}\mathbb{B}$, corresponds $\theta(a,b) \vee \ker h \in \operatorname{Con}\mathbb{A}$. If θ' is the complement of $\theta(a,b)$ in $\operatorname{Con}\mathbb{A}$, then the image of $\theta' \vee \ker h$ under η is the complement of $\theta(h(a), h(b))$ in $\operatorname{Con}\mathbb{B}$. ∎

Proposition 2.4. *Any CD and PCC variety is Sub SS.*

Proof: By [6; 4.13], for any variety $\mathcal{V}$, CD+PCC ⇔ EDPC+SS. By [6; 4.5], EDPC ⇒ CEP. Thus, every nontrivial subalgebra of a simple algebra is simple. ∎

Theorem 2.5. *For a variety $\mathcal{V}$, the following are equivalent:*

(**1**) *$\mathcal{V}$ is a DD-variety.*

(**2**) *$\mathcal{V}$ is CD, PCC and PCI.*

(**3**) *$\mathcal{V}$ is PCC and $\mathbb{F}_{\mathcal{V}}(3)$ is CD and PCI.*

(**4**) *Every n-generated ($n \le 3$) subalgebra of a subdirectly irreducible algebra of $\mathcal{V}$ is in $\operatorname{Si}\mathcal{V}$, and $\mathbb{F}_{\mathcal{V}}(3)$ is CD, PCC and PCI.*

Proof: (**1**)⇔(**2**) [15; Th. DD].

(**2**)⇒(**3**) Obvious.

(**3**)⇒(**4**) By [10; 6.3], $\mathcal{V}$ is CD. By Proposition 2.4, $\mathcal{V}$ is Sub SS.

(**4**)⇒(**1**) By [10; 6.3], $\mathcal{V}$ is CD. We want to show the existence of a DD-term on every $\mathbb{A} \in \operatorname{Si}\mathcal{V}$.

Let $\mathbb{F} = \mathbb{F}_{\mathcal{V}}(\{x,y,z\})$ be a $\mathcal{V}$-free algebra freely generated by $\{x,y,z\}$. Let γ be the complement of $\theta(x,y)$ ($\gamma \ne \nabla$ exists by PCC, and is unique by CD). By PCI, there exists $q = q(x,y,z) \in F$ such that

(**i**) $\Big(z, q(x,y,z)\Big) \in \gamma$,

(**ii**) $\Big(q(x,y,z), x\Big) \in \theta(x,y)$.

Let $\mathbb{A} \in \mathcal{V}$, $a, c \in A$, and $\varphi : \mathbb{F} \to \mathbb{A}$ be the homomorphism defined by $a = \varphi(x) = \varphi(y)$ and $c = \varphi(z)$. As $\theta(x, y) \subseteq \ker \varphi$, we must have $(q, x) \in \ker \varphi$, by (ii), i.e., $q(a, a, c) = \varphi(q(x, y, z)) = \varphi(x) = a$.

Now, let $\mathbb{A} \in \operatorname{Si} \mathcal{V}$ be nontrivial. For any $a, b, c \in A$, with $a \neq b$, let $\psi \colon \mathbb{F} \to \mathbb{A}$ be the homomorphism defined by $\psi(x) = a$, $\psi(y) = b$, $\psi(z) = c$. By the hypothesis, $\psi(\mathbb{F}) \in \operatorname{Si} \mathcal{V}$, since $\psi(\mathbb{F}) \in S(\mathbb{A})$ is n-generated, $n \leq 3$. Moreover, $\psi(\mathbb{F})$ is a simple algebra since, by PCC, $\mathbb{F}$ is SS. Thus $\nabla = \theta(a, b) \in \operatorname{Con} \psi(\mathbb{F})$. By Proposition 2.3, $(\psi \times \psi)(\gamma) \in \operatorname{Con} \psi(\mathbb{F})$ is the complement of $\theta(a, b)$, so $(\psi \times \psi)(\gamma) = \Delta$. Hence, by (i), $c = \psi(z) = \psi(q) = q(a, b, c)$. This shows that q is a DD-term on $\operatorname{Si} \mathcal{V}$. ∎

Corollary 2.6. *For a variety $\mathcal{V}$ with CEP, the following are equivalent:*

(i) *$\mathcal{V}$ is a DD-variety.*

(ii) *$\mathbb{F}_{\mathcal{V}}(3)$ is CD, PCC and PCI.*

The corresponding characterizations of D-varieties (including that given in [15; Th. D] or [6; 4.16]) are next given.

Theorem 2.7. *For a variety $\mathcal{V}$, t.f.a.e.:*

(1) *$\mathcal{V}$ is a D-variety.*

(2) *$\mathcal{V}$ is CD, PCC and CP.*

(3) *$\mathcal{V}$ is PCC and $\mathbb{F}_{\mathcal{V}}(3)$ is CD and CP.*

(4) *Every n-generated, $n \leq 3$, subalgebra of a subdirectly irreducible of $\mathcal{V}$ is in $\operatorname{Si} \mathcal{V}$, and $\mathbb{F}_{\mathcal{V}}(3)$ is CD, PCC and CP.*

Corollary 2.8. *For a variety $\mathcal{V}$ with CEP, the following are equivalent:*

(i) *$\mathcal{V}$ is a D-variety.*

(ii) *$\mathbb{F}_{\mathcal{V}}(3)$ is CD, PCC and CP.*

Remark. The results of this section were presented, without proof, in a talk held at the Conference LOGICA98 [17], in honor to late Professor António Aniceto Monteiro. In this talk, equational bases for the variety $\mathcal{PDD}$ ($\mathcal{PD}$) and for its least non trivial subvariety were also presented (the later by using (1.3)). □

3 – On results related to some properties of DD-varieties

The proof that any DD-variety $\mathcal{V}$ has PCC, given in [7; 3.8], is based in a proposition concerning CD varieties. First it is observed that $D_1(x,y,z,t) = q(x,y,z)$, $D_2(x,y,z,t) = q(x,y,t)$ is a pair of terms such that (P): for any $\mathbb{A} \in \mathrm{Si}\,\mathcal{V}$ and $a,b,c,d \in A$, $D_1(a,b,c,d) = D_2(a,b,c,d) \Leftrightarrow a = b$ or $c = d$. Proposition 2.9 in [1] ensures that such terms form a pair of principal intersection terms, i.e., for any $\mathbb{A} \in \mathcal{V}$, and any $a,b,c,d \in A$, $\theta(a,b) \cap \theta(c,d) = \theta(q(a,b,c),\, q(a,b,d))$. This fact was then used to deduce that, for any algebra $\mathbb{A} \in \mathcal{V}$ and any $a,b \in A$, $\gamma(a,b) \cap \theta(a,b) = \Delta$. So $\gamma(a,b)$ and $\theta(a,b)$ are complements of each other.

An alternate way of showing that $\theta(a,b) \cap \gamma(a,b) = \Delta$, without needing congruence distributivity, follows as a corollary of (1)⇒(2) in either of the following two propositions. The definition of $\gamma(a,b)$ by equations is crucial in their proofs.

Proposition 3.1. *Let $\mathcal{V}$ be a variety and $\{D_i(x,y,z,t),\ D_{i'}(x,y,z,t)\colon\ i \in I\}$ be any family of quaternary terms. Then, t.f.a.e.:*

(**1**) *For any $\mathbb{A} \in \mathrm{Si}\,\mathcal{V}$ and any $a,b,c,d \in A$,*

$$D_i(a,b,c,d) = D_{i'}(a,b,c,d) \ \ \text{for all } i \in I \quad \Longleftrightarrow \quad a = b \ \text{ or } \ c = d\ .$$

(**2**) *For any $\mathbb{A} \in \mathcal{V}$ and any $a,b \in A$,*

$$\gamma(a,b) = \Big\{(x,y)\colon\ D_i(a,b,x,y) = D_{i'}(a,b,x,y) \ \ \text{for all } i \in I\Big\} \in \mathrm{Con}\,\mathbb{A}\ ,$$
$$\gamma(a,a) = \nabla \quad \text{and} \quad \theta(a,b) \cap \gamma(a,b) = \Delta\ .$$

Proof: (**1**)⇒(**2**) Let $\mathbb{A} \in \mathcal{V}$ and $a,b \in A$. To check that $\gamma(a,b) \in \mathrm{Con}\,\mathbb{A}$, it suffices to use (1), observing that $(x,y) \in \gamma(a,b)$ iff, for every onto homomorphism $h\colon \mathbb{A} \to \mathbb{B}$, where $\mathbb{B} \in \mathrm{Si}\,\mathcal{V}$, $(h(x),h(y)) \in \gamma(h(a),h(b))$.

Obviously (1) implies $\gamma(a,a) = \nabla$. As $\theta(a,a) = \Delta$, then (2) holds whenever $a = b$. Assume now that $a \neq b$. For any c.m.i. $\rho \in \mathrm{Con}\,\mathbb{A}$, we have that

$$\begin{aligned}\theta(a,b) \not\subseteq \rho \ \Leftrightarrow\ (a,b) \notin \rho \ &\Leftrightarrow\ \widetilde{a} \neq \widetilde{b}, \quad \text{where } \widetilde{a} \in A/\rho \text{ denotes the } \rho\text{-class of } a\\ &\Leftrightarrow\ \gamma(\widetilde{a},\widetilde{b}) = \Delta, \quad \text{by (1), since } \mathbb{A}/\rho \in \mathrm{Si}\,\mathcal{V}\\ &\Leftrightarrow\ \gamma(a,b) \subseteq \rho\ .\end{aligned}$$

Hence, $\theta(a,b) \cap \gamma(a,b) = \Delta$.

(**2**)⇒(**1**) Let $\mathbb{A} \in \mathcal{V}$ and $a,b \in A$. By the reflexivity of $\gamma(a,b)$, $D_i(a,b,c,c) = D_{i'}(a,b,c,c)$ for all $c \in A$ and all $i \in I$. As $\gamma(a,a) = \nabla$, then $D_i(a,a,c,d) = D_{i'}(a,a,c,d)$ for all $c,d \in A$ and all $i \in I$.

Assume now that $\mathbb{A} \in \mathrm{Si}\,\mathcal{V}$, $a, b, c, d \in A$, $a \neq b$ and $D_i(a,b,c,d) = D_{i'}(a,b,c,d)$ for all $i \in I$. By (2), $\theta(a,b) \cap \gamma(a,b) = \Delta$. As $\theta(a,b) \neq \Delta$, then $\gamma(a,b) = \Delta$, and $c = d$. ∎

Proposition 3.2. *Let $\mathcal{V}$ be a variety for which there exists a family of quaternary terms $\{D_i(x,y,z,t), D_{i'}(x,y,z,t)\colon\ i \in I\}$ such that, for any $\mathbb{A} \in \mathcal{V}$ and any $a, b \in A$,*

$$\gamma(a,b) \;=\; \Big\{(x,y)\colon\ D_i(a,b,x,y) = D_{i'}(a,b,x,y), \quad \text{for all } i \in I\Big\} \;\in\; \mathrm{Con}\,\mathbb{A}\ ,$$
$$\theta(a,b) \vee \gamma(a,b) = \nabla\ .$$

Then t.f.a.e.:

(**1**) *For any $\mathbb{A} \in \mathcal{V}$ and $a, b \in A$, with $a \neq b$, $(a,b) \notin \gamma(a,b)$.*

(**2**) *For any $\mathbb{A} \in \mathcal{V}$ and $a, b \in A$, $\theta(a,b) \cap \gamma(a,b) = \Delta$.*

(**3**) *For any $\mathbb{A} \in \mathrm{Si}\,\mathcal{V}$, and $a, b, c, d \in A$,*

$$D_i(a,b,c,d) = D_{i'}(a,b,c,d) \ \text{ for all } i \in I \quad \Longleftrightarrow \quad a = b \ \text{ or } \ c = d\ .$$

Proof: (**1**)⇒(**2**): Let $\mathbb{A} \in \mathcal{V}$ and $a, b \in A$. If $a = b$, then the claim is obvious. Assume now that $a \neq b$. By [16] $\mathbb{A}$ is semisimple, so Δ is the intersection of maximal congruences. For any maximal congruence ρ, we have that

$$\begin{aligned}(a,b) \notin \rho \ \Leftrightarrow\ \widetilde{a} \neq \widetilde{b} \ \text{ in } A/\rho \ \Leftrightarrow\ & \gamma(\widetilde{a},\widetilde{b}) \neq \nabla \quad && \text{since, by (1), } (\widetilde{a},\widetilde{b}) \notin \gamma(\widetilde{a},\widetilde{b}) \\ \Leftrightarrow\ & \gamma(\widetilde{a},\widetilde{b}) = \Delta && \text{since } \mathbb{A}/\rho \text{ is simple}\ .\end{aligned}$$

The remaining runs like in (1)⇒(2) of Proposition 3.1.

(**2**)⇒(**3**): The same proof as in (2)⇒(1) in Proposition 3.1.

(**3**)⇒(**1**): Let $\mathbb{A} \in \mathcal{V}$, and $a, b \in A$, with $a \neq b$. We want to show that $(a,b) \notin \gamma(a,b)$. If $a \neq b$, there exists an onto homomorphism $h : \mathbb{A} \to \mathbb{B}$, with $\mathbb{B} \in \mathrm{Si}\,\mathcal{V}$, such that $h(a) \neq h(b)$. By (3), $D_i(h(a),h(b),h(a),h(b)) \neq D_{i'}(h(a),h(b),h(a),h(b))$, for some $i \in I$. Then, for some $i \in I$, $D_i(a,b,a,b) \neq D_{i'}(a,b,a,b)$. ∎

In [3], Blok and Pigozzi sharpened [1; 2.9], showing that, if in a variety $\mathcal{V}$ the meet of principal congruences of any of its algebras is a compact congruence constructed in a uniform way from the generators of the principal congruences, through a finite family of quaternary terms, then $\mathcal{V}$ is CD. Proposition 3.4 below extends [1; 2.9] and [3; Th. 1.3] by dropping the finiteness of that family of terms.

The proof relies on a characterization of distributive congruence lattices relying on the one given in [11].

Lemma 3.3. *Let* $\mathrm{Con}\,\mathbb{A}$ *be the congruence lattice of an algebra* $\mathbb{A}$*. T.f.a.e.:*

(**1**) $\mathrm{Con}\,\mathbb{A}$ *is distributive;*

(**2**) *For every* $\rho, \alpha, \beta \in \mathrm{Con}\,\mathbb{A}$, ρ *c.m.i., if* $\alpha \wedge \beta \leq \rho$ *then either* $\alpha \leq \rho$ *or* $\beta \leq \rho$.

(**3**) *For every* $a, b, c, d \in A$, *and* $\rho \in \mathrm{Con}\,\mathbb{A}$, *where* ρ *c.m.i., if* $\theta(a,b) \wedge \theta(c,d) \leq \rho$ *then either* $\theta(a,b) \leq \rho$ *or* $\theta(c,d) \leq \rho$.

Proof: (**1**)⇔(**2**): See [11; Lemma 2.1].

(**2**)⇒(**3**): Obvious.

(**3**)⇒(**2**): Let $\alpha = \bigvee(\theta(a_j, b_j)\colon j \in J)$, $\beta = \bigvee(\theta(c_k, d_k)\colon k \in K)$ and $\alpha \wedge \beta \leq \rho$. Assume that $\alpha \not\leq \rho$. Then, for some $j_1 \in J$, $\theta(a_{j_1}, b_{j_1}) \not\leq \rho$. However, for all $k \in K$, $\theta(a_{j_1}, b_{j_1}) \wedge \theta(c_k, d_k) \leq \alpha \wedge \beta \leq \rho$. By (3), $\theta(c_k, d_k) \leq \rho$, for all $k \in K$. Hence $\beta \leq \rho$. ∎

Proposition 3.4. *For any variety* $\mathcal{V}$*, t.f.a.e.:*

(**1**) $\mathcal{V}$ *is CD and there exists a family of terms* $\{D_i(x,y,z,w),\, D_{i'}(x,y,z,w)\colon i \in I\}$ *such that, (P) for any* $\mathbb{A} \in \mathrm{Si}\,\mathcal{V}$,

$$D_i(a,b,c,d) = D_{i'}(a,b,c,d) \ \text{ for all } i \in I \quad \Longleftrightarrow \quad a = b \ \text{ or } \ c = d\ .$$

(**2**) *There exists a family of terms* $\{D_i(x,y,z,w),\, D_{i'}(x,y,z,w)\colon i \in I\}$*, such that for all* $\mathbb{A} \in \mathcal{V}$ *and* $a, b, c, d \in A$,

$$\theta(a,b) \cap \theta(c,d) \ = \bigvee_{i \in I} \Big(\theta\Big(D_i(a,b,c,d), D_{i'}(a,b,c,d)\Big)\Big)\ .$$

Proof: We first note that for any $\mathbb{A} \in \mathcal{V}$, any $a, b, c, d \in A$, and any $\rho \in \mathrm{Con}\,\mathbb{A}$,

$$\bigvee_{i \in I} \theta\Big(D_i(a,b,c,d),\, D_{i'}(a,b,c,d)\Big) \ \leq\ \rho \quad \Longleftrightarrow$$
$$\Longleftrightarrow \quad D_i(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = D_{i'}(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) \quad \text{for all} \ \ i \in I, \ \ \text{in } \mathbb{A}/\rho\ .$$

(**1**)⇒(**2**): Let $\mathbb{A} \in \mathcal{V}$ and $a, b, c, d \in A$. We will show that, for any c.m.i. $\rho \in \mathrm{Con}\,\mathbb{A}$,

$$\theta(a,b) \cap \theta(c,d) \leq \rho \quad \Longleftrightarrow \quad \bigvee_{i \in I} \Big(\theta\Big(D_i(a,b,c,d),\, D_{i'}(a,b,c,d)\Big)\colon i \in I\Big) \ \leq\ \rho\ .$$

Let $\rho \in \operatorname{Con}\mathbb{A}$ be c.m.i.. Then, we have that

$$\bigvee_{i\in I} \theta\Big(D_i(a,b,c,d),\, D_{i'}(a,b,c,d)\Big) \le \rho \iff$$
$$\iff D_i(\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d}) = D_{i'}(\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d}), \quad \text{for all } i \in I$$
$$\iff \widetilde{a}=\widetilde{b} \text{ or } \widetilde{c}=\widetilde{d} \quad \text{in } A/\rho, \text{ by (P)}$$
$$\iff \theta(a,b) \le \rho \text{ or } \theta(c,d) \le \rho$$
$$\iff \theta(a,b) \cap \theta(c,d) \le \rho, \quad \text{by CD}\,.$$

(**2**)⇒(**1**): We will show that $\mathcal{V}$ is CD by using Lemma 3.3. Assume that $\mathbb{A} \in \mathcal{V}$, $a,b,c,d \in A$, and $\rho \in \operatorname{Con}\mathbb{A}$ is c.m.i.. Then

$$\theta(a,b) \wedge \theta(c,d) = \bigvee_{i\in I} \theta\Big(D_i(a,b,c,d),\, D_{i'}(a,b,c,d)\Big) \le \rho \iff$$
$$\iff D_i(\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d}) = D_{i'}(\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d}), \quad \text{for all } i \in I, \text{ in } A/\rho$$
$$\iff \theta(\widetilde{a},\widetilde{b}) \wedge \theta(\widetilde{c},\widetilde{d}) = \bigvee \theta\Big(D_i(\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d}),\, D_{i'}(\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d})\Big) = \Delta$$
$$\iff \theta(\widetilde{a},\widetilde{b}) = \Delta \text{ or } \theta(\widetilde{c},\widetilde{d}) = \Delta \quad \text{since } A/\rho \in \operatorname{Si}\mathcal{V}$$
$$\iff \theta(a,b) \le \rho \text{ or } \theta(c,d) \le \rho\,.$$

Arguments in the previous proof ensure the second claim (P). ∎

If $\mathcal{V}$ is a DD-variety and q a DD-term, then, for any $\mathbb{A}\in\mathcal{V}$ and any $a,b,x,y\in A$, $(x,y) \in \theta(a,b) \Leftrightarrow x = q(x,y,q(a,b,x))$ and $y = q(x,y,q(a,b,y))$ (implicit in [7; pg. 92]), i.e., $\mathcal{V}$ has EDPC (see [5] or [12] for definition). In the following proposition the same conclusion is obtained under weaker conditions.

Proposition 3.5. *Let $\mathcal{V}$ be a variety having a term $t(x,y,z)$ for which I in Theorem 2.1 holds, and (P) for any $\mathbb{A} \in \operatorname{Si}\mathcal{V}$, $t(a,b,c) = t(a,b,d) \Leftrightarrow a = b$ or $c = d$. Then $\mathcal{V}$ has EDPC. More precisely, for any $\mathbb{A} \in \mathcal{V}$, and any $a,b,x,y \in A$,*

$$(x,y) \in \theta(a,b) \iff$$
$$\iff t(x,y,x) = t\Big(x,y,t(a,b,x)\Big) \quad \text{and} \quad t(x,y,y) = t\Big(x,y,t(a,b,y)\Big)\,.$$

Proof: Let $\mathbb{A}\in\mathcal{V}$ and $a,b\in A$. We will show that, for each c.m.i. congruence ρ

$$(a,b) \in \rho \iff$$
$$\iff \theta = \Big\{(x,y)\colon\, t(x,y,x) = t\Big(x,y,t(a,b,x)\Big),\; t(x,y,y) = t\Big(x,y,t(a,b,y)\Big)\Big\} \subseteq \rho\,,$$

noting that $\theta \in \operatorname{Con}\mathbb{A}$, by (P).

($\Rightarrow$) Let $(a,b)\in\rho$, i.e., $\widetilde{a}=\widetilde{b}$ in $\mathbb{A}/\rho\in \mathrm{Si}\,\mathcal{V}$. If $(x,y)\in\theta$, then

$$t(\widetilde{x},\widetilde{y},\widetilde{x}) = t\Big(\widetilde{x},\widetilde{y},t(\widetilde{a},\widetilde{b},\widetilde{x})\Big)\,, \qquad t(\widetilde{x},\widetilde{y},\widetilde{y}) = t\Big(\widetilde{x},\widetilde{y},t(\widetilde{a},\widetilde{b},\widetilde{y})\Big)$$

which implies, by (P), $t(\widetilde{x},\widetilde{y},\widetilde{x}) = t(\widetilde{x},\widetilde{y},\widetilde{y})$ and $\widetilde{x}=\widetilde{y}$, i.e., $(x,y)\in\rho$.

($\Leftarrow$) By I, $t(a,b,a)=t(a,b,t(a,b,a))$, $t(a,b,b)=t(a,b,t(a,b,b))$, i.e., $(a,b)\in\theta\subseteq\rho$. ∎

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