PORTUGALIAE MATHEMATICA
Vol. 58 Fasc. 3 – 2001
Nova Série

# A ONE-DIMENSIONAL FREE BOUNDARY PROBLEM ARISING IN COMBUSTION THEORY *

S.N. Antontsev, A.M. Meirmanov and V.V. Yurinsky

*Presented by J.F. Rodrigues*

**Abstract:** The free boundary problem considered in this paper arises in the mathematical theory of combustion. It consists in finding two functions $p^{\pm}(x,t)$ defined in their respective domains $\Pi^{\pm}_T = \bigcup_{0<t<T} \Pi^{\pm}(t)$, with $\Pi^-(t) = \{-1 < x < R(t)\}$ and $\Pi^+(t) = \{R(t) < x < 1\}$, that are separated by the free boundary $\Gamma_T = \{x = R(t),\ t \in (0,T)\}$. In $\Pi^{\pm}_T$, the functions satisfy heat equations with different heat capacities, and on the free boundary they obey the conjugation conditions

$$p^+(x,t) = p^-(x,t) = 0\,, \qquad \frac{\partial p^+(x,t)}{\partial x} - \frac{\partial p^-(x,t)}{\partial x} = \beta\,, \qquad x = R(t)\ .$$

Typically, the free boundary can be viewed as a model of the flame front separating the burnt and unburned domains, and $p^{\pm}$ are temperatures in these domains.

The article is dedicated to the study of the problem of existence of global-in-time classical solutions, the large-time asymptotic behavior of such solutions, and the comparison principle.

It includes some remarks on the modification of the above problem where the conjugation conditions on the free boundary specify not the jump of the temperature gradient, but the jump of its square.

*Received*: June 19, 1999; *Revised*: March 2, 2000.

* Work supported by project PRAXIS XXI 2/2.1/MAT/53/94 “Modelos Matemáticos com Fronteiras Livres”.

## 1 – Introduction

The one-dimensional free boundary problem considered in this paper arises in the mathematical theory of combustion. The model problem is that of finding the free boundary $\Gamma_T = \{x = R(t),\ t \in (0,T)\}$ that separates the domains

$$\Pi^\pm_T = \bigcup_{0<t<T} \Pi^\pm(t) ,$$
$$\Pi^-(t) = \Big\{-1 < x < R(t)\Big\} ,$$
$$\Pi^+(t) = \Big\{R(t) < x < 1\Big\} ,$$

and the functions $p^\pm(x,t)$ defined in the respective domains $\Pi^\pm$. These functions satisfy the equations

$$(1.1) \qquad L^\pm p^\pm \equiv c^\pm \frac{\partial p^\pm}{\partial t} - \frac{\partial^2 p^\pm}{\partial^2 x} = 0 \qquad \text{in } \Pi^\pm_T ,$$

$$(1.2) \qquad p^+(x,t) = p^-(x,t) = 0 \qquad \text{on } \Gamma_T = \Big\{x = R(t),\ t \in (0,T)\Big\} ,$$

$$(1.3) \qquad \frac{\partial p^+(x,t)}{\partial x} - \frac{\partial p^-(x,t)}{\partial x} = \beta \qquad \text{on } \Gamma_T = \Big\{x = R(t),\ t \in (0,T)\Big\} ,$$

$$(1.4) \qquad p^\pm(\pm 1, t) = \pm\alpha^\pm , \qquad t \in (0,T) ,$$

$$(1.5) \qquad p^\pm(x,0) = p^\pm_0(x) , \qquad x \in (-1,1), \quad R(0) = R_0 ,$$

where $R_0 \in (-1,1)$ and $c^\pm$, $\beta$, $\pm\alpha$, are some positive constants.

In the above, $p(x,t)$ is interpreted as scaled temperature, and the free boundary $\Gamma_T$ is its smooth level line $p(x,t) = 0$, $(\pm p^\pm(x,t) > 0,\ (x,t) \in \Pi^\pm_T)$, corresponding to the flame front that separates the burnt region from the unburned one.

Condition (1.3) takes into account the heat released by the combustion process: essentially, it states that the amount of heat released locally is proportional to the area of the flame front. For more information on the physical background of the above models and recent results, we refer the reader to [ANT92, BL91, L91, BH1, BH2, BLS92, CLW97, LW97, VA96, LVW97, LVW00].

Below, we investigate the global-in-time existence of classical solutions, the large-time asymptotic behavior of the solutions, and the comparison principle for the problem (1.1)–(1.5). These considerations constitute the body of the article.

Another version of problem (1.1)–(1.5) that can be studied in a similar manner is one with condition (1.3) replaced by

$$\left(\frac{\partial p^+(x,t)}{\partial x}\right)^2 - \left(\frac{\partial p^-(x,t)}{\partial x}\right)^2 = \beta \quad \text{on} \;\; \Gamma_T = \Big\{x = R(t),\; t \in (0,T)\Big\} \; . \tag{1.6}$$

The alterations that should be made in the arguments to treat the solutions to equations (1.1), (1.2), (1.4), (1.5), and (1.6) instead of (1.3) are indicated in a separate section.

## 2 – Main results

We assume that

$$p_0^-(x) < 0\,, \qquad p_0^+(x) > 0\,, \qquad \frac{dp_0^\pm(x)}{dx} \geq \alpha_0 > 0 \tag{2.1}$$

and for some $\lambda > 0$

$$p_0^\pm(x) \in H^{2+\lambda}(\Pi^\pm(0))\,, \qquad \|p_0^\pm\|_{H^{2+\lambda}(\Pi^\pm(0))} \leq \alpha_1 \; . \tag{2.2}$$

We also assume that the standard compatibility conditions are satisfied:

$$\begin{gathered} p_0^\pm = 0, \;\; \frac{dp_0^+}{dx} - \frac{dp_0^-}{dx} = \beta \quad \text{on} \;\; x = R(0) \;, \\ \frac{d^2 p_0^\pm(\pm 1, t)}{d^2 x} = 0 \;, \end{gathered} \tag{2.3}$$

$$\frac{p_{0xx}^+}{c^+\, p_{0x}^+} = \frac{p_{0xx}^-}{c^-\, p_{0x}^-} \quad \text{on} \;\; x = R(0) \; . \tag{2.4}$$

It is easy to verify that the function

$$p_\infty^\pm(x) \;=\; \alpha^\pm \, \frac{x - R_\infty}{1 \pm R_\infty} \; . \tag{2.5}$$

is the unique solution of the corresponding stationary problem (1.1)–(1.4). Here the number $R_\infty$ is determined uniquely from the equation

$$g(R_\infty) \;\equiv\; \frac{\alpha^+}{1 - R_\infty} - \frac{\alpha^-}{1 + R_\infty} - \beta \;=\; 0 \;, \tag{2.6}$$

and it is easy to see that

$$g'(\tau) \;\equiv\; \frac{\alpha^+}{(1-\tau)^2} + \frac{\alpha^-}{(1+\tau)^2} \;>\; 0\,, \qquad \tau \in (-1,1), \quad g(\pm) = \pm\infty \; .$$

**Theorem 2.1.** *Assume that conditions (2.1)–(2.4) are satisfied. Then the problem (1.1)–(1.5) has a classical solution $p^{\pm}(x,t)$, defined for all $t>0$, which is unique and has the following properties:*

$$p^{\pm}(x,t) \in H^{2+\lambda,\frac{2+\lambda}{2}}(\overline{\Pi}^{\pm}_{\infty})\,, \qquad R(t) \in H^{1+\lambda}(0,\infty)\;, \tag{2.7}$$

*and there exist positive constants $C_0$, $\lambda_0$ such that*

$$\left|\ln \frac{\partial p^{\pm}}{\partial x}(x,t)\right| \le C_0\,, \qquad 0 \le \frac{\partial p^{\pm}}{\partial x}(x,t)\;. \tag{2.8}$$

$$|R(t)-R_{\infty}| + |p^{\pm}(x,t) - p^{\pm}_{\infty}(x)| \;\le\; C_0\, e^{-\lambda_0 t}\;. \tag{2.9}$$

The proof of the theorem includes two preparatory steps.

At the first step, we introduce, following [MEI92, MPS97], the von Mises variables and reduce the original initial problem with free boundary to one where the boundary of the domain is known. To justify this transformation, we use the properties of the solution specified in (2.8).

At the second step, we prove some comparison principles and a priori estimates for the solution which are valid for any finite time $t$.

The proof of the theorem, which establishes that the solution exists globally in time, is based on these estimates and the local existence results of the paper [AG94].

## 3 – Behavior of solutions and comparison principles

### 3.1. Equivalent problem in von Mises variables

For the classical solutions of problem (1.1)–(1.5), the maximum principle is valid in its strict form:

- the function $p^-(x,t)$ attains its minimum on the boundary $x=-1$, and its derivative is positive there, $(\partial/\partial x)\,p^-(-1,t)>0$;
- $p^-(x,t)$ attains its maximum at the boundary $x=R(t)$ and $(\partial/\partial x)\,\cdot$ $\cdot\, p^-(R(t),t)>0$ at this point.

It follows, after (2.1) is taken into consideration, that

$$\frac{\partial p^-}{\partial x}(x,t) > 0 \quad \text{in} \;\; \Pi^+_T\;.$$

A similar argument holds true for $p^+(x,t)$ in $\Pi^+_T$.

Thus, in the domains where $(\partial/\partial x)\, p^{\pm} > 0$ it is possible to change the variables to

$$t = t\,, \quad y = p^{\pm}(x,t), \quad x \in \Pi_T^{\pm}\,. \tag{3.1}$$

In the new variables, the images of $\Pi_T^{\pm}$ are the known domains

$$\Omega_T^{\pm} \;=\; \Omega^{\pm} \times (0,T)$$

with $\Omega^- = (-\alpha^-, 0)$ and $\Omega^+ = (0, \alpha^+)$. The unknown boundary $x = R(t)$ is transformed into the straight line $y = 0$.

The new unknown functions are

$$u^{\pm}(y,t) = x\,, \quad (y,t) \in \Omega_T^{\pm}\,. \tag{3.2}$$

for which

$$\frac{\partial p^{\pm}}{\partial x} = \frac{1}{u_y^{\pm}}\,, \quad \frac{\partial p^{\pm}}{\partial t} = -\frac{u_t^{\pm}}{u_y^{\pm}}\,. \tag{3.3}$$

These functions satisfy the equations

$$c^{\pm}\,\frac{\partial u^{\pm}}{\partial t} + \frac{\partial}{\partial y}\left(\frac{1}{u_y^{\pm}}\right) = 0 \quad \text{for} \;\; (y,t) \in \Omega_T^{\pm}\,, \tag{3.4}$$

and on the image of the free boundary the equations

$$u^+ = u^-\,, \qquad \left(\frac{\partial u^+}{\partial y}\right)^{-1} - \left(\frac{\partial u^-}{\partial y}\right)^{-1} = \beta \quad \text{for} \;\; y = 0\,. \tag{3.5}$$

The boundary conditions for the new unknown functions are

$$u^{\pm}(\pm\alpha^{\pm}, t) \;=\; \pm\beta^{\pm}\,, \tag{3.6}$$

and the initial conditions

$$u^{\pm}(y,0) = u_0^{\pm}(y)\,, \quad y \in \Omega^{\pm}\,. \tag{3.7}$$

In the formulas above, $\beta^{\pm} = 1$ if the original problem is considered in the domain $\Pi = (-1,1)$. In the general case, the values of $\beta^{\pm}$ should satisfy the inequality $-\beta^- < \beta^+$, being otherwise arbitrary.

### 3.2. A priori estimates

Assume that the positive functions

$$w^{\pm}(y,t) = \frac{1}{u_y^{\pm}} > 0 \tag{3.8}$$

are regular in the respective domains $\Omega_T^{\pm}$ where they satisfy the equation

$$c^{\pm} \frac{\partial w^{\pm}}{\partial t} = (w^{\pm})^2 \frac{\partial^2 w^{\pm}}{\partial y^2}, \qquad (y,t) \in \Omega_T^{\pm} . \tag{3.9}$$

On the boundary $y = 0$, they obey equation (1.3) in the form

$$w^+ - w^- = \beta, \qquad y = 0 , \tag{3.10}$$

and the equation

$$\frac{1}{c^+} \frac{\partial w^+}{\partial y} - \frac{1}{c^-} \frac{\partial w^-}{\partial y} = 0, \qquad y = 0 , \tag{3.11}$$

which follows from equation (3.4) and condition (3.5) after this latter is differentiated in the time variable.

Finally, for $y = \pm\alpha^{\pm}$ equations (3.4) and (1.4) imply that

$$\frac{\partial w^{\pm}}{\partial y} = 0, \qquad y = \pm\alpha^{\pm} , \tag{3.12}$$

and the initial condition is transformed into

$$w^{\pm}(y,0) = w_0^{\pm}(y), \qquad y \in \Omega^{\pm} . \tag{3.13}$$

The function $w^{\pm}$ cannot attain its maximum or minimum at the boundaries $y = \pm\alpha^{\pm}$ by condition (3.12).

If the functions $w^+$ and $w^-$ attain their maximal values at $y = 0$, then by (3.10) they would attain maximum simultaneously.

Let us assume that both functions $\omega^{\pm}$ attain absolute maximum at a point $(0, t^*)$. Then

$$\frac{\partial \omega^+}{\partial y}(0,t^*) > 0 \quad \text{and} \quad \frac{\partial \omega^-}{\partial y}(0,t^*) < 0 ,$$

which is impossible because of condition (3.12).

For this reason, we can conclude that $w^{\pm}$ cannot attain maximum for $y = 0$ later than at the initial moment.

Consequently, the positive maximum of $w^+$ or $w^-$ is attained at a point with $t = 0$, where both functions are bounded from the above by a constant because of condition (2.1); the above-mentioned constant depends on the initial data only.

Thus, there exists a positive constant $C_1$ such that

$$w^\pm(y,t) \le C_1 \ . \tag{3.14}$$

The minimum of the function $w^+(y,t)$ is easy to estimate because on the boundary $y = 0$ there is inequality $w^+(0,t) \ge \beta > 0$.

In the general case, it is impossible to estimate the minimum of $w^-(y,t)$ because if

$$\min w^+(y,0) < \beta \ ,$$

then $w^-(y,t)$ can attain its maximal value for $y = 0$, where $(\partial/\partial y)\, w^- < 0$. This does not contradict condition (3.11) because at this point $w^+(y,t)$ does not attain its minimal value (at this point $w^+ > \beta$).

The above shows that an additional argument is needed to estimate $w^-(y,t)$ from below.

We show first that $|u_t^\pm|$ is bounded.

**Lemma 3.1.** *For all solutions of the problem (1.1)–(1.5) which satisfy (2.1), the following estimate holds true: if* $u^\pm(y,t) \in H^{2+\lambda,\frac{2+\lambda}{2}}(\overline{\Omega}_T^\pm)$, *then*

$$\left|\frac{\partial u^\pm(y,t)}{\partial t}\right| = \left|\frac{u_{yy}^\pm}{(u_y^\pm)^2}\right| \le C_2 = \max\left(\max_{y\in\Omega^+} \frac{u_{0yy}^+}{(u_{0y}^+)^2}, \ \max_{y\in\Omega^-} \frac{u_{0yy}^-}{(u_{0y}^-)^2}\right) . \tag{3.15}$$

**Proof:** We introduce the functions

$$q_h^\pm = \frac{u^\pm(y,t+h) - u^\pm(y,t)}{h}$$

which satisfy the equations

$$c^\pm \frac{\partial q_h^\pm}{\partial t} - \frac{\partial}{\partial y}\left(a^\pm \frac{\partial q_h^\pm}{\partial y}\right) = 0 \qquad \text{in } \ \Omega_T^\pm \ , \tag{3.16}$$

$$q_h^+ = q_h^- \,, \quad a^+ \frac{\partial q_h^+}{\partial y} = a^- \frac{\partial q_h^-}{\partial y} \qquad \text{on } \ y = 0 \ , \tag{3.17}$$

$$q_h^\pm(\pm\alpha^\pm, t) = 0 \,, \qquad q_h^\pm(y,0) = q_{0,h}^\pm(y,0) \ . \tag{3.18}$$

We multiply equation (3.16) by $(q_h^{\pm})^{2k-1}$, $k = 1, 2, ..,$ and integrate by parts. We take into account (3.17) and (3.18), and arrive at the identity

$$(3.19)\qquad \frac{1}{2k}\,\frac{d}{dt}\left(c^-\int_{\Omega^-}(q_h^-)^{2k}\,dy \,+\, c^+\int_{\Omega^+}(q_h^+)^{2k}\,dy\right) \,+$$

$$+\;(2\,k-1)\left(\int_{\Omega^-}a^-(q_h^-)^{2k-2}\left(\frac{\partial q_h^-}{\partial y}\right)^2 dy \,+\int_{\Omega^-}a^+(q_h^+)^{2k-2}\left(\frac{\partial q_h^+}{\partial y}\right)^2 dy\right) \;=\; 0\;,$$

which implies that for $k = 1, 2, ...$

$$Y_k(t)\;\equiv\;\left(c^-\int_{\Omega^-}(q_h^-)^{2k}\,dy \,+\, c^+\int_{\Omega^+}(q_h^+)^{2k}\,dy\right)^{1/2k} \;\le\; Y_k(0)\;,$$

and, after the passage to the limit as $k\to\infty$,

$$\sup_{(y,t,h)}\left|\frac{u(y,t+h)-u(y,t)}{h}\right| \;\le\; Y_\infty(t)\;\le\; Y_\infty(0)\;\le\; C_2\;.$$

This estimate proves (3.15). ∎

**Remark 3.1.** Assume that $u_t^{\pm}\in H^{2+\lambda,\frac{2+\lambda}{2}}(\Pi_T^{\pm})$. In this case, inequality (3.15) can be obtained using the maximum principle for the function $u_t^{\pm}$.

Indeed, it is easily seen that the function $v^{\pm}(y,t) = u_t^{\pm}$ solves the following problem:

$$(3.20)\qquad c^{\pm}\,\frac{\partial v^{\pm}}{\partial t} \;=\; \frac{v_{yy}^{\pm}}{(u_y^{\pm})^2} - \frac{2\,u_{yy}^{\pm}}{(u_y^{\pm})^3}\,v_y^{\pm} \;=\; \frac{\partial}{\partial y}\left(\frac{v_y^{\pm}}{(u_y^{\pm})^2}\right), \qquad (y,t)\in\Omega_T^{\pm}\;,$$

$$(3.21)\qquad v^+ = v^-\;,\qquad \frac{v_y^+}{(u_y^+)^2} = \frac{v_y^-}{(u_y^-)^2}\quad \text{on } y=0\;,$$

$$(3.22)\qquad v^{\pm}(\pm\alpha^{\pm},t) = 0\;,$$

$$(3.23)\qquad v^{\pm}(y,0) = v_0^{\pm}(y,0)\,,\qquad y\in\Omega^{\pm}\;.$$

Because of condition (3.21), the function $v^{\pm}$ cannot attain its absolute extremum for $y = 0$. Consequently

$$|v^{\pm}(y,t)|\;\le\;\max_{y\in\Omega^{\pm}}|v^{\pm}(y,0)|\;.\;\square$$

**Lemma 3.2.** *Let $u_1^\pm(y,t)$ and $u_2^\pm(y,t)$ be two solutions of problem (3.4)–(3.6) such that*

$$u_1^\pm(y,t) \le u_2^\pm(y,t) \quad \textit{if} \;\; y = \pm\alpha^\pm \;\; \textit{and} \;\; t = 0 \; .$$

*Then*

$$u_1^\pm(y,t) \le u_2^\pm(y,t) \,, \qquad \forall\,(y,t) \in \Omega_T^\pm \; . \tag{3.24}$$

**Proof:** To prove the lemma, it suffices to consider the difference

$$U^\pm(y,t) \;=\; u_1^\pm - u_2^\pm \; .$$

This difference satisfies a homogeneous parabolic equation in $\Pi_T^\pm$,

$$LU^\pm \;\equiv\; c^\pm \,\frac{\partial U^\pm}{\partial t} - A^\pm\,\frac{\partial^2 U^\pm}{\partial y^2} - \frac{\partial A^\pm}{\partial y}\,\frac{\partial U^\pm}{\partial y} \;=\; 0 \;,$$

with

$$A^\pm = \frac{1}{u_{1y}^\pm\, u_{2y}^\pm} > 0 \;,$$

and it is non-negative for $y = \pm\alpha^\pm$ and $t = 0$. On the boundary $y = 0$ it satisfies the conditions

$$U^+ = U^- \,, \qquad \frac{1}{u_{1y}^+\, u_{2y}^+}\,\frac{\partial U^+}{\partial y} \;=\; \frac{1}{u_{1y}^-\, u_{2y}^-}\,\frac{\partial U^-}{\partial y} \; . \tag{3.25}$$

It follows from the above conditions that neither $U^+(y,t)$ nor $U^-(y,t)$ can attain a negative minimum at a point with $y = 0$. Consequently, for all $(y,t) \in \Pi_T^\pm$

$$U^\pm(y,t) \ge 0 \; . \;\blacksquare$$

**Remark 3.2.** We can derive an estimate for $R(t)$ as follows.

Choose $u_2^\pm(y,t)$ as the solution of problem (3.4)–(3.8), and choose $u_1^\pm(y)$ as the stationary solution of problem (3.4)–(3.5) such that

$$u_1^-(-\alpha^-) = -1 \,, \qquad u_{1y}^-(y) = a > 0 \;,$$

where the number $a$ is small enough. In this case,

$$u_1^- \;=\; 1 + a\,(y + \alpha^-) \,, \qquad u_1^+(y) \;=\; -1 + a\biggl(\alpha^- + \frac{y}{1 + a\,\beta}\biggr) \; . \tag{3.26}$$

It is easily seen that if $a$ is chosen small enough

$$u_1^-(y) < u_0^-(y) \;,$$

and if in addition

$$u_1^+(\alpha^+) < u_0(0) \ ,$$

then it follows from the monotonicity of the functions $u_0^+(y)$ and $u_1^\pm(y)$ that

$$u_0^+(y) > u_0(0)\,, \quad u_1^+(y) < u_1^+(\alpha^+)\,, \quad u_0^+(y) > u_1^+(y) \ .$$

Thus,

$$u_1^\pm(y,t) < u^\pm(y,t) \ .$$

In particular, for some $\delta > 0$

$$-1 \ < \ -1+\delta \ \le \ R_\infty^- \ = \ u_1^-(0) \ < \ u^-(0,t) \ = \ R(t) \ \le \ 1-\delta \ < \ 1 \ .$$

The upper bound is derived in a completely analogous way:

$$u^+(0,t) \ = \ R(t) \ < \ u_2^+(0) \ = \ R_\infty^2 \ < \ 1 \ . \ \square$$

We now apply the above considerations in the case when $u_1^\pm$ is the stationary solution of problem (1.3)–(1.5), so that $u_1^-(-\alpha^-) = -1$ and $u_1^+(\alpha^+) = u_0(0)$. We conclude that

$$-1 \ < \ R_\infty^1 \ = \ u_1^-(0) \ = \ u_1^+(0) \ \le \ u^-(0,t) \ = \ u^+(0,t) \ = \ R(t) \ . \tag{3.27}$$

Let us now choose $u_2^\pm$ as the stationary solution to problem (1.3)–(1.6). This function satisfies the conditions $u_2^-(-\alpha^-) = u_0(0)$ and $u_2^+(\alpha^+) = 1$. The result is that

$$R(t) \ = \ u^+(0,t) \ = \ u^-(0,t) \ \le \ u_2^+(0) \ = \ u_2^-(0) \ = \ R_\infty^2 \ < \ 1 \ . \tag{3.28}$$

Thus,

$$-1 \ < \ -1+\delta \ \le \ R(t) \ \le \ 1-\delta \ < \ 1\,, \quad t \in [0,\infty) \ . \tag{3.29}$$

### 3.3. The lower bound for $p_x(x,t) = w^- = 1/u_y^-$

To estimate $w^-(y,t)$ from below, it is more convenient to return to the original variables $(x,t)$. In these variables, the unknown boundary $x = R(t)$ satisfies inequalities (3.29). The estimate of (3.15) with $y = 0$ yields the inequality

$$\left|\frac{dR}{dt}\right| = |\dot{R}(t)| = \left|\frac{\partial u^\pm}{\partial t}\right| \le C_2 \ . \tag{3.30}$$

Using (3.29) and the simplest barrier function, it is easy to show that

$$\frac{\partial p^-}{\partial x}(-1,t) \;=\; w^-(-\alpha^-,t) \;\geq\; C_3 \;>\; 0\;,$$

where the constant $C_3$ depends only on the data of the problem. To estimate $(\partial/\partial x)\, p^-$ from below on the unknown boundary $x = R(t)$, we use the variables

$$t = t\,, \qquad z = \frac{x+1}{R+1}$$

which reduce the equation for the new unknown function

$$h(z,t) = \, p^-(x,t)$$

to the form

$$c^- \,\frac{\partial h}{\partial t} - \frac{1}{(1{+}R)^2}\,\frac{\partial^2 h}{\partial z^2} - \frac{z}{(1{+}R)}\,\dot R\,\frac{\partial h}{\partial z} \;=\; 0\,, \qquad 0{<}z{<}1, \quad 0{<}t{<}T\;. \tag{3.31}$$

The boundary conditions and the initial condition in these new variables are as follows:

$$h(0,t) = -\alpha^-, \quad h(1,t) = 0\,, \qquad 0{<}t{<}T\;, \tag{3.32}$$

$$h(z,0) = h_0(z)\,, \qquad 0{<}z{<}1\;, \tag{3.33}$$

where

$$\frac{dh_0}{dz} \;=\; h_0'(z) \;\geq\; (1{+}R_0)\,\alpha_0 \;>\; 0\,, \qquad h_0\!\left(\frac{1+x}{1+R_0}\right) \;=\; p_0^-(x)\;.$$

Consider the function

$$\Phi(z) \;=\; \frac{A}{2\,C_2}\left(e^{-2C_2} - e^{-2C_2 z}\right)\,.$$

For a sufficiently small positive number $A$, it satisfies the inequalities

$$\Phi(0) > h(0,t) = -\alpha^-\,, \qquad \Phi(z) \geq h_0(z)\;.$$

Since $\Phi(1) = 0$, the estimate

$$h(z,t) \leq \Phi(z)$$

follows from the maximum principle and the inequality

$$L(h-\Phi) \equiv \left(\frac{\partial}{\partial t} - \frac{1}{(1+R)^2}\frac{\partial^2}{\partial z^2} - \frac{z}{(1+R)}\dot{R}\frac{\partial}{\partial z}\right)(h-\Phi) < 0 , \tag{3.34}$$

which is valid for all $z \in (0,1)$ and $t \in (0,T)$. In combination with the condition $h(1,t) = \Phi(1) = 0$, the above estimate signifies that

$$\frac{1}{(1+R)}\frac{\partial p^-}{\partial x}(R,t) = \frac{\partial h}{\partial z}(1,t) \geq \Phi'(1) = A\,e^{-C_2} .$$

Thus, if (3.34) is satisfied, then

$$\frac{\partial p^-}{\partial x}(R,t) \geq (1+R)\,A\,e^{-C_2} \geq \delta\,A\,e^{-C_2} . \tag{3.35}$$

The validity of inequality (3.34) is verified directly:

$$L(h-\Phi) = \frac{1}{(1+R)^2}\Big(\Phi'' + \dot{R}\,z(1+R)\,\Phi'\Big) < \frac{1}{(1+R)^2}\left(\Phi'' + 2\,C_2\,\Phi'\right) = 0 .$$

We state the result obtained as a lemma.

**Lemma 3.3.** *Assume that*

$$u^{\pm}(y,t) \in H^{2+\lambda,\frac{2+\lambda}{2}}(\overline{\Pi})^{\pm}_T$$

*is the classical solution of problem (3.4)–(3.7) (or that stated in (A.3), (3.4)–(3.6), and (A.3)). Then*

$$\frac{1}{C_1} \leq u^{\pm}_y(y,t) = \frac{1}{p^{\pm}_x} \leq C_1 < \infty \tag{3.36}$$

*and*

$$|\dot{R}|, \quad |u^{\pm}_t| = \left|\frac{p^{\pm}_t}{p^{\pm}_x}\right|, \quad |u^{\pm}_{yy}| = \left|\frac{p^{\pm}_{xx}}{(p^{\pm}_x)^3}\right|, \quad |p^{+}_{xx}| \leq C_2 , \tag{3.37}$$

*with constants $C_1$ and $C_2$ which depend only on $\alpha_0$ and $\alpha_1$ from conditions (2.1), (2.2).*

**Remark 3.3.** It is easy to see that estimates (3.37) imply the inequalities

$$\Big|u^{\pm}(y,t+\tau) - u^{\pm}(y,t)\Big| \leq C_2\,|\tau| , \quad (y,t) \in \overline{\Omega}^{\pm}_T ,$$

$$\Big|u^{\pm}_y(y+h,t) - u^{\pm}_y(y,t)\Big| \leq C_2\,|h| , \quad (y,t) \in \overline{\Omega}^{\pm}_T .$$

Hence $u^{\pm}_y \in C^{\alpha}(\overline{\Omega}^{\pm}_T)$ according to [LSU88, Lemma 3.1, Ch.II, §2]. □

As the next step, we prove the estimate

$$\|u^{\pm}\|_{H^{2+\lambda,\frac{2+\lambda}{2}}(\overline{\Pi}_T^{\pm})} \le C_3(\alpha_1,\alpha_2,T) \tag{3.38}$$

for any finite $T$.

To this end, we introduce the new variables

$$t = t\,, \qquad \xi = c^{\pm}\, y \quad \text{if} \quad \pm\, y > 0\ .$$

Put, moreover,

$$a(v,\xi) \;=\; c^{-}(v-\beta)^2\left(1 - H(\xi)\right) \;+\; c^{+}\, v^2\, H(\xi)\ ,$$

where

$$H(\xi) = \frac{1}{2}\left(1 + \frac{\xi}{|\xi|}\right), \qquad \xi_0^- < \xi < \xi_0^+\ ,$$
$$\xi_0^- = -\alpha^- c^-\,, \qquad \xi_0^+ = \alpha^+ c^+\ ,$$

and

$$v(\xi,t) = \begin{cases} w^-(y,t) + \beta & \text{if}\ \ y < 0, \\ w^+(y,t) & \text{if}\ \ y > 0\ . \end{cases} \tag{3.39}$$

According to conditions (3.10) and (3.11), the function $v(\xi,t)$ and its derivative $(\partial/\partial\xi)\, v(\xi,t)$ are continuous in the domain

$$Q_T = Q\times(0,T)\,, \qquad Q = \left\{\xi\colon\ \xi_0^- < \xi < \xi_0^+\right\}\ .$$

In the domain $Q_T$, they satisfy, in the usual sense, the equation

$$\frac{\partial v}{\partial t} = a(v,\xi)\, \frac{\partial^2 v}{\partial \xi^2}\,, \qquad (\xi,t) \in Q_T, \quad \xi \ne 0\ , \tag{3.40}$$

at all points not belonging to the line $\xi = 0$, where the coefficient $a(v,\xi)$ has a jump. Note in this connection that according to the definition of the function $a(v,\xi)$ and the estimate (3.36) there exists a constant $C_4 = C_4(c^{\pm},\alpha^{\pm},\beta,C_2)$ such that

$$\frac{1}{C_4} \;\le\; a(v,\xi) \;\le\; C_4\ . \tag{3.41}$$

The function $v(\xi,t)$ satisfies also the following boundary and initial conditions:

$$v(\xi_0^{\pm},t)=0\,,\qquad t\in(0,T)\ , \tag{3.42}$$

$$v(\xi,0)=v_0(\xi)\,,\qquad \xi\in Q\ . \tag{3.43}$$

Here, condition (3.42) results from (3.12).

The given function $v_0(\xi)\in W_2^1(Q)$ is supposed to admit the estimate

$$\|v_0\|_{2,Q}^1\ \le\ C_0\ .$$

According to [LSU88], the problem (3.40), (3.42), (3.43) has a solution $v(\xi,t)$ in the class $W_2^{2,1}(Q_T)$; it is easy to prove the energy relation

$$\int_Q v_\xi^2(\xi,t)\,d\xi\ +\int_0^T\!\!\int_Q\Big(a\,v_{\xi\xi}^2+\frac{1}{a}\,v_t^2\Big)\,d\xi\,dt\ =\ \int_Q v_\xi^2(\xi,0)\,d\xi\ . \tag{3.44}$$

Note in this connection that the derivative $r(\xi,t)=v_\xi(\xi,t)$ resolves the problem

$$\frac{\partial r}{\partial t}=\frac{\partial}{\partial\xi}\Big(a\,\frac{\partial r}{\partial\xi}\Big)\,,\qquad (\xi,t)\in Q_T\ , \tag{3.45}$$

$$r(\xi_0^{\pm},t)=0\,,\qquad t\in(0,T)\ , \tag{3.46}$$

$$r(\xi,0)=r_0(\xi)=v_{0\xi}(\xi)\,,\qquad \xi\in Q\ . \tag{3.47}$$

According to [LSU88], the problem (3.45)–(3.47), has a solution $r(\xi,t)$ from the space $W_2^{1,0}(Q_T)\cap H^{\lambda,\frac{\lambda}{2}}(\overline{Q}_T)$ such that

$$\|r\|_{H^{\lambda,\frac{\lambda}{2}}(\overline{Q}_T)}\le\ C_5\Big(C_4,\ \|r_0\|_{H^{\lambda,\frac{\lambda}{2}}(\overline{Q})}\Big)\ . \tag{3.48}$$

Using Lemma 3.1, formulae (3.4), (3.8), (3.39), and the relation

$$r^{\pm}(\xi,t)\ =\ v_\xi^{\pm}\ =\ -\frac{u_{yy}^{\pm}}{u_y^{\pm}\,c^{\pm}}\ , \tag{3.49}$$

we establish that

$$\Big(\|u_t^{\pm}\|_{H^{\lambda,\frac{\lambda}{2}}(\overline{\Omega}_T^{\pm})};\ \|u^{\pm}\|_{H^{2+\lambda,\frac{2+\lambda}{2}}(\overline{\Omega}_T^{\pm})}\Big)\le\ C_6\Big(C_2,\ C_4,\ \|u_0^{\pm}\|_{H^{2+\lambda}(\overline{\Omega}(0))},\ T\Big)\ . \tag{3.50}$$

Using formula (3.3) and returning to the functions $p^{\pm}(x,t)$, we come to the estimate

$$\Big(\|R\|_{H^{1+\lambda/2}([0,T])};\ \|p^{\pm}\|_{H^{2+\lambda,\frac{1+\lambda}{2}}(\overline{\Pi}_T^{\pm})}\Big)\le\ C_7\Big(C_6,\ \|p_0^{\pm}\|_{H^{2+\lambda}(\overline{\Pi}(0))},\ T\Big)\ . \tag{3.51}$$

We state the result obtained as a lemma.

**Lemma 3.4.** *Assume that $p^{\pm}(x,t)$ (or, in another form, $u^{\pm}(y,t)$) are the classical solutions of problem (1.1)–(1.5), (or, respectively, (3.4)–(3.7)). Then the estimates (3.50) and (3.51) hold true.*

Next, we prove the existence of solution to the problem (1.1)–(1.5), (or, respectively, (3.4)–(3.7)). According to the paper [AG94], the problem (3.4)–(3.7)) has a classical solution up to a small time $t_1 > 0$ ($t_1$ is determined by $\|u^{\pm}(y,0)\|$).

Using estimate (3.50), the solution $u^{\pm}$ can be continued to any finite interval $t \in (0,T)$.

### 3.4. Comparison principles

**Lemma 3.5.** *If $p^{\pm}(x,t)$ and $p_1^{\pm}$ are two solutions of problem (1.1)–(1.5), then the functions $p(x,t)$, $p_1(x,t)$ (defined as $p = p^{\pm}$, $p_1 = p_1^{\pm}$ in $\Pi^{\pm}(t)$) are the generalized solutions, in the space $W_2^{1,0}(\Pi_T)$, of the equation*

$$\frac{\partial \tilde{c}(p)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x} - \beta\, H(p)\right) \tag{3.52}$$

*with the corresponding initial (1.5) and boundary conditions (1.4) on the lines $x = \pm 1$; here*

$$\tilde{c}(p) = c^{\pm}\, p \quad \text{if} \quad \pm\, p > 0\ ,$$

$$H(p) = \begin{cases} 0 & \text{if } \ p < 0, \\ 1 & \text{if } \ p > 0\ , \end{cases} \qquad \text{and} \qquad H(0) \in [0,1]\ .$$

*If on the boundaries $x = \pm 1$ and at the initial moment $t = 0$*

$$p(x,t) \le p_1(x,t)\ ,$$

*then everywhere in $\Pi_T$*

$$p(x,t) \le p_1(x,t)\ . \tag{3.53}$$

**Proof:** A simple calculation shows that each classical solution of problem (3.52), (1.4), (1.5) is a generalized solution of equation (3.52) where $p = p^{\pm}$ if $\pm\, x > 0$ and

$$\tilde{c}(p) = \begin{cases} c^{+}p^{+} & \text{if } \ p^{+} > 0, \\ c^{-}p^{-} & \text{if } \ p^{-} < 0\ . \end{cases}$$

This solution satisfies the boundary conditions (1.4) and the initial condition (1.5).

The existence of the unique generalized solution of problem (3.52) is proved by standard methods [MEI92]. It is convenient to state the problem for the new unknown function

$$g(x,t) = \tilde{c}[p(x,t)] \ .$$

This function satisfies the equation

$$\frac{\partial g}{\partial t} = \frac{\partial}{\partial x}\Big(\kappa(g)\,\frac{\partial g}{\partial x} - \beta\,H(g)\Big) \ , \tag{3.54}$$

as well as the boundary and initial conditions

$$g(\pm 1,t) = \pm\beta^{\pm} \ , \qquad g(x,0) = g_0 \ , \tag{3.55}$$

where $\kappa(g) = [\tilde{c}\,'(p)]^{-1}$ and

$$\min\left[\frac{1}{c^-}, \frac{1}{c^+}\right] \le \kappa(g) \le \max\left[\frac{1}{c^-}, \frac{1}{c^+}\right] .$$

It is easy to verify that

$$g \in L_2(0,T;H^1(\Pi)) \, , \qquad \frac{\partial g}{\partial t} \in L_2(0,T;H^{-1}(\Pi)) \ ,$$

i.e., that the problem (3.54), (3.55) does indeed have a generalized solution in the stipulated class; moreover,

$$p^{\pm}(x,t) \le p_1^{\pm}(x,t)$$

and

$$|p^{\pm}|_{\Pi_T^{\pm}}^{(2+\lambda)} \le C \ . \tag{3.56}$$

In terms of the original function $p(x,t)$, the generalized solution to the problem (1.4), (1.5), (3.52) is the limit of the solution $p^{\varepsilon}(x,t)$ of the regularized equation

$$\frac{\partial \tilde{c}_{\varepsilon}(p^{\varepsilon})}{\partial t} = \frac{\partial}{\partial x}\Big(\frac{\partial p^{\varepsilon}}{\partial x} - \beta\,H_{\varepsilon}(p^{\varepsilon})\Big) \ , \tag{3.57}$$

where $\widetilde{c}_\varepsilon, H_\varepsilon \in C^\infty$ and

$$\lim_{\varepsilon\to 0} \widetilde{c}_\varepsilon(p) = \widetilde{c}(p)\,, \qquad \lim_{\varepsilon\to 0} H_\varepsilon(p) = H(p)\ . \tag{3.58}$$

Assume that $p_1(x,t)$ is another generalized solution of the problem (1.4), (1.5), (3.52), so that $p(x,t)\le p_1(x,t)$ on the boundaries $x=\pm 1$ and at the initial time $t=0$. Then $p_1$ is the limit of the solution $p_1^\varepsilon(x,t)$ of equation (3.57), so

$$p^\varepsilon(x,t) \le\ p_1^\varepsilon(x,t)$$

on the boundaries $x=\pm 1$ and at time $t=0$.

It is easily seen that the difference $\widetilde{p} = p_1^\varepsilon - p_1$ satisfies in $\Pi_T$ the homogeneous parabolic equation and is non-negative on the boundaries $x=\pm 1$ and at the initial time $t=0$. For this reason, by the maximum principle

$$p^\varepsilon(x,t) \le\ p_1^\varepsilon(x,t)\,, \qquad \forall\,(x,t)\in \Pi_T\ .$$

Passing to the limit as $\varepsilon\to 0$, we obtain the desired estimate (3.53). It is obvious that this estimate guarantees the uniqueness both of the generalized and classical solutions to the problem (1.1)–(1.5). ∎

### 3.5. Asymptotic behavior

It is evident, after (3.3) and (3.36) are taken into account, that inequality (2.9) would be proved if an analogous one were proved for the functions $u^\pm(y,t)$.

Consider the stationary solution of problem (3.4)–(3.6)

$$u_\infty^\pm(y)\ =\ R_\infty + \frac{1+R_\infty}{\alpha^\pm}\ y\ ,$$

where the number $R_\infty$ is determined uniquely from equation (2.6).

For the difference

$$U^\pm(y,t)\ =\ u^\pm(y,t) - u_\infty^\pm(y)\ ,$$

we obtain the following problem:

$$c^{\pm}\, U_t^{\pm} = \frac{\partial}{\partial y}\Big( A^{\pm}\, \frac{\partial U^{\pm}}{\partial y} \Big) , \qquad (y,t) \in \Omega_\infty^{\pm} \ , \tag{3.59}$$

$$U^+ = U^- \ , \qquad A^+\, \frac{\partial U^+}{\partial y} = A^-\, \frac{\partial U^-}{\partial y} \quad \text{on} \ \ y = 0 \ , \tag{3.60}$$

where

$$A^{\pm} = \frac{1}{u_y^{\pm}\, u_{\infty y}^{\pm}} , \qquad |\ln A^{\pm}| \le C_8(C_6) \ .$$

Multiplying equation (3.59) by $U^{\pm}$ and integrating by parts, we arrive at the energy relation

$$\frac{d}{2\,dt} \int_{\Omega^{\pm}} c^{\pm}(U^{\pm})^2\, dy + \int_{\Omega^{\pm}} A^{\pm} \Big( \frac{\partial U^{\pm}}{\partial y} \Big)^2 dy \ = \ 0$$

and, consequently, to the estimate

$$\sup_{t\in(0,\infty)} \int_{\Omega^{\pm}} c^{\pm}(U^{\pm})^2\, dy + \int_0^{\infty} \int_{\Omega^{\pm}} A^{\pm} \Big( \frac{\partial U^{\pm}}{\partial y} \Big)^2 dy\, dt \ \le \ C_9 \ .$$

Using the inequality

$$Y(t) \ \equiv \ \int_{\Omega^{\pm}} c^{\pm}(U^{\pm})^2\, dy \ \le \ C_{10} \int_{\Omega^{\pm}} A^{\pm} \Big( \frac{\partial U^{\pm}}{\partial y} \Big)^2 dy \ ,$$

where $C_{10} = C_{10}(c^{\pm}, \alpha^{\pm})$, we obtain the ordinary differential inequality for the function $Y(t)$,

$$\frac{dY(t)}{dt} + \frac{2}{C_{10}}\, Y(t) \ \le \ 0 \ ,$$

and the estimate

$$Y(t) \le Y(0)\, e^{-C_{11} t} , \qquad C_{11} = 2/C_{10} \ . \tag{3.61}$$

Using (3.36), we can write

$$|U^{\pm}(y,t)|^2 \ \le \ 2 \Big( \int_{\Omega^{\pm}} (U^{\pm})^2\, dy \Big)^{1/2} \Big( \int_{\Omega^{\pm}} \Big( \frac{\partial U^{\pm}}{\partial y} \Big)^2 dy \Big)^{1/2} \le \ C_{12}\, Y^{1/2}(t) \ . \tag{3.62}$$

Combining (3.61) and (3.62), we see that

$$\max_{y\in\Omega^{\pm}} |U^{\pm}(y,t)|^2 \ \le \ C_{12}\, Y^{1/2}(t) \ \le \ C_{13}\, e^{-2\lambda_0 t} \ .$$

The last estimate proves (2.9).

## Appendix A – Alterations for the case of prescribed jump of squared temperature gradient

In this appendix we provide a brief exposition of the alterations that should be made in the arguments of the article if the problem considered is stated in equations (1.1)–(1.2), (1.4)–(1.5), and (1.6) (which replaces (1.3)).

Let us begin with stationary solutions. It is easy to verify that the function

$$p^{\pm}_{\infty}(x) \;=\; \alpha^{\pm}\,\frac{x-R_\infty}{1\pm R_\infty} \tag{A.1}$$

is the unique solution of the corresponding stationary problem (1.1)–(1.2), (1.4), (1.6). Here the number $R_\infty$ is uniquely defined from the equation

$$g(R_\infty)\;\equiv\;\left(\frac{\alpha^+}{1-R_\infty}\right)^2-\left(\frac{\alpha^-}{1+R_\infty}\right)^2-\beta\;=\;0\;, \tag{A.2}$$

and it is easily seen that

$$g'(\tau)\;=\;\frac{2\,(\alpha^+)^2}{(1-\tau)^3}+\frac{(\alpha^-)^2}{(1+\tau)^3}\;>\;0\,,\qquad \tau\in(-1,1),\quad g(\pm)=\pm\infty\;.$$

Note that when (1.6) is used instead of (1.3), condition (3.5) assumes the form

$$u^+=u^-\;,\qquad \left(\frac{\partial u^+}{\partial y}\right)^{-2}-\left(\frac{\partial u^-}{\partial y}\right)^{-2}=\beta\;. \tag{A.3}$$

All the arguments of subsection 3.1 should be repeated almost verbatim, but the problem considered is changed to (3.4), (A.3), (3.6), and (3.7).

In subsection 3.2, we introduce the new functions

$$w^{\pm}(y,t)=\left(\frac{1}{u^{\pm}_y}\right)^2 \tag{A.4}$$

which solve the following boundary problem (cf. (3.8)–(3.13)):

$$c^{\pm}\,w^{\pm}_t\;=\;w^{\pm}\,w^{\pm}_{yy}-\frac{1}{2}\,(w^{\pm}_y)^2\;, \tag{A.5}$$

$$w^+-w^-=\beta\,,\qquad \frac{1}{c^+}\,\frac{w^+_y}{(w^+)^{3/2}}-\frac{1}{c^-}\,\frac{w^-_y}{(w^-)^{3/2}}=0\quad\text{on}\quad y=0\;, \tag{A.6}$$

$$\frac{\partial w^{\pm}}{\partial y}(\pm\alpha,t)=0\;, \tag{A.7}$$

$$w^{\pm}(y,0)=w^{\pm}_0(y)\;. \tag{A.8}$$

Using an argument quite similar to that of subsection 3.2, we establish the estimates

$$w^{\pm}(y,t) \le C_2 \,, \quad w^{+}(y,t) \ge \beta \,.$$

In the proof of Lemma 3.1 (see Remark 3.1) for the problem (3.20)–(3.23), the only alteration that is necessary is changing conditions (3.21) to the following ones:

$$v^{+} = v^{-} \,, \quad \frac{v^{+}_{y}}{(u^{+}_{y})^{3}} = \frac{v^{-}_{y}}{(u^{-}_{y})^{3}} \,. \tag{A.9}$$

The remaining part of the proof is repeated without change.

In the proof of estimate for $R(t)$ (see Remark 3.2), the only alteration is changing equation (3.26) to

$$u^{+}_{1}(y) \;=\; -1 + a\left(\alpha + \frac{y}{\sqrt{1+a^{2}\,\beta}}\right) \;;$$

the rest of the proof is repeated without change.

In subsection 3.3, the proof of the estimate in (3.35) does not specify the form of the boundary condition at $x = R(t)$. Hence the assertions of Lemmas 3.1–3.3 and Remark 3.3 are also valid in the case considered here.

Other considerations of the article can be realized in a similar manner.

## REFERENCES

**[AG94]** Andreucci, D. and Gianni, R. – Classical solutions to a multidimensional free boundary problem arising in combustion theory, *Commun. Partial Differ. Equations,* 5–6 (1994), 803–826.

**[ANT92]** An Ton Bui – On a free boundary problem arising in the study of flame propagation, *Math. Meth. Appl. Sci.,* 15 (1992), 479–493.

**[L91]** Larrouturou, B. – *Mathematics contribute to the progress of combustion science*, in "Fluid Dynamical Aspects of Combustion Theory", Pitman Res. Notes Math. Ser. 223, pp. 251-267, 1991.

**[BL91]** Beresticky, H. and Larrouturou, B. – *Quelques aspects mathématiques de la propagation des flammes prémelangées*, in "Nonlinear Partial Differential Equations and Their Applications", Lect. Coll. de France Semin., Paris/Fr. 1987-88, Vol. X, Pitman Res. Notes Math. Ser. 220, pp. 65-129, 1991.

**[BH1]** Bertsch, M. and Hilhorst, D. – *On a free boundary problem for Burgers equation: the large time behaviour* (to appear).

**[BH2]** Bertsch, M.; Hilhorst, D. and Schmidt-Lainé, Cl. – *The well-posedness of a free boundary problem for Burgers equation* (to appear).

**[BLS92]** Brauner, C.M.; Linardi, A. and Schmidt-Lané, Cl. – Stability of travelling waves with interface conditions, *Nonlinear Analysis TMA,* 19 (1992), 455–474.

**[CLW97]** Cafarelli, L.A.; Lederman, C. and Wolanski, N. – Pointwise and viscosity solutions for the limit of a two phase parabolic singular perturbation problem. In *Free Boundary Problems, Theory and Applications, Herakleion, Crete, Greece, 8–14 June, 1997. Abstracts,* p. 61.

**[LSU88]** Ladyzhenskaya, O.A.; Solonnikov, V.A. and Ural'ceva, N.N. – *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, vol. 23, American Mathematical Society, Providence, Rhode Island, 1988.

**[LW97]** Lederman, C. and Wolanski, N. – Viscosity solutions and regularity of the free boundary for the limit of an elliptic two phase singular perturbation problem. In *Free Boundary Problems, Theory and Applications, Herakleion, Grete, Greece, 8–14 June, 1997. Abstracts,* p. 64.

**[MEI92]** Meirmanov, A.M. – *The Stefan Problem*, De Gruyter Expositions in Mathematics, vol. 3, Walter de Gruyter, Berlin–New York, 1992.

**[MPS97]** Meirmanov, A.M.; Pukhnachev, V.V. and Shmarev, S.I. – *Evolution Equations and Lagrangian Coordinates*, De Gruyter Expositions in Mathematics, vol. 24, Walter de Gruyter, Berlin–New York, 1997.

**[VA96]** Vázquez, J.L. – *The free boundary problem for the heat equation with fixed gradient condition*, in "Free Boundary Problems, Theory and Applications" – FBP'95 Proceedings (M. Niezgódka and P. Strzelecki, Eds.), Pitman Research Notes Math. Sciences, #363, Longman, 1996.

**[LVW97]** Lederman, C.; Vázquez, J.L. and Wolanski, N. – Uniqueness of solution to a free boundary problem from combstion, *Trans. Amer. Math. Soc.* (to appear).

**[LVW00]** Lederman, C.; Vázquez, J.L. and Wolanski, N. – A mixed semilinear parabolic problem in a non-cylindrical space-time domain, *Diff. Int. Eqns.* (to appear).

S.N. Antontsev, A.M. Meirmanov, V.V. Yurinsky,
Departamento de Matemática/Informática, Universidade da Beira Interior,
Convento de Santo António, 6201-001 Covilhã – PORTUGAL