PORTUGALIAE MATHEMATICA
Vol. 58 Fasc. 4 – 2001
Nova Série

EXISTENCE OF L^∞ ENTROPY SOLUTIONS FOR A REACTING EULER SYSTEM *

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Abstract: In this Note, we give an existence result of L^∞ solutions for an isentropic Euler system with a source term. Our method follows the analysis of DiPerna and uses the compensated compactness theory developed by Murat and Tartar.

1 – Introduction

We are interested in the Cauchy problem for an isentropic inhomogeneous Euler system describing the flow of two reactive species. This simple two-phase flow model has been already studied in e.g. [14] without source terms. In a quasi-conservative form, it reads:

$$\begin{cases} \partial_t\rho + \partial_x(\rho\, u) = 0 \\ \partial_t(\rho\, c) + \partial_x(\rho\, u\, c) = \rho\, g(c) \\ \partial_t(\rho\, u) + \partial_x(\rho\, u^2 + p) = 0\ , \end{cases} \tag{1}$$

where $\rho \geq \alpha > 0$, $c \in [0,1]$, $u \in \mathbb{R}$ are respectively the global density of the considered mixture, the concentration of one of the phases appearing in the flow, and the common velocity. The pressure law p depends in general on both variables (ρ, c). The first equation expresses the conservation of the global mass. The second one describes the evolution of one of the mass fractions according to a reaction process modelled by the source term $g \in C^1(\mathbb{R})$. At last, the third one is

Received: November 7, 2000.
Keywords: Compensated compactness; Euler equations; Source terms.
* Partially supported by TMR project HCL#ERBFMRXCT960033 and the French Atomic Energy Commission (CEA), Department of Mechanics and Technology.

concerned with the global mixture momentum. For reasonable pressure laws and if no vacuum appears, this inhomogeneous system is strictly hyperbolic, [12]. Our goal is to study the Cauchy problem for (1) through its Lagrangian formulation:

$$\begin{cases} \partial_t v - \partial_x u = 0 \\ \partial_t u + \partial_x p(v,c) = 0 \\ \partial_t c = g(c) \end{cases}, \qquad x \in \mathbb{R}, \quad t > 0 \;. \tag{2}$$

The variable $v \in \mathbb{R}_*^+$ stands thus for the specific volume of the mixture. It is known that this change of variables is one-to-one even for discontinuous functions, [15]. This last system is shown to be strictly hyperbolic under the hypotheses (3) on its pressure law $p(v,c)$. Its entropies can be easily derived from those of the classical p-system (see Lemma 1 and (9)). Then we consider a sequence of viscous approximations (10) for which we present some sufficient conditions to ensure the existence of a positively invariant region, [12] for all $\varepsilon > 0$: see (11), (12) in Lemmas 2 and 3. If no vacuum appears, this provides an L^∞ bound and we can apply the theory of compensated compactness, [10, 13]. Indeed, we perform a change of variables in (10) (see (15)) in order to weaken the coupling of its unknowns and it is therefore possible to conclude invoking a compactness theorem of DiPerna, [8]: see Theorem 1. This result extends towards the original Eulerian system (1) by means of [15]. At last, we give some remarks and extensions, especially when a relaxation process is involved, [4] (see Lemma 4), or in in the case of more complex pressure laws. We refer to [1, 2, 6, 7] for some recent contributions to such inhomogeneous problems, see also [3, 5] for other studies in an homogeneous case. Part of this work appears in the Ph.D. thesis [9].

2 – Elementary properties and entropies

We make the following hypotheses on the pressure law, [12]:

$$p \in C^2(\mathbb{R}_*^+ \times [0,1]; \mathbb{R}_*^+)\,, \qquad p_v < 0 \;\text{ and }\; p_{vv} > 0 \;, \tag{3}$$

and we note

$$U = \begin{pmatrix} v \\ u \\ c \end{pmatrix}, \qquad F(U) = \begin{pmatrix} -u \\ p(v,c) \\ 0 \end{pmatrix}, \qquad S(U) = \begin{pmatrix} 0 \\ 0 \\ g(c) \end{pmatrix},$$

where F is the flux function, S the source term. The problem (2) rewrites in condensed form:

$$\begin{cases} \partial_t U + \partial_x F(U) = S(U) \\ U(\cdot, 0) = U_0 = (v_0, u_0, c_0) \ . \end{cases}$$

Our hypotheses ensure its strict hyperbolicity since the Jacobian matrix is:

$$\nabla F = \begin{pmatrix} 0 & -1 & 0 \\ p_v & 0 & p_c \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{4}$$

Its eigenvalues are: $\Lambda(U) = \{-\sqrt{-p_v(v,c)}; 0; \sqrt{-p_v(v,c)} \stackrel{\text{def}}{=} \lambda\}$. The right eigenvectors associated to 0, $-\lambda$, λ are:

$$\vec{R}_0 = \begin{pmatrix} p_c \\ 0 \\ \lambda^2 \end{pmatrix} \ , \qquad \vec{R}_{-1} = \begin{pmatrix} 1 \\ \lambda \\ 0 \end{pmatrix} \ , \qquad \vec{R}_1 = \begin{pmatrix} -1 \\ \lambda \\ 0 \end{pmatrix} \ . \tag{5}$$

This implies that two characteristic fields of (2) are *genuinely nonlinear*, [12], i.e. $\langle \nabla\lambda, \vec{R}_{\pm 1}\rangle_{\mathbb{R}^3} \neq 0$. The last one is obviously *linearly degenerate* since it is static. We will also make use of the left eigenvectors of ∇F:

$$\vec{L}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ , \qquad \vec{L}_{-1} = \begin{pmatrix} \lambda^2 \\ \lambda \\ -p_c \end{pmatrix} \ , \qquad \vec{L}_1 = \begin{pmatrix} -\lambda^2 \\ \lambda \\ p_c \end{pmatrix} \ . \tag{6}$$

We want now to point out the particular structure of its entropies. At this level, we can consider the homogeneous version ($g \equiv 0$) of (2) and remark that if one gives as an initial data some uniform concentration $c_0(x) \equiv C_0 \in [0,1]$, one solves in fact the following system:

$$\begin{cases} \partial_t v - \partial_x u = 0 \\ \partial_t u + \partial_x p(v, C_0) = 0 \ . \end{cases} \tag{7}$$

Lemma 1. *Under the assumptions (3), any entropy-entropy flux pair of the p-system (7) $(\eta, q) \in C^2(\mathbb{R}_*^+ \times \mathbb{R})$ defines an entropy-entropy flux pair $(\tilde{\eta}, \tilde{q}) \in C^2(\mathbb{R}_*^+ \times \mathbb{R} \times [0,1])$ for the system (2). This means that one has for all U:*

$$\nabla\tilde{q}(U) = \nabla\tilde{\eta}(U)\,.\nabla F(U) \ . \tag{8}$$

Proof: Let $C_0 \in [0,1]$: under the assumptions (3), the map

$$V \colon\ (v,u) \mapsto \Big(p(v, C_0), u\Big)$$

is a diffeomorphism of $\mathbb{R}_*^+ \times \mathbb{R}$ onto itself. Let (η, q) be an entropy-entropy flux pair of the p-system (7). We can consider $(\bar{\eta}, \bar{q})$ such that

$$\begin{cases} \eta = \bar{\eta} \circ V \\ q = \bar{q} \circ V \end{cases}$$

with

$$\begin{cases} \nabla \eta = \dfrac{\partial V}{\partial U} \nabla \bar{\eta} \\ \nabla q = \dfrac{\partial V}{\partial U} \nabla \bar{q} \end{cases} \quad \text{and} \quad \left| \frac{\partial V}{\partial U} \right| = \begin{vmatrix} p_v & 0 \\ 0 & 1 \end{vmatrix} < 0 \ .$$

With the notation $f(v,u) = (-u, p(v, C_0))$, the entropy flux satisfies: $\nabla \eta . \nabla f = \nabla q$. This is equivalent to:

$$\begin{cases} q_v = \eta_u \cdot p_v \\ q_u = -\eta_v \ . \end{cases}$$

By identification, we get:

$$\begin{cases} p_v \cdot \bar{q}_v = \bar{\eta}_u \cdot p_v \\ \bar{q}_u = -\bar{\eta}_v \cdot p_v \end{cases} \quad \text{which leads to} \quad \begin{cases} \bar{q}_v = \bar{\eta}_u \\ \bar{q}_u = -\bar{\eta}_v \cdot p_v \ . \end{cases}$$

One defines:

$$\begin{cases} \widetilde{q}(v,u,c) = \bar{q}\Big(p(v,c), u\Big) \stackrel{\text{def}}{=} \bar{q} \circ W(v,u,c) \\ \widetilde{\eta}(v,u,c) = \bar{\eta}\Big(p(v,c), u\Big) \stackrel{\text{def}}{=} \bar{\eta} \circ W(v,u,c) \end{cases}$$

where

$$\begin{aligned} W\colon \ \mathbb{R}_*^+ \times \mathbb{R} \times [0,1] &\to \mathbb{R}_*^+ \times \mathbb{R} \\ (v,u,c) &\mapsto (p(v,c), u) \end{aligned} \ .$$

and we have:

$$\nabla \widetilde{q} = \begin{pmatrix} \bar{q}_v \cdot p_v \\ \bar{q}_u \\ \bar{q}_v \cdot p_c \end{pmatrix} \quad \text{and} \quad \nabla \widetilde{\eta} = \begin{pmatrix} \bar{\eta}_v \cdot p_v \\ \bar{\eta}_u \\ \bar{\eta}_v \cdot p_c \end{pmatrix} \ .$$

From the preceding relations on $\nabla \eta$ and ∇q, we get:

$$\nabla \widetilde{q} = \begin{pmatrix} \bar{\eta}_u \cdot p_v \\ -\bar{\eta}_v \cdot p_v \\ \bar{\eta}_u \cdot p_c \end{pmatrix} \quad \text{which gives} \quad \nabla \widetilde{q} = \begin{pmatrix} \widetilde{\eta}_u \cdot p_v \\ -\widetilde{\eta}_v \\ \widetilde{\eta}_u \cdot p_c \end{pmatrix} \ .$$

So we end up with $\nabla \widetilde{q} = \nabla \widetilde{\eta} . \nabla F$ and thus we are done. ∎

This formulation allows one to recover for instance the entropy proposed in [14] from the one computed in [12] (p. 399):

$$\eta(v,u) = \frac{u^2}{2} - \int^v p(s)\,ds \quad \text{gives: } \widetilde{\eta}(v,u,c) = \frac{u^2}{2} - \int^v p(s,c)\,ds\ . \tag{9}$$

3 – Some amplitude uniform estimates

We need now a uniform bound on a sequence of viscous approximations of (2); i.e. on the solutions U_ε, $U_\varepsilon(\cdot,0) = U_0 \in L^2 \cap L^\infty(\mathbb{R})$ and $\varepsilon > 0$ of the regularized system

$$\begin{cases} \partial_t v^\varepsilon - \partial_x u^\varepsilon = \varepsilon\,\partial_{xx} v^\varepsilon \\ \partial_t u^\varepsilon + \partial_x p(v^\varepsilon, c^\varepsilon) = \varepsilon\,\partial_{xx} u^\varepsilon\ , \qquad x \in \mathbb{R},\quad t > 0\ . \\ \partial_t c^\varepsilon = g(c^\varepsilon) + \varepsilon\,\partial_{xx}\, c^\varepsilon \end{cases} \tag{10}$$

The usual way to derive an L^∞ estimate for (10) is to use the notion of *positively invariant region* due to Chueh, Conley and Smoller (see [12] for detailed results).

Definition 1. We call *positively invariant region* a closed subset Σ in the states space such that any solution U^ε of (10) having its initial data U_0 contained in Σ remains therein for all $t > 0$. □

Generally, Σ is expressed as an intersection of "half-spaces":

$$\Sigma \;=\; \bigcap_{i=1}^{3} \Big\{ U \in \mathbb{R}_*^+ {\times} \mathbb{R} {\times} [0,1] \;\text{ such that } G_i(U) \le 0 \Big\}$$

where the G_i are C^2 convex and such that $\nabla G_i \neq \vec{0}$.

To apply this theory to the system (10), we must require that the Jacobian matrix of the fluxes (4) for (2) admits left eigenvectors (6) which can be expressed as gradients of some convex Riemann invariants. We are about to look for pressure laws for which this condition can be satisfied.

Lemma 2. *In the case $g \equiv 0$, it is sufficient for the pressure $p(v,c)$ to satisfy*

$$p(v,c) = f\Big(v + K(c)\Big) \quad \textit{where} \quad \begin{cases} f' < 0\,, \qquad f'' > 0, \\ K \in C^2([0,1])\,, \qquad K'' \le 0\ , \end{cases} \tag{11}$$

in order to ensure the existence of a positively invariant region Σ for (10).

Proof: According to the expression of the left eigenvectors (6), difficulties come from $\vec{L}_{\pm 1}$, especially the term p_c (except if it is zero, since in this last case, classical results [8, 12] apply). We are about to work out $\vec{L}_{-1}$ that we rewrite the following way:

$$\vec{L}_{-1} = \begin{pmatrix} -y \,.\, p_v \\ y \,.\, \sqrt{-p_v} \\ -y \,.\, p_c \end{pmatrix} \qquad \text{where} \quad y(v,u,c) \in C^1 \quad \text{and} \quad y \neq 0 \;.$$

We want to derive pressure laws for which there holds: $\operatorname{curl}(\vec{L}_{-1}) = \vec{0}$:

$$\operatorname{curl}(\vec{L}_{-1}) \;=\; \begin{vmatrix} \partial_v & -y \,.\, p_v \\ \partial_u & y \,.\, \sqrt{-p_v} \\ \partial_c & -y \,.\, p_c \end{vmatrix} \;=\; \begin{pmatrix} -p_v \,.\, y_u - (y \,.\, \sqrt{-p_v})_v \\ (y \,.\, \sqrt{-p_v})_c + p_c \,.\, y_u \\ -p_{vc} \,.\, y - p_v \,.\, y_c + p_{cv} \,.\, y + p_c \,.\, y_v \end{pmatrix} = \vec{0} \;.$$

The last line gives: $p_c \,.\, y_v = p_v \,.\, y_c$, which is satisfied by $y = a(u) \,.\, p(v,c)$. We plug this expression in the first two equations:

$$a'(u) \,.\, p \,.\nabla p + a(u)\, \nabla (p \,.\, \sqrt{-p_v}) \;=\; \vec{0} \;.$$

To remove any dependence in $u \mapsto a(u)$, we use the fact that $p_c \neq 0$:

$$\frac{p_v}{p_c} \;=\; -\frac{p_v \,.\, \sqrt{-p_v} - (p_{vv}/2\,\sqrt{-p_v})}{p_c \,.\, \sqrt{-p_v} - (p_{vc}/2\,\sqrt{-p_v})} \;.$$

This gives:

$$\frac{p_v}{p_c} = \frac{p_{vv}}{p_{vc}} \implies \frac{\partial}{\partial v}\left(\frac{p_v}{p_c}\right) = 0 \;.$$

And we get: $p_c = k(c) \,.\, p_v$. We introduce now a function $K\colon [0,1] \to \mathbb{R}$ such that $K'(c) = k(c)$. It is clear that the expression

$$p(v,c) \;=\; f\Big(v + K(c)\Big)$$

satisfies our criteria: $p_c(v,c) = k(c)\, f'(v + K(c)) = k(c)\, p_v(v,c)$.

It remains to check the convexity of the functions $U \mapsto G_i(U)$ according to this pressure law. This is about to give the condition on the second derivative of $c \mapsto K(c)$. We introduce:

$$G_{\pm 1}(v,u,c) = \pm u - \int^{v+K(c)} \lambda(s)\, ds \,, \quad \nabla G_{\pm 1} = \begin{pmatrix} -\lambda \\ \pm 1 \\ -k(c)\, \lambda \end{pmatrix} \qquad \text{with} \quad \lambda = \sqrt{-f'} \;.$$

Then we have:

$$\lambda\,.\nabla G_{-1} = -\vec{L}_{-1} = \begin{pmatrix} f' \\ -\sqrt{-f'} \\ k(c)\,f' \end{pmatrix} \quad \text{and} \quad \lambda\,.\nabla G_{1} = \vec{L}_{1} = \begin{pmatrix} f' \\ \sqrt{-f'} \\ k(c)\,f' \end{pmatrix} .$$

The assumptions on the derivatives of f ensure that $-\lambda_v = \frac{f''}{2\sqrt{-f'}} > 0$. So the matrix

$$D^2 G_{\pm 1} = \begin{pmatrix} -\lambda_v & 0 & -k(c)\,\lambda_v \\ 0 & 0 & 0 \\ -k(c)\,\lambda_v & 0 & -K''\,\lambda - k(c)^2\,\lambda_v \end{pmatrix}$$

is positive definite: $\forall\,\vec{X} = (x, y, z) \in \mathbb{R}^3$, we have:

$$\left\langle \vec{X}, D^2 G_{\pm 1}(U)\,.\,\vec{X} \right\rangle_{\mathbb{R}^3} = (-\lambda_v)\left(x + k(c)\,z\right)^2 + \left(-K''(c)\,\lambda\right) z^2 .$$

The convexity conditions are satisfied in the case where $K'' \leq 0$. ∎

Lemma 3. *In addition to (11), if one assumes $g \in C^1(\mathbb{R})$ and:*

(12) $$\forall\, c \in [0,1]\,, \qquad K'(c) \leq 0, \quad g(c) \leq 0 \;\; \textit{and} \;\; g(0) = 0 .$$

Then there exists a positively invariant region for the regularized system (10):

(13) $$\Sigma = \left\{ U = (v,u,c) \in (\mathbb{R}_*^+ \times \mathbb{R} \times [0,1]); \; \begin{bmatrix} u \leq C_1 + \displaystyle\int^{v+K(c)} \sqrt{-f'(s)}\;ds \\ u \geq C_2 - \displaystyle\int^{v+K(c)} \sqrt{-f'(s)}\;ds \end{bmatrix} \right\} .$$

Proof: First, we notice that the diffusion matrix of (10) is the identity matrix. Let $\bar{U} \in \partial\Sigma$, i.e. $G_i(\bar{U}) = 0$ for $i = \pm 1$ or $i = 0$:

– $i = 0$, i.e. $\vec{L}_0(U) = \nabla G_0(U)$. From the expression of $\vec{L}_0$, we see that $G_0(U) = c$ is convenient and the scalar product $\langle \nabla G_0(\bar{U}), S(\bar{U}) \rangle_{\mathbb{R}^3}$ is non-positive in $c = 1$ and $c = 0$.

– $i = \pm 1$: we rewrite $\vec{L}_{\pm 1}$ according to (11):

$$\vec{L}_{\pm 1}(U) = \begin{pmatrix} -\sqrt{-f'(v + K(c))} \\ \pm 1 \\ -k(c)\,\sqrt{-f'(v + K(c))} \end{pmatrix} .$$

So $G_{\pm 1}(U) = \pm u - \int^{v+K(c)} \sqrt{-f'(s)}\,ds$. On the boundary, we have $c \in [0,1]$ and $u = C \pm \int^{v+K(c)} \sqrt{-f'(s)}\,ds$. We have to ensure that:

$$\left\langle \nabla G_{\pm 1}(\bar{U}), S(\bar{U}) \right\rangle_{\mathbb{R}^3} = -g(c)\,.\,k(c)\,.\,\sqrt{-f'(v + K(c))} \;\leq\; 0 .$$

It is thus sufficient that $g(c) \leq 0$ since $K'(c)$ is negative according to (12). ∎

Under the restrictions (11), (12), the "mechanical entropy" $U \mapsto \widetilde{\eta}(U)$, (9), is strictly convex.

We thus see that even if one assumes the restriction on the pressure law (11), one can derive invariant regions for (10) only in the case of *sink terms* of constant sign for which λ, the sound speed in the mixture, decreases. The severe conditions on the right-hand side come from the fact that *the rectangles of the plane* $(v + K(c), u)$ *are not invariant* by (10).

In [4], Chen, Levermore et Liu studied a p-system endowed with a relaxation term on the velocity equation of the form $f(v) - u$. The same way, they assume the sub-characteristic condition $|p'(v)| - f'(v)^2 \geq 0$ in order to preserve the usual invariant region (see e.g. [12]) of this system.

4 – Compactness and existence of solutions

We can now state our main result concerning the existence of L^∞ solutions for (2) which is related to case 1 in Theorem 2.1, [3].

Theorem 1. *Let $(v_0, u_0) \in L^2 \cap L^\infty(\mathbb{R})$, and $c_0 \in L^2 \cap BV(\mathbb{R})$ with $c_0 \in [0,1]$. Under the hypotheses (3), (11), (12), the solutions of (10) satisfying $|v^\varepsilon| \leq M < +\infty$ converge almost everywhere (up to the extraction of a subsequence) as $\varepsilon \to 0$ towards a weak solution $L^\infty([0,T] \times \mathbb{R})$, $T \in \mathbb{R}_*^+$ of (2) which satisfies in the sense of distributions:*

$$\partial_t \widetilde{\eta}(U) + \partial_x \widetilde{q}(U) \;\leq\; \nabla\widetilde{\eta}(U) \,.\, S(U) \;. \tag{14}$$

Proof: We notice that the change of variables

$$U = (v, u, c) \;\mapsto\; \widetilde{U} = (w = v + K(c), u, c) \tag{15}$$

is a diffeomorphism which weakens the coupling in (10). We get:

$$\begin{cases} \partial_t w^\varepsilon - \partial_x u^\varepsilon - k(c^\varepsilon)\, \partial_t c^\varepsilon = \varepsilon\, \partial_{xx} v^\varepsilon \\ \partial_t u^\varepsilon + \partial_x p(w^\varepsilon) = \varepsilon\, \partial_{xx} u^\varepsilon \\ \partial_t c^\varepsilon = g(c^\varepsilon) + \varepsilon\, \partial_{xx}\, c^\varepsilon \;. \end{cases} \tag{16}$$

The hypotheses (3), (11), (12) ensure the existence of a positively invariant region for the regularized system (16). We first show the compactness of the sequence $(w^\varepsilon, u^\varepsilon)$.

We have: $\partial_{xx} w^\varepsilon = \partial_{xx} v^\varepsilon + \partial_x (k(c^\varepsilon)\, \partial_x c^\varepsilon)$. The first equation rewrites:

$$\partial_t w^\varepsilon - \partial_x u^\varepsilon \;=\; \varepsilon \Big(\partial_{xx} w^\varepsilon - k'(c^\varepsilon) \,.\, (\partial_x c^\varepsilon)^2 \Big) + k(c^\varepsilon) \,.\, g(c^\varepsilon) \ .$$

Let $\eta(w,u)$ be a C^2 entropy for the usual p-system, $q(w,u)$ its associated entropy flux. Following DiPerna [8], we plan to show:

$$\partial_t \eta(w^\varepsilon, u^\varepsilon) + \partial_x q(w^\varepsilon, u^\varepsilon) \quad \text{compact in } \ H^{-1}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+_*) \ . \tag{17}$$

The classical theory applies directly to the terms $\varepsilon\, \partial_{xx} w^\varepsilon$, $\varepsilon\, \partial_{xx} u^\varepsilon$. We are left with:

$$\eta_w(w^\varepsilon, u^\varepsilon) \,.\, \Big[k(c^\varepsilon) \,.\, g(c^\varepsilon) - \varepsilon\, k'(c^\varepsilon) \,.\, (\partial_x c^\varepsilon)^2 \Big] \ ,$$

in which c^ε is solution of $\partial_t c^\varepsilon = g(c^\varepsilon) + \varepsilon\, \partial_{xx} c^\varepsilon$.

Since we have L^∞ bounds on w^ε, u^ε, c^ε, the regularity of η and k gives the same type of bounds for $\eta_w(w^\varepsilon, u^\varepsilon)$ and $k(c^\varepsilon)$. Now, since $g(0) = 0$, $g(c^\varepsilon) \in L^2(\mathbb{R}\times[0,T])$, the term $\eta_w(w^\varepsilon, u^\varepsilon) \,.\, k(c^\varepsilon) \,.\, g(c^\varepsilon)$ is uniformly bounded in L^2; it is therefore compact in H^{-1}.

Concerning $\eta_w(w^\varepsilon, u^\varepsilon) \,.\, \varepsilon\, k'(c^\varepsilon) \,.\, (\partial_x c^\varepsilon)^2$, we have to study the equation on c^ε. We multiply it by c^ε and integrate on $\mathbb{R}\times[0,T]$. Since $c^\varepsilon(\cdot,t) \in H^1(\mathbb{R})$ is zero at infinity for $t \in [0,T]$:

$$\varepsilon \int_{\mathbb{R}\times[0,T]} c^\varepsilon \,.\, \partial_{xx} c^\varepsilon(x,t)\ dx\, dt \;=\; -\varepsilon \int_{\mathbb{R}\times[0,T]} (\partial_x c^\varepsilon)^2(x,t)\ dx\, dt \;\le\; 0 \ ,$$

and Gronwall's lemma implies that:

$$\| c^\varepsilon(\cdot, T) \|^2_{L^2(\mathbb{R})} \;\le\; \| c_0 \|^2_{L^2(\mathbb{R})} \exp\Big(2 \,.\, \| g' \|_{L^\infty} \,.\, T \Big) \ .$$

Plugging this estimate in the equation, we obtain:

$$\varepsilon \int_{\mathbb{R}\times[0,T]} (\partial_x c^\varepsilon)^2(x,t)\ dx\, dt \;\le\; \| c_0 \|^2_{L^2(\mathbb{R})} \exp\Big(2 \,.\, \| g' \|_{L^\infty} \,.\, T \Big) \ .$$

The lemma of Murat [10] ensures that (17) holds true and the compactness theorem of DiPerna [8] guarantees that $(w^\varepsilon, u^\varepsilon) \to (w,u)$ strongly in all the $L^{p<\infty}$ and almost everywhere up to the extraction of a subsequence. The strong convergence of the sequence c^ε can be obtained directly using Helly's compactness principle. ∎

Theorem 1 extends to the system (1) thanks to the results of [15]. For certain pressure laws, no vacuum state can appear as soon as the initial data has a bounded specific volume, see [12].

5 – Remarks and extensions

This method cannot handle the case of a right-hand side of the type $g(\rho, c)$ since one cannot treat simultaneously the limit $\varepsilon \to 0$ by means of DiPerna's theorem together with a fixed point algorithm in order to handle the coupling. Nevertheless, we give here an extension of a stability result (see [9], p. 189) in the case where (1) is endowed with a *relaxation term*, following [4].

Lemma 4. *Let $g(\rho, c) = \mu(\rho) - c$ where $\mu \in C^1(\mathbb{R}^+; [0,1])$ is strictly monotone and satisfies the condition $\partial_c p(\rho, c) \,.\, \partial_\rho \mu(\rho) \le 0$ for all $(\rho, c) \in \mathbb{R}^+_* \times [0,1]$. Then under the hypotheses (3), (11), (12), the following C^2 function:*

$$\eta^\circ(\rho, \rho c, \rho u) \;=\; \frac{\rho u^2}{2} + \rho\left(\int_{\mu^{-1}(c)}^{\rho} \frac{p(s,c)}{s^2}\, ds + \int^{\mu^{-1}(c)} \frac{p(s,\mu(s))}{s^2}\, ds\right) \tag{18}$$

is a strictly convex entropy for (1) and satisfies $\partial_{\rho c}\eta^\circ(\rho, \rho c, \rho u) . g(\rho, c) \le 0$.

Proof: We have to show that $\partial_{\rho c}\eta^\circ(\rho, \rho c, \rho u)\,(\mu(\rho) - c) \le 0$. We therefore write:

$$\partial_{\rho c}\eta^\circ(\rho, \rho c, \rho u) \;=\; \frac{\partial_c \eta^\circ(\rho, c, \rho u)}{\rho} \;.$$

And we get:

$$\partial_{\rho c}\eta^\circ(\rho, \rho c, \rho u) \;=\; \partial_c\left[\int_{\mu^{-1}(c)}^{\rho} \frac{p(s,c)}{s^2}\, ds \;+\; \int^{\mu^{-1}(c)} \frac{p(s,\mu(s))}{s^2}\, ds\right] .$$

We carry out the differentiations inside the integral and at the endpoints of the summation interval:

$$\begin{aligned} \partial_{\rho c}\eta^\circ(\rho, \rho c, \rho u) \;=\; &\int_{\mu^{-1}(c)}^{\rho} \partial_c\left(\frac{p(s,c)}{s^2}\right) ds \\ &- \partial_c \mu^{-1}(c) \,.\, \frac{p(\mu^{-1}(c), c)}{\mu^{-1}(c)^2} \;+\; \partial_c \mu^{-1}(c) \,.\, \frac{p(\mu^{-1}(c), c)}{\mu^{-1}(c)^2} \;. \end{aligned} \tag{19}$$

And the last two terms in (19) cancel. By the smoothness of p, one can use the mean-value theorem to find the sign of the first integral of (19). The restriction on the signs of $\partial_c p(\rho, c)$ and $\partial_\rho \mu(\rho)$ allows to conclude that $\partial_{\rho c}\eta^\circ(\rho, \rho c, \rho u) . g(\rho, c) \le 0$. We check that η is a strictly convex entropy following the classical ways, see e.g. [14]. ∎

This work is closely related to [3] where Béreux, Bonnetier and LeFloch studied the homogeneous Euler equations in the case where the closure law is such that *the Lagrangian sound speed is a function of the pressure only*. This means:

$$\rho^2\,\partial_\rho(p)(\rho,S) = a(p)(\rho,S)$$

where S is the physical entropy of the gas dynamics system. In this context, this restriction can be expressed:

$$\partial_\rho(p) = -\frac{\partial_v(p)}{\rho^2}, \quad \text{i.e.} \quad -\,\partial_v(p) = a(p)\ .$$

According to the sign conventions for $\partial_v(p)$, this differential equation gives:

$$\partial_v\Big(A(p)\Big) = 1 \quad \text{and} \quad \rho(v,S) = A^{-1}\Big(\sigma + K(S)\Big)\ .$$

In [14], I. Toumi derives a (non-convex) invariant region for the Riemann problem applied to the system (1) in the homogeneous case and with a diphasic vapour/water pressure law of the type:

$$p(\rho,c) = \left(\frac{1.6\,\rho\,(1-c)}{1.6-\rho\,c}\right)^{\gamma}, \qquad \gamma > 1\ .$$

We notice that it is not of the form prescribed by (11). An alternative for the study of such a problem in the context of bounded variations functions could follow the approach of Glimm [1, 6, 12]. See also [11] for a study of a two-phase flow system endowed with a special pressure law using the Glimm scheme.

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