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REAL ALGEBRAIC CURVES AND REAL ALGEBRAIC FUNCTIONS

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Abstract: In this paper we consider real generic holomorphic functions $f\colon \mathcal{C} \to \mathbf{P}^1(\mathbf{C})$, where $\mathcal{C}$ is a compact connected Riemann surface of genus g. f is said to be generic if all the critical values have multiplicity one and it is real if and only if there exists an antiholomorphic involution σ acting on $\mathcal{C}$ such that for all z in $\mathcal{C}$, $f \circ \sigma(z) = \overline{f(z)}$. It is possible to give a combinatoric description of the monodromy of the unramified covering obtained by restricting f to $\mathcal{C} - f^{-1}(B)$, where B is the set of critical values of f. In this paper we want to describe the topological type of the antiholomorphic involution σ of the Riemann surface $\mathcal{C}$ that gives the real structure, once we know the monodromy graph of f. More precisely, we give a lower bound on the number of connected components of the fixed point locus of σ in terms of the monodromy graph, in the case in which f has all real critical values. Moreover, we are able to determine the exact number of the fixed components of σ in terms of the monodromy graph, when the monodromy graph satisfies some suitable properties.

Introduction

Let $f\colon \mathcal{C} \to \mathbf{P}^1(\mathbf{C})$ be a non constant holomorphic function of degree d, where $\mathcal{C}$ is a compact connected Riemann surface of genus g. We assume that there exists an antiholomorphic involution σ on $\mathcal{C}$ and that f is a real holomorphic map, that is $f \circ \sigma(z) = \overline{f(z)}$, $\forall\, z \in \mathcal{C}$.

In [Fr1] it is given a combinatorial description of the monodromies of generic real algebraic functions.

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The purpose of this paper is to understand the topological type of the involution σ on $\mathcal{C}$ from the monodromy graph of the real algebraic function $f\colon \mathcal{C} \to \mathbf{P}^1$.

Let Σ be a compact connected Riemann surface of genus g, let $M(\Sigma)$ denote the set of all the complex structures of Σ.

Consider a curve with a complex structure $X \in M(\Sigma)$, X can be embedded in a projective space $\mathbf{P}^n(\mathbf{C})$ in such a way that the image $\mathcal{C}$ of X in the projective space is defined by a finite number of polynomial equations, i.e. $\mathcal{C}$ is a complex algebraic curve.

If $\mathcal{C}$ is defined by real polynomials, then $\mathcal{C}$ is invariant under complex conjugation in $\mathbf{P}^n(\mathbf{C})$. Complex conjugation induces an antiholomorphic involution on X, $\sigma : X \to X$.

Viceversa, if X is a Riemann surface together with an antiholomorphic involution σ, it is possible to choose a pluricanonical embedding of X into $\mathbf{P}^n(\mathbf{C})$ in such a way that σ is the involution induced by complex conjugation. So X is a complex algebraic curve defined by real polynomials.

An antiholomorphic involution $\sigma\colon X \to X$ is induced by an orientation reversing involution of the topological surface $\sigma\colon \Sigma \to \Sigma$.

Definition 0.1. Two orientation reversing involutions $\sigma\colon \Sigma \to \Sigma$, $\tau\colon \Sigma \to \Sigma$ are said to be *of the same topological type* if there exists a homeomorphism $f\colon \Sigma \to \Sigma$ such that $\tau = f \circ \sigma \circ f^{-1}$.

Equivalently we could say that σ and τ are of the same topological type iff $\Sigma/\langle\sigma\rangle$ is homeomorphic to $\Sigma/\langle\tau\rangle$, where $\langle\sigma\rangle$ is the group generated by $\sigma\colon \Sigma \to \Sigma$. □

Let $\sigma\colon \Sigma \to \Sigma$ be an orientation reversing involution, let Σ_σ be the fixed-point set of σ and let ν be the number of connected components of Σ_σ.

Definition 0.2. The orientation reversing involution σ of the surface Σ is said to be of type $(g, \nu, a = 0)$ if $\Sigma - \Sigma_\sigma$ is not connected. Otherwise σ is of type $(g, \nu, a = 1)$. We say that a real algebraic curve is of type $(g, \nu, 0)$, or $(g, \nu, 1)$ according to the type of its topological model. □

Remark 0.3. If $a(\sigma) = 0$ and Σ_σ is the fixed point locus of σ then the number of components of $\Sigma - \Sigma_\sigma$ is two. □

In fact two adjacent connected components A, B of $\Sigma - \Sigma_\sigma$ are homeomorphic, since they are exchanged by σ. σ fixes the common boundary of $\bar{A}$ and $\bar{B}$, so the closure of $A \cup B$ is a compact subvariety of Σ, hence it concides with Σ.

Theorem 0.4. *Let $\nu \geq 0$ be an integer. Assume that (g, ν, a) is the topological type of a real algebraic curve. If $a = 1$, then $0 \leq \nu \leq g$, if $a = 0$, then $1 \leq \nu \leq g + 1$ and $\nu \equiv g + 1 \pmod 2$. These are the only restrictions for the topological types of real algebraic curves of genus g.* ∎

The second inequality, $1 \leq \nu \leq g + 1$, is a famous theorem of Harnack, the rest was proved by Klein and Weichhold (see [Kl], [We]).

One computes that there are $[(3g + 4)/2]$ topological types of orientation reversing involutions of a genus g surface.

In this paper we consider a real generic algebraic function $f\colon \mathcal{C} \to \mathbf{P}^1$ with all real critical values and we want to determine the topological invariants $a(\sigma)$ and $\nu(\sigma)$ of the antiholomorphic involution σ acting on the Riemann surface $\mathcal{C}$.

In this contest we would like to mention a paper of Natanzon ([Na2]) in which he studies the Hurwitz space of isomorphism classes of holomorphic mappings $f\colon \mathcal{C} \to \mathbf{P}^1$ both in the complex and in the real case, and in which, in particular in the real case he gives a complete description of the topological invariants of the antiholomorphic involution acting on $\mathcal{C}$ and of the real map f.

In [Fr1] we give a complete description of all the monodromy graphs of generic real algebraic functions: if $\mathcal{C}$ is a compact (connected) Riemann surface of genus g and $f\colon \mathcal{C} \to \mathbf{P}^1(\mathbf{C})$ is an algebraic function of degree d we say that f is generic if all its critical values have multiplicity one. In fact we generalize the results obtained in [C-F] in the case of polynomial mappings, i.e. holomorphic mappings $f\colon \mathbf{P}^1(\mathbf{C}) \to \mathbf{P}^1(\mathbf{C})$ such that there exists a point p in the target for which $f^{-1}(p)$ consists of only one point.

The problem of giving a topological classification of polynomial mappings was reduced by Davis and Thom (see [Da], [Th]) to a combinatorial problem, more precisely in 1957 C. Davis ([Da]) showed that for each choice of n distinct real numbers there is a real polynomial of degree $(n+1)$ having those as critical values, and a similar question was asked for complex polynomials. Thom in 1965 ([Th]) observed that by Riemann's existence theorem the answer is that for each choice of n distinct complex numbers and an equivalence class of admissible monodromy, there exists exactly one polynomial, up to affine transformations in the source, having those points as critical values and the given monodromy.

The case in which f is rational, $f^{-1}(\infty)$ has cardinality 2 and f is complex has been treated by Arnold in [Ar4] where he counts all the monodromy graphs in the complex case. In this case the monodromy graphs have the same number n of edges and vertices and the product of the transpositions corresponding to the edges consists of two cycles of lengths p and q with $p+q=n$, where p and q are the orders of the poles of the Laurent polynomial f. In this paper Arnold gives a formula that counts such monodromy graphs depending on p and q.

The general (complex) case in which f has only ∞ as non generic critical value has been recently solved by Goryunov and Lando (see [G-L]). Hurwitz in 1891 published a conjecture giving the number of topological types of rational functions on $\mathbf{P}^1(\mathbf{C})$ with fixed orders of poles and fixed critical values, assuming that all the critical values except ∞ were generic (see [Hu]). Goryunov and Lando gave a proof of the conjecture using, as Arnold already did for Laurent polynomials, properties of the Lyashko–Looijenga mapping.

We describe now in detail the structure of the paper: the first section contains results of [Fr1], namely we consider a compact (connected) Riemann surface $\mathcal{C}$ of genus g and an algebraic function of degree d, $f\colon \mathcal{C} \to \mathbf{P}^1(\mathbf{C})$. We assume that f is generic, then in particular, by Hurwitz's formula we know that the number of critical values of f is $2g+2d-2$.

If $\mathcal{B}$ is a finite set and 0 and ∞ are not in $\mathcal{B}$, by Riemann's existence theorem one knows that there is a bijection between

(**1**) the set of conjugacy classes of homomorphisms $\mu\colon\ \pi_1(\mathbf{P}^1(\mathbf{C})-\mathcal{B},0) \to \mathcal{S}_d$ such that $Im(\mu)$ is a transitive subgroup, and for a given basis $\gamma_1,...,\gamma_n$ of $\pi_1(\mathbf{C}-\mathcal{B},0)$, $\mu(\gamma_i)=\sigma_i$ is a transposition, the product $\sigma_1...\sigma_n = id$

and

(**2**) the set of equivalence classes of algebraic maps f of degree d that are generic with branch set equal to $\mathcal{B}$.

If f is real then $\mathcal{B}$ is selfconjugate and viceversa if $\mathcal{B}$ is selfconjugate then f is real if and only if complex conjugation on $\mathbf{P}^1(\mathbf{C})$ lifts to $\mathcal{C}$.

Once and for all we fix the canonical geometric basis of $\pi_1(\mathbf{P}^1(\mathbf{C})-\mathcal{B},0)$ described in [C-F] (see Fig. 1). Then as in [B-C], we associate to the class, modulo inner automorphisms of $\mathcal{S}_d$, of the monodromy of an algebraic generic function $f\colon \mathcal{C} \to \mathbf{P}^1(\mathbf{C})$ of degree d a connected graph with d vertices and $2g+2d-2$ labeled edges in such a way that the vertices correspond to the points in the fiber $f^{-1}(0)$, the edge labeled by i connects two vertices iff they are interchanged by $\mu(\gamma_i)$.

We state as in [Fr1] all the necessary conditions that a graph must satisfy in order to be the monodromy graph of a generic real algebraic function of degree d from a compact Riemann surface of genus g. Then we state the theorem of [Fr1] where we prove that the conditions that we have found, are also sufficient.

The reduced graph $\mathcal{G}_{red}$ of the monodromy graph $\mathcal{G}$ of f is defined to be the graph obtained from $\mathcal{G}$ in this way: for every two vertices we remove some edges of $\mathcal{G}$ in such a way that there remains only one edge connecting the two vertices.

A polygon contained in $\mathcal{G}$ is said to be odd if it has an odd number of edges.

We obtain the two following results

Theorem 0.5. *Let $\mathcal{C}$ be a real smooth algebraic curve of genus g, $\sigma\colon \mathcal{C} \to \mathcal{C}$ be the antiholomorphic involution that gives the real structure, assume that $\nu(\sigma) \neq 0$, let $f\colon \mathcal{C} \to \mathbf{P}^1_{\mathbf{C}}$ be a generic real algebraic function of degree $d \geq 2$ $(f \circ \sigma(x) = \overline{f(x)},\ \forall\, x \in \mathcal{C})$.*

Assume that all the critical values of f are real and positive (if they are not positive it suffices to perform a base point change).

Let $\mathcal{G}$ be the monodromy graph of f. Assume furthermore that any two polygons contained in $\mathcal{G}_{red}$ have no common edges. Set $\sigma_* : H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z}) \to H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})$. Then

$$\dim(\sigma + identity)_*\Big(H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})\Big) \;=\; \#\{odd\ polygons\ in\ \mathcal{G}_{red}\}\ .$$

So

$$\nu(\sigma) \;=\; g + 1 - \#\{odd\ polygons\ in\ \mathcal{G}_{red}\}\ .$$

In particular, if $\mathcal{G}_{red}$ does not contain any polygon, $\nu(\sigma) = g + 1$. ∎

Theorem 0.6. *Let $\mathcal{C}$ be a real smooth algebraic curve of genus g, $\sigma\colon \mathcal{C} \to \mathcal{C}$ be the antiholomorphic involution that gives the real structure, assume that $\nu(\sigma) \neq 0$, let $f\colon \mathcal{C} \to \mathbf{P}^1_{\mathbf{C}}$ be a generic real algebraic function of degree $d \geq 2$ $(f \circ \sigma(x) = \overline{f(x)},\ \forall\, x \in \mathcal{C})$.*

Assume that all the critical values of f are real and positive (if they are not positive it suffices to perform a base point change).

Let $\mathcal{G}$ be the monodromy graph of f. Then

$$\dim(\sigma + identity)_*\Big(H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})\Big) \;\leq\; \rho(\mathcal{G}_{red})\ ,$$

where $\rho(\mathcal{G}_{red})$ is the minimal number of polygons of $\mathcal{G}_{red}$ whose union is the union of all the polygons of $\mathcal{G}_{red}$. So $\nu(\sigma) \geq g + 1 - \rho(\mathcal{G}_{red})$. ∎

1 – Monodromies of generic real algebraic functions

In this section we explain some results that are in [Fr1]. Let $\mathcal{C}$ be a compact (connected) Riemann surface of genus g; a necessary condition for an algebraic function $f: \mathcal{C} \to \mathbf{P}^1$ to be real is that the branch locus $\mathcal{B}$ of f must be selfconjugate. Therefore the critical values of f are k real critical values $w_1, ..., w_k$ and m pairs $(v_i, \bar{v_i})$ of complex conjugate critical values where v_i is in the upper half plane.

If $\mathcal{B}$ is selfconjugate f is real if and only if complex conjugation on $\mathbf{P}^1$ lifts to $\mathcal{C}$. This means that complex conjugation fixes the monodromy class $[\mu]$ (obviously we have chosen a basis of π_1).

We suppose that 0 and ∞ are not critical values and we choose a geometric basis of $\pi_1(\mathbf{P}^1 - \mathcal{B}, 0)$ by taking $\gamma_1, ..., \gamma_k$ loops around w_i and pairs of selfconjugate loops $(\delta_i, \bar{\delta}_i)$ around $(v_i, \bar{v_i})$, where δ_i is in the upper half plane (see Fig. 1). The circles around the critical values in the loops are performed counterclockwise.

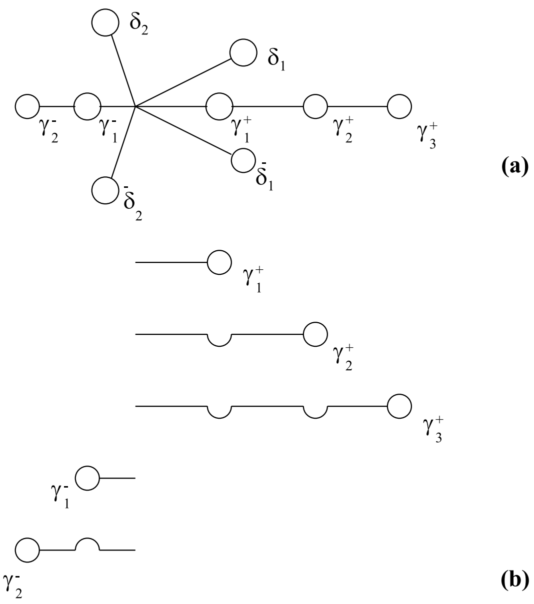

Fig. 1

Suppose now that $f: \mathcal{C} \to \mathbf{P}^1$ is a generic algebraic function. We partition the set $\mathcal{B}$ of critical values into two subsets: the set of negative critical values ${w_s}^- < ... < {w_1}^- < 0$ and the set of positive critical values $0 < {w_1}^+ < ... < {w_r}^+$. With the choice of a geometric basis for $\pi_1(\mathbf{P}^1(\mathbf{C}) - \mathcal{B}, 0)$ as in [C-F] (see Fig. 1) we have: $\bar{\gamma}_i^+ = (\gamma_1^+)^{-1} (\gamma_2^+)^{-1} ... (\gamma_{i-1}^+)^{-1} (\gamma_i^+)^{-1} {\gamma_{i-1}}^+ ... {\gamma_1}^+$, analogously for $\bar{\gamma_i}^-$.

We can therefore conclude that μ is the monodromy of a real algebraic function if and only if there exists a permutation α of period 2 (induced by complex conjugation on $f^{-1}(0)$) such that, if $\tau_i=\mu(\gamma_i^+)$, $\tau_i'=\mu(\gamma_i^-)$, $\nu_j=\mu(\delta_j)$, $\bar{\nu_j}=\mu(\bar{\delta_j})$, $\rho_{i-1}=\tau_1\tau_2...\tau_{i-1}$, $\rho'_{i-1}=\tau'_1...\tau'_{i-1}$, we have:

$$\alpha\,\tau_i\,\alpha=\rho_{i-1}\,\tau_i\,\rho_{i-1}^{-1}\,,\qquad \alpha\,\tau_i'\,\alpha=\rho'_{i-1}\,\tau_i'\,\rho_{i-1}'^{-1}\,,\qquad \alpha\,\nu_j\,\alpha=\bar{\nu_j}\ . \tag{$**$}$$

Let now $\mathcal{G}$ be the monodromy graph of f, $\mathcal{E}^+$ be the subgraph of $\mathcal{G}$ with the edges labeled by the w_j^+'s. Analogously we define $\mathcal{E}^-$. Let $\mathcal{S}_i$ be the subgraph of $\mathcal{G}$ with the edges labeled by those w_j^+'s with $j\le i$. In the same way we define $\mathcal{S}'_i$; finally let $\mathcal{G}^*$ be the subgraph of $\mathcal{G}$ with labels given by the ν_j's and by the $\bar{\nu_j}$'s. For a subgraph $\mathcal{S}$, we define $\mathrm{supp}(\mathcal{S})$ as the union of the vertices of $\mathcal{S}$.

Remark 1.1. If we order the loops δ_i's of the basis of π_1 in increasing order starting from the first that we meet if we move counterclockwise with respect to the positive direction of $\mathbf{R}$, we have:

$$\gamma_r^+...\gamma_1^+\,\delta_1...\delta_m\,\gamma_s^-...\gamma_1^-\,\bar{\delta_m}...\bar{\delta_1}=id\ ,$$

$$\bar{\nu_1}...\bar{\nu_m}\,\tau_1'...\tau_s'\,\nu_m...\nu_1\,\tau_1...\tau_r=id\,,\qquad r+s+2m=2d+2g-2\ .\ \square$$

Remark 1.2. Every vertex of $\mathcal{G}$ is contained in at least two edges. $\square$

Proof: If there exists one vertex which is contained in only one edge of $\mathcal{G}$, this is moved only by one transposition, thus the product of all the transpositions corresponding to the edges of $\mathcal{G}$ is not the identity. ∎

Consider now the ordering on the labels of the edges induced by the natural ordering on $\mathbf{R}$ of the critical values, i.e. $\sigma_1=\tau_s'$, ..., $\sigma_s=\tau_1'$, $\sigma_{s+1}=\tau_1$, ..., $\sigma_{s+r}=\tau_r$.

To state the necessary conditions that $\mathcal{G}$ must satisfy in order to be the monodromy graph of a generic real algebraic function, we need some technical definitions.

Definition 1.3. Let $\mathcal{T}$ be a subgraph of $\mathcal{E}=\mathcal{E}^+\cup\mathcal{E}^-$. Let v be a vertex of $\mathcal{T}$. Let $(k_1,...,k_t)\in\mathbf{N}^t$ with $k_i<k_{i+1}\ \forall\,i=1,...,t-1$, such that $\sigma_{k_1},...,\sigma_{k_t}$ are the transpositions that correspond to all the edges of $\mathcal{T}$ that contain v. We say that $\mathcal{T}$ is **saturated** in v if $\forall\,m$ such that σ_m is a transposition corresponding to an edge of $\mathcal{E}$ containing v, then either $m<k_1$, or $m>k_t$, or $m\in\{k_1,...,k_t\}$.

$\mathcal{T}$ is said to be **saturated** if it is saturated in every vertex v. $\square$

Definition 1.4. A **triod** is a graph with three edges and 4 vertices with respective valences 3, 1, 1, 1. □

Definition 1.5. An **order degenerate saturated triod** is a saturated graph which is associated to the following transpositions: $\sigma_i = (a, b) = \sigma_k$, $\sigma_j = (b, c)$, $i < j < k$, $c \neq a$ (see Fig. 2(a)). □

Definition 1.6.

- A **3-path** is a graph made of three consecutive distinct edges.
- A three-path with labeled edges is said to be **snake** if for the labeled edge in the middle the labels of the neighboring edges are either both greater or both smaller than its label (see Fig. 2(b)).
- A **non-degenerate triangle** is the graph associated to the following transpositions: $\sigma_i = (a, b)$, $\sigma_j = (b, c)$, $\sigma_k = (c, a)$, with a, b, c all distinct (see Fig. 2(c)). □

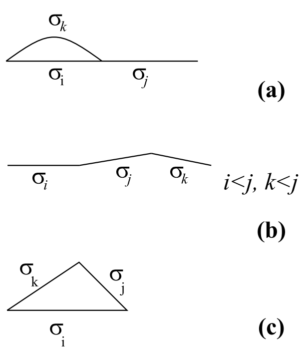

Fig. 2

Let us now suppose that all the critical values are real and positive, so $\tau_i = \sigma_i$ $\forall\, i$. With simple computations, using $(**)$, (see [Fr1]) one can prove the following lemmas.

Lemma 1.7. (See [Fr1]) *There don't exist saturated triods $\mathcal{T}$ in $\mathcal{E}$ and there don't exist order degenerate saturated triods.* ■

Lemma 1.8. (See [Fr1]) *Any saturated tree-path is snake and there doesn't exist any non-degenerated saturated triangle.* ■

Observe that in [C-F] we proved that the monodromy graphs that are associated to real generic polynomials are only linear snakes, that means that there are no triods, and that every three path is snake. In this case, since the graph is a tree, the condition of non existence of triods is equivalent to the condition of non existence of saturated triods, so 1.7 and 1.8 give generalizations of the necessary conditions found for polynomials in [C-F].

Lemma 1.9. (See [Fr1]) *Let $\mathcal{G}$ be the monodromy graph of a real generic algebraic function $f\colon \mathcal{C} \to \mathbf{P}^1(\mathbf{C})$ of degree $d \geq 2$, where $\mathcal{C}$ is a Riemann surface of genus g. If $g = 0$ and $\mathcal{T}$ is a subgraph of $\mathcal{G}$ made of two vertices and r edges that connect the two vertices, then $r \leq 2$. If $g > 0$ and $\mathcal{T}$ is a saturated subgraph of $\mathcal{G}$ made of two vertices and r edges that connect the two vertices, then $r \leq 2 + 2g$, $\forall\, d \geq 2$.* ∎

The proof of Lemma 1.9, which we will not reproduce, is made in two steps: at first we prove the statement with simple geometric methods in the case of genus $g = 0$, then we use induction on g.

Now we would like to generalize the results of 1.7 and 1.8, but we need the following

Definition 1.10. Let a be a vertex of a graph $\mathcal{G}$ with labeled edges. We say that a set of indices $\{i_0, ..., i_r,\ i_0 < i_1 < ... < i_r\}$ is a **saturated angle** in a if $a \in \mathrm{supp}(\sigma_{i_s})$, $\forall\, s = 0, ..., r$, and if $\forall\, j \geq 1\ i_j = \min\{m > i_{j-1} |\ a \in \mathrm{supp}(\sigma_m)\}$. □

Recall that we are now assuming $\sigma_i = \tau_i$, $\forall\, i$. Using $(**)$ we can show the following two lemmas that generalize respectively 1.7 and 1.8.

Lemma 1.11. (See [Fr1]) *Let $\mathcal{G}$ be as above. Let $\tau_h = (b, c)$. Let $h < r_1 < r_2 < ... < r_k < i$ be a saturated angle in b and suppose that $\tau_{r_1} \neq \tau_h$, $\tau_{r_i} = \tau_{r_1} = (b, d)$ $\forall\, i = 1, ..., k$, $d \neq c$, $\tau_i = (a, b)$, $a \neq d$.*

Then k is even. ∎

Lemma 1.12. (See [Fr1]) *Let $\tau_h = (c, d)$, $h < r_1 < r_2 < ... < r_m$ a saturated angle in d, $\tau_{r_1} = \tau_{r_2} = ... = \tau_{r_m} = (d, b)$, $b \neq c$. Let $i = \min\{r > r_m |\ b \in \mathrm{supp}(\tau_r)\}$, $\tau_i = (b, a)$, $a \neq d$. Then m is even.* ∎

We are now ready to define on a graph $\mathcal{G}$ that satisfies these necessary conditions an involution α.

Theorem 1.13. (See [Fr1]) *Let d, g be two integers, $d \geq 3$, $g \geq 0$. Let $\mathcal{G}$ be a connected graph with d vertices and $2d+2g-2$ edges with labels $\tau_1, ..., \tau_{2d+2g-2}$ that verifies the necessary conditions 1.1, 1.2, 1.9, 1.11, 1.12. Then there is a canonical procedure which associates to $\mathcal{G}$ an involution α.* ∎

We don't give the proof of the theorem, which is in [Fr1], but we describe how α is defined, since we will use its definition in the sequel.

Every vertex b is in the support of two edges of $\mathcal{E}^+$.

Let $i = \min\{h|\ b \in \mathrm{supp}(\tau_h)\}$, $j = \min\{k > i|\ b \in \mathrm{supp}(\tau_k)\}$, $\tau_i = (a, b)$.

1. $\tau_i \neq \tau_j$. We define $\alpha(b) = a$.

2. $\tau_i = \tau_j = (a, b)$.

(**a**) $\exists\, h > j|\ b \in \mathrm{supp}(\tau_h)$ and $\tau_h \neq \tau_i$. Choose $h = \min\{k > j|\ b \in \mathrm{supp}(\tau_k)$, $\tau_k \neq \tau_i\}$. Let $i = r_1 < r_2 < ... < r_t < h$ be a saturated angle in b. By assumption $\tau_{r_k} = (a, b)\ \forall\, k = 1, ..., t$.
If t is even we define $\alpha(b) = b$.
If t is odd we define $\alpha(b) = a$.

(**b**) $\forall\, h > j$ s.t. $b \in \mathrm{supp}(\tau_h)$, $\tau_h = \tau_j$. Since $d \geq 3$ and $\mathcal{G}$ is connected, there exists $h|\ a \in \mathrm{supp}(\tau_h)$ and $\tau_h \neq \tau_i$.

(**i**) $i = \min\{k|\ a \in \mathrm{supp}(\tau_k)\}$ and $a \in \mathrm{supp}(\tau_h)$, $h > i$, $\tau_h \neq \tau_i$. Take $h = \min\{k > i|\ a \in \mathrm{supp}(\tau_k),\ \tau_k \neq \tau_i\}$. Let $i = r_1 < r_2 < ... < r_t < h$ be a saturated angle in a, then by assumption $\tau_{r_j} = (a, b)\ \forall\, j = 1, ..., t$.
Then if t is even we define $\alpha(b) = b$.
If t is odd we define $\alpha(b) = a$.

(**ii**) $a \in \mathrm{supp}(\tau_h)$ with $h < i$. Then $\tau_h \neq \tau_i$ since $b \notin \mathrm{supp}(\tau_k)$ with $k < i$.
We define $\alpha(b) = b$.

Let us now suppose that the critical values of f are real but not necessarily positive.

Remark 1.14. Suppose that $\tau_1 = \sigma_l$, that we change base point in $\pi_1(\mathbf{C} - \mathcal{B}, 0)$ and that we choose as base point the point a on the real axis such that $w_1^+ < a < w_2^+$. Then if we call $\hat{\sigma}_1, ..., \hat{\sigma}_m$ the totally ordered set of transpositions that we get, we have: $\hat{\sigma}_i = \sigma_i$, for $i \geq l$; $\hat{\sigma}_j = \sigma_l\, \sigma_j\, \sigma_l$, for $j \leq l - 1$. □

In order to give the general statement we introduce the following

Definition 1.15. An **odd multiple triod** is a graph whose edges correspond to the transpositions: $\sigma_h = (b, c)$, $\sigma_{r_1} = ... = \sigma_{r_k} = (b, d)$, $c \neq d$, $\sigma_i = (a, b)$, $a \neq d$, where $h < r_1 < ... < r_k < i$ is a saturated angle in b and k is odd (e.g. a non degenerate or an order degenerate triod).

An **odd multiple path** is a graph whose edges correspond to the transpositions $\sigma_h = (c, d)$, $\sigma_{r_1} = ... = \sigma_{r_m} = (d, b)$, $\sigma_i = (b, a)$ where $b \neq c$, $a \neq d$, $h < r_1 < ... < r_m$ is a saturated angle in d, $i = \min\{r > r_m | b \in \mathrm{supp}(\sigma_r)\}$ and m is odd (e.g. a non-degenerate triangle or a non snake 3-path). ▫

Definition 1.16. A multiple-bond-snake pair is a pair $((\sigma_1, ..., \sigma_m), l)$, where $\sigma_1, ..., \sigma_m$ is a totally ordered set of transpositions, $l \in \{1, ..., m+1\}$ satisfying the following:

1. $\sigma_m \sigma_{m-1} ... \sigma_l \, \sigma_1 ... \sigma_{l-1} = id$;

2. if $\mathcal{G}$ is the associated graph, then $\mathcal{G}$ doesn't contain any odd multiple triod;

3. $\mathcal{G}$ doesn't contain any odd multiple path;

4. $\mathcal{G}$ satisfies 1.9. ▫

Definition 1.17. The admissible operation is the following operation:

$$((\sigma_1, ..., \sigma_m), l) \;\to\; (\sigma_l \sigma_1 \sigma_l, ..., \sigma_l \sigma_{l-1} \sigma_l, \sigma_l, ..., \sigma_m, l+1) \;. \;\square$$

Then one can prove the following

Proposition 1.18. (See [Fr1]) *The admissible operation carries the set of multiple-bond-snake pairs to itself.* ▪

Finally in section 2 of [Fr1] we prove the following

Theorem 1.19. (See [Fr1]) *Let g, d be two integers such that $d \geq 2$, $g \geq 0$, let $\mathcal{G}$ be a connected graph with d vertices and $2g+2d-2$ edges that is associated to the multiple-bond-snake pair $(\sigma_1, ..., \sigma_{2d+2g-2}, l = 1)$ (since $l = 1$, $\tau_i = \sigma_i$, $\forall\, i$). Let α be the canonical involution provided by 1.13, then α satisfies $(**)$, therefore $\mathcal{G}$ is the monodromy graph of a generic real algebraic function $f \colon \mathcal{C} \to \mathbf{P}^1(\mathbf{C})$ of degree d whose critical values are all real and positive, where $\mathcal{C}$ is a compact Riemann surface of genus g.*

2 – Topological types of involutions

Let $f\colon \mathcal{C} \to \mathbf{P}^1_{\mathbf{C}}$ be a real generic algebraic function, let $\sigma\colon \mathcal{C} \to \mathcal{C}$ be the antiholomorphic involution that gives the real structure, i.e. $f \circ \sigma(x) = \overline{f(x)}$, $\forall\, x \in \mathcal{C}$.

We would like to describe the topological invariants of the involution σ, once we know the monodromy graph of f. On this subject we would like to mention a paper of Natanzon (see [Na2]) in which he studies both the complex and the real Hurwitz space.

We would like now to understand from the monodromy graph of a real generic algebraic function $f\colon \mathcal{C} \to \mathbf{P}^1$ the topological type of the antiholomorphic involution $\sigma\colon \mathcal{C} \to \mathcal{C}$.

We will partially solve this problem in the case in which f has only real critical values by computing the action of the involution σ on $H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})$, using Reidemeister–Schreier presentation of $\pi_1(\mathcal{C})$.

Theorem 2.1. *Let $\mathcal{C}$ be a real smooth algebraic curve of genus g, $\sigma\colon \mathcal{C} \to \mathcal{C}$ be the antiholomorphic involution that gives the real structure, assume that $\nu(\sigma) \neq 0$, let $f\colon \mathcal{C} \to \mathbf{P}^1_{\mathbf{C}}$ be a generic real algebraic function of degree $d \geq 2$ ($f \circ \sigma(x) = \overline{f(x)}$, $\forall\, x \in \mathcal{C}$).*

Assume that all the critical values of f are real.

Let $\mathcal{G}$ be the monodromy graph of f, set $\sigma_*\colon H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z}) \to H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})$. If $\mathcal{G}$ doesn't contain any (non degenerate) polygon as a subgraph, then

$$\sigma_* \equiv \mathit{identity} : H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z}) \to H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z}) \ ,$$

so $a(\sigma) = 0$, $\nu(\sigma) = g + 1$.

Proof: We assume that all the critical values of f are positive, the general case follows easily by base point change in $\pi_1(\mathbf{P}^1 - \{\mathit{critical\ values\ of}\ f\})$.

By the hypothesis we have made on the graph $\mathcal{G}$, we know that there exists a vertex b such that if $r_1, ..., r_t$ are the labels of all the edges that pass through b, then $\tau_{r_1} = \tau_{r_2} = ... = \tau_{r_t} = (b, a)$.

Let $i = \min\{h|\ b \in \mathrm{supp}(\tau_h)\}$, $j = \min\{k > i|\ b \in \mathrm{supp}(\tau_k)\}$.

We immediately see from the hypothesis on b that the first case in 1.13 can be excluded. We therefore assume that $\tau_i = \tau_j$, i.e. we restrict our attention to case two of 1.13.

In order to find the action of σ_* on $H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})$, we give a presentation of $H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})$ using the Reidemeister–Schreier method (see e.g. [M-K-S]).

Observe that, since the graph doesn't contain any polygon and the product $\tau_1\tau_2 ... \tau_{2d+2g-2} = id$, for each pair of two vertices $\{a, b\}$, there are at least two edges that connect $\{a, b\}$.

First of all we find a presentation of $\pi_1(\mathcal{C} - f^{-1}(B), b)$ where B is the set of critical values of f, $b \in f^{-1}(0)$ corresponds to the vertex of the graph with the properties described above, then we have to take the quotient for some relations to determine $\pi_1(\mathcal{C}, b)$. Finally we take the quotient for the commutators and we consider everything in $\mathbf{Z}/2\mathbf{Z}$, in order to determine $H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})$.

Note that the order in which we perform these operations is important, since we cannot abelianize before we have found a Reidemeister–Schreier presentation of the fundamental group.

In order to apply the Reidemeister–Schreier method we must find a Schreier system of representatives for $\pi_1(\mathbf{P}^1 - B, 0)$ modulo $f_*\pi_1(\mathcal{C} - f^{-1}(B), b)$. We choose a maximal tree contained in $\mathcal{G}$ in this way: for each two vertices that are connected by an edge, we choose the edge with the smallest label.

In our situation we have $\tau_i = \tau_j = \tau_{r_3} = ... = \tau_{r_t} = (a, b)$, $\tau_h = (a, c)$ with $c \neq b$ and where h is the minimum of the labels of the edges that connect a and c (see Fig. 3).

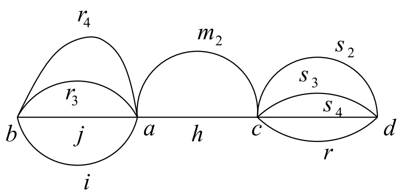

Fig. 3

A Schreier system of representatives for $\pi_1(\mathbf{P}^1 - B, 0)$ modulo $f_*\,\pi_1(\mathcal{C} - f^{-1}(B), b)$ is the following:

$$L_0 = 1\,, \qquad L_1 = \gamma_i\,, \qquad L_2 = \gamma_i\,\gamma_h\,, \qquad L_3 = \gamma_i\,\gamma_h\,\gamma_r\,, \qquad ...$$

and so on following the maximal tree that we have chosen.

After some easy computations we see that a set of generators for $H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})$ is given as follows:

$$g_1 = \gamma_i\,\gamma_j\,, \qquad g_2 = \gamma_i\,\gamma_{r_3}\,, \qquad g_3 = \gamma_i\,\gamma_{r_4}\,, \qquad \cdots, \qquad g_{t-1} = \gamma_i\,\gamma_{r_t}\ ;$$

let $m_1 = h < m_2 < ... < m_k$ be a saturated angle in a and $\tau_{m_s} = (a, c)$, $\forall\, s = 1, ..., k$, we continue the list of generators as follows:

$$g_t = \gamma_i\,\gamma_h\,\gamma_{m_2}\,\gamma_i^{-1}\,, \qquad g_{t+1} = \gamma_i\,\gamma_h\,\gamma_{m_3}\,\gamma_i^{-1}\,, \qquad \cdots, \qquad g_{t+k-2} = \gamma_i\,\gamma_h\,\gamma_{m_k}\,\gamma_i^{-1}\ ;$$

now if r is the minimum of the labels of the edges that connect c with a vertex d different from a and b, and $s_1 = r < s_2 < ... < s_l$ is a saturated angle in c and $\tau_{s_j} = (c, d)$, $\forall\, j = 1, ..., l$, then we continue the list of generators as follows:

$$g_{t+k-1} = \gamma_i\, \gamma_h\, \gamma_r\, \gamma_{s_2}\, \gamma_h^{-1}\, \gamma_i^{-1}\,, \quad g_{t+k} = \gamma_i\, \gamma_h\, \gamma_r\, \gamma_{s_3}\, \gamma_h^{-1}\, \gamma_i^{-1}\,, \quad \dots\,,$$
$$g_{t+k+l-3} = \gamma_i\, \gamma_h\, \gamma_r\, \gamma_{s_l}\, \gamma_h^{-1}\, \gamma_i^{-1}\ .$$

The complete list of generators of $H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})$ is made by going on in this way using all the edges of the maximal tree that we have chosen above.

Notice that we must take the quotient by some relations. More precisely the relations are the following:

$$R_1 = \gamma_{t_1}\,, \quad R_2 = \gamma_{t_2}\,, \quad \dots, \quad R_n = \gamma_{t_n}\ ,$$

where $\{\gamma_{t_1}, ..., \gamma_{t_n}\}$ is the set of all the loops such that the corresponding transpositions $\{\tau_{t_1}, ..., \tau_{t_n}\}$ do not move b;

$$R_{n+1} = \gamma_i^2, \quad R_{n+2} = \gamma_j^2, \quad R_{n+3} = \gamma_{r_3}^2, \quad \dots\,,$$

$$R_{n+t} = \gamma_{r_t}^2, \quad R_{n+t+1} = \gamma_i\gamma_h^2\gamma_i^{-1}, \quad R_{n+t+2} = \gamma_i\gamma_{m_2}^2\gamma_i^{-1}, \quad R_{n+t+3} = \gamma_i\gamma_{m_3}^2\gamma_i^{-1}, \quad \dots\,,$$

$$R_{n+t+k} = \gamma_i\gamma_{m_k}^2\gamma_i^{-1}, \quad R_{n+t+k+1} = \gamma_i\gamma_h\gamma_i\gamma_h^{-1}\gamma_i^{-1}, \quad R_{n+t+k+2} = \gamma_i\gamma_h\gamma_j\gamma_h^{-1}\gamma_i^{-1},$$
$$R_{n+t+k+3} = \gamma_i\gamma_h\gamma_{r_3}\gamma_h^{-1}\gamma_i^{-1}, \quad \dots\,,$$

$$R_{n+k+2t} = \gamma_i\gamma_h\gamma_{r_t}\gamma_h^{-1}\gamma_i^{-1}, \quad R_{n+k+2t+1} = \gamma_i\gamma_h\gamma_r^2\gamma_h^{-1}\gamma_i^{-1},$$
$$R_{n+k+2t+2} = \gamma_i\gamma_h\gamma_{s_2}^2\gamma_h^{-1}\gamma_i^{-1}, \quad R_{n+k+2t+3} = \gamma_i\gamma_h\gamma_{s_3}^2\gamma_h^{-1}\gamma_i^{-1}, \quad \dots\,,$$

$$R_{n+k+2t+l} = \gamma_i\gamma_h\gamma_{s_l}^2\gamma_h^{-1}\gamma_i^{-1}, \quad R_{n+k+2t+l+1} = \gamma_i\gamma_h\gamma_r\gamma_i\gamma_r^{-1}\gamma_h^{-1}\gamma_i^{-1},$$
$$R_{n+k+2t+l+2} = \gamma_i\gamma_h\gamma_r\gamma_j\gamma_r^{-1}\gamma_h^{-1}\gamma_i^{-1}, \quad R_{n+k+2t+l+3} = \gamma_i\gamma_h\gamma_r\gamma_{r_3}\gamma_r^{-1}\gamma_h^{-1}\gamma_i^{-1},$$
$$\dots\,,$$

$$R_{n+k+3t+l} = \gamma_i\gamma_h\gamma_r\gamma_{r_t}\gamma_r^{-1}\gamma_h^{-1}\gamma_i^{-1}, \quad R_{n+k+3t+l+1} = \gamma_i\gamma_h\gamma_r\gamma_h\gamma_r^{-1}\gamma_h^{-1}\gamma_i^{-1},$$
$$R_{n+k+3t+l+2} = \gamma_i\gamma_h\gamma_r\gamma_{m_2}\gamma_r^{-1}\gamma_h^{-1}\gamma_i^{-1}, \quad R_{n+k+3t+l+3} = \gamma_i\gamma_h\gamma_r\gamma_{m_3}\gamma_r^{-1}\gamma_h^{-1}\gamma_i^{-1},$$
$$\dots\,,$$

$$R_{n+k+3t+l+k} = \gamma_i\gamma_h\gamma_r\gamma_{m_k}\gamma_r^{-1}\gamma_h^{-1}\gamma_i^{-1}\ ,$$

and so on using the edges of the maximal tree that we have chosen above.

Finally there is also the relation

$$S_1 = \gamma_1\, \gamma_2 \dots \gamma_{2d+2g-2}\ ,$$

that comes from $\pi_1(\mathbf{P}^1 - B, 0)$ and all the relations conjugated to S_1 by the paths of the maximal tree:

$$\begin{aligned}
S_2 &= (\gamma_i)\,\gamma_1\,\gamma_2 \ldots \gamma_{2d+2g-2}\,(\gamma_i^{-1}) \ , \\
S_3 &= (\gamma_i\,\gamma_h)\,\gamma_1\,\gamma_2 \ldots \gamma_{2d+2g-2}\,(\gamma_h^{-1}\,\gamma_i^{-1}) \ , \\
S_4 &= (\gamma_i\,\gamma_h\,\gamma_r)\,\gamma_1\,\gamma_2 \ldots \gamma_{2d+2g-2}\,(\gamma_r^{-1}\,\gamma_h^{-1}\,\gamma_i^{-1}) \ ,
\end{aligned}$$

and so on. Finally we must abelianize and consider everything in $\mathbf{Z}/2\mathbf{Z}$, i.e. $g_j = g_j^{-1} \ \forall\, j$.

So we start now with case 2.

Case 2.(a) is excluded by the hypothesis on b. In fact there cannot exist $h > j$ such that $b \in \mathrm{supp}(\tau_h)$ and $\tau_h \neq \tau_i$.

Therefore we treat case 2.(b), that is we assume that $\forall\, h > j$ such that $b \in \mathrm{supp}(\tau_h)$, then $\tau_h = \tau_j$ and there exists h such that $a \in \mathrm{supp}(\tau_h)$ and $\tau_h \neq \tau_i$.

We assume at first that we are under the hypothesis of case 2.(b)(i), that is $i = \min\{k|\ a \in \mathrm{supp}(\tau_k)\}$, $a \in \mathrm{supp}(\tau_h)$, $h > i$, $\tau_h \neq \tau_i$. Let $i = r_1 < j = r_2 < r_3 < \ldots < r_t < h$ be a saturated angle in a such that $\tau_i = \tau_j = \tau_{r_3} = \ldots = \tau_{r_t}$. If t is even then $\sigma(b) = b$, if t is odd then $\sigma(b) = a$.

Suppose that t is even, so $\sigma(b) = b$.

Recall that $\sigma_*(\gamma_n) = \gamma_1^{-1}\gamma_2^{-1}\ldots\gamma_{n-1}^{-1}\gamma_n^{-1}\gamma_{n-1}\ldots\gamma_2\gamma_1 \ \ \forall\, n$.

Thus, using the relations written above we get:

$$\begin{aligned}
\sigma_*(g_1) &= \sigma_*(\gamma_i\gamma_j) \\
&= (\gamma_1^{-1}\gamma_2^{-1}\ldots\gamma_{i-1}^{-1}\gamma_i^{-1}\gamma_{i-1}\ldots\gamma_2\gamma_1) \\
&\quad .\,(\gamma_1^{-1}\gamma_2^{-1}\ldots\gamma_{i-1}^{-1}\gamma_i^{-1}\gamma_{i+1}^{-1}\ldots\gamma_{j-1}^{-1}\gamma_j^{-1}\gamma_{j-1}\ldots\gamma_{i+1}\gamma_i\gamma_{i-1}\ldots\gamma_2\gamma_1) \\
&= \gamma_1^{-1}\gamma_2^{-1}\ldots\gamma_{i-1}^{-1}\gamma_i^{-2}\gamma_{i+1}^{-1}\ldots\gamma_{j-1}^{-1}\gamma_j^{-1}\gamma_{j-1}\ldots\gamma_{i+1}\gamma_i\gamma_{i-1}\ldots\gamma_2\gamma_1 \\
&= (\gamma_1^{-1})(\gamma_2^{-1})\ldots(\gamma_{i-1}^{-1})(\gamma_i^{-2})(\gamma_{i+1}^{-1})\ldots(\gamma_{j-1}^{-1})(\gamma_j^{-2})(\gamma_j\gamma_i^{-1})(\gamma_i\gamma_{j-1}\gamma_i^{-1}) \\
&\quad \ldots(\gamma_i\gamma_{i+1}\gamma_i^{-1})(\gamma_i^{2})(\gamma_{i-1})\ldots(\gamma_1) \\
&= (\gamma_j\gamma_i^{-1}) = g_1^{-1} = g_1 \ .
\end{aligned}$$

Analogously we have: $\sigma_*(g_2) = \sigma_*(\gamma_i\gamma_{r_3}) = g_2$. Similarly one sees that $\sigma_*(g_s) = \sigma_*(\gamma_i\gamma_{r_{s+1}}) = g_s \ \forall\, s = 1, \ldots, t-1$.

With a similar computation we obtain: $\sigma_*(g_t) = \sigma_*(\gamma_i\gamma_h\gamma_{m_2}\gamma_i^{-1}) = g_t$.

Analogously one sees that $\sigma_*(g_{t+s}) = g_{t+s} \ \forall\, s$. Therefore $\sigma_* = Id$.

We consider now case 2.(b)(i) with t odd, so that $\sigma(b) = a$.

Observe that since $\tau_{2g+2d-2}\tau_{2g+2d-1}...\tau_1(a) = a$, and $\mathcal{G}$ doesn't contain any polygon, there must exist an odd number of edges connecting a and b with labels greater than h.

Therefore we have $i = r_1 < j = r_2 < r_3 < ... < r_t < h < m_1 < m_2 < ... < m_s$, a saturated angle in a where $\tau_i = \tau_j = \tau_{r_3} = ... = \tau_{r_t} = \tau_{m_1} = ... = \tau_{m_s} = (a, b)$, $\tau_h = (a, c)$ with $c \neq b$ and both t and s are odd.

Now since $\tau_{2g+2d-2}\tau_{2g+2d-1}...\tau_1(b) = b$, and $\mathcal{G}$ doesn't contain any polygon, there must exist $h < p_1 < p_2 < ... < p_m < m_1 < ... < m_s$, a saturated angle in a where m is odd, $\tau_h = \tau_{p_1} = ... = \tau_{p_m} = (a, c)$ (see Fig. 4).

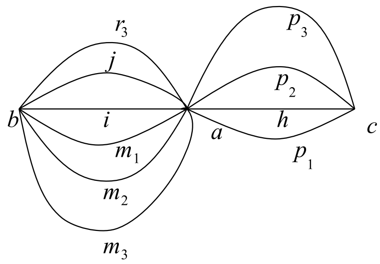

Fig. 4

A set of generators for $\pi_1(\mathcal{C}, b)$ can be found as we have explained above.

Now, if we let σ_* act on $\pi_1(\mathcal{C}, b)$ we have to take in the target $a = \sigma(b)$ as base point. Thus we have another maximal tree and therefore another set of generators and relations for $\pi_1(\mathcal{C}, a)$.

We only write the set of generators, the relations can be found as above.

We call the generators of $\pi_1(\mathcal{C}, b)$ g_i's as before, the generators of $\pi_1(\mathcal{C}, a)$ in the target ψ_i's.

$$\psi_1 = \gamma_i\gamma_j = g_1, \quad \psi_2 = \gamma_i\gamma_{r_3} = g_2, \quad \psi_3 = \gamma_i\gamma_{r_4} = g_3, \quad ..., \quad \psi_{t-1} = \gamma_i\gamma_{r_t} = g_{t-1} \ ,$$
$$\psi_t = \gamma_h\gamma_{p_1}, \quad \psi_{t+1} = \gamma_h\gamma_{p_2}, \quad ..., \quad \psi_{t+m-1} = \gamma_h\gamma_{p_m} \ ,$$
$$\psi_{t+m} = \gamma_i\gamma_{m_1}, \quad \psi_{t+m+1} = \gamma_i\gamma_{m_2}, \quad ..., \quad \psi_{t+m+s-1} = \gamma_i\gamma_{m_s} \ .$$

And so on following the maximal tree.

We perform now σ_*.

$$\begin{aligned} \sigma_*(g_1) &= \sigma_*(\gamma_i\gamma_j) \\ &= (\gamma_1^{-1}\gamma_2^{-1}...\gamma_{i-1}^{-1}\gamma_i^{-1}\gamma_{i-1}...\gamma_1) \\ &\quad .(\gamma_1^{-1}...\gamma_{i-1}^{-1}\gamma_i^{-1}\gamma_{i+1}^{-1}...\gamma_{j-1}^{-1}\gamma_j^{-1}\gamma_{j-1}...\gamma_{i+1}\gamma_i\gamma_{i-1}...\gamma_1) \ = \end{aligned}$$

$$\begin{aligned} &= (\gamma_1^{-1})...(\gamma_{i-1}^{-1})(\gamma_i^{-2})(\gamma_{i+1}^{-1})...(\gamma_{j-1}^{-1})(\gamma_j^{-2})(\gamma_j\gamma_i^{-1})(\gamma_i\gamma_{j-1}\gamma_i^{-1}) \\ &\quad ...(\gamma_i\gamma_{i+1}\gamma_i^{-1})(\gamma_i^2)(\gamma_{i-1})...(\gamma_1) \\ &= \gamma_j\gamma_i^{-1} = \psi_1 . \end{aligned}$$

In the same way one sees that $\sigma_*(g_j) = \psi_j \ \forall j = 1, ..., t-1$.

Analogously we can compute the image of the other generators and we obtain: $\sigma_*(\gamma_i\gamma_h\gamma_{p_1}\gamma_i^{-1}) = \psi_t$.

Observe that since we have performed a base point change in the target, we have $Id_*(\gamma_i\gamma_h\gamma_{p_1}\gamma_i^{-1}) = \gamma_h\gamma_{p_1} = \psi_t$.

With a similar computation it is easy to see that

$$\sigma_*(\gamma_i\gamma_h\gamma_{p_l}\gamma_i^{-1}) = \gamma_h\gamma_{p_l} = Id_*(\gamma_i\gamma_h\gamma_{p_l}\gamma_i^{-1}), \qquad \forall l = 1, ..., m .$$

The image of the last group of generators can be computed analogously and we get: $\sigma_*(\gamma_i\gamma_{m_1}) = \psi_t\psi_{t+1}...\psi_{t+m-1}\psi_{t+m}$.

Now we show, using the relations, that $\psi_t\psi_{t+1}...\psi_{t+m-1} = 1$.

In fact recall that

$$\gamma_1\gamma_2...\gamma_{2g+2d-2} = 1 ,$$
$$\gamma_i\gamma_1\gamma_2...\gamma_{2g+2d-2}\gamma_i^{-1} = 1 .$$

So we have to write these two relations in terms of the generators $\psi_j's$.

$$\begin{aligned} 1 &= \gamma_1\gamma_2...\gamma_{2g+2d-2} \\ &= (\gamma_1)...(\gamma_{i-1})(\gamma_i\gamma_{i+1}\gamma_i^{-1})...(\gamma_i\gamma_{j-1}\gamma_i^{-1})(\gamma_i\gamma_j)(\gamma_{j+1})...(\gamma_{r_3}\gamma_i^{-1}) \\ &\quad ...(\gamma_{r_t}\gamma_i^{-1})(\gamma_i\gamma_{r_t+1}\gamma_i^{-1})...(\gamma_i\gamma_h\gamma_i^{-1})...(\gamma_i\gamma_{p_1}\gamma_i^{-1})...(\gamma_i\gamma_{p_m}\gamma_i^{-1}) \\ &\quad ...(\gamma_i\gamma_{m_1})(\gamma_{m_1+1})...(\gamma_i\gamma_{m_s})(\gamma_{m_s+1})...(\gamma_{2d+2g-2}) \\ &= \psi_1\psi_2...\psi_{t-1}\psi_{t+m}\psi_{t+m+1}...\psi_{t+m+s-1} . \end{aligned}$$

$$\begin{aligned} 1 &= \gamma_i\gamma_1\gamma_2...\gamma_{2g+2d-2}\gamma_i^{-1} \\ &= (\gamma_i\gamma_1\gamma_i^{-1})...(\gamma_i\gamma_{i-1}\gamma_i^{-1})(\gamma_i^2)(\gamma_{i+1})...(\gamma_{j-1})(\gamma_j\gamma_i^{-1})(\gamma_i\gamma_{j+1}\gamma_i^{-1}) \\ &\quad ...(\gamma_i\gamma_{r_3})(\gamma_{r_3+1})...(\gamma_i\gamma_{r_t})(\gamma_{r_t+1})...(\gamma_{h-1})(\gamma_h\gamma_{h+1}\gamma_h^{-1})...(\gamma_h\gamma_{p_1})(\gamma_{p_1+1}) \\ &\quad ...(\gamma_{p_2}\gamma_h^{-1})(\gamma_h\gamma_{p_2+1}\gamma_h^{-1})...(\gamma_h\gamma_{p_m})(\gamma_{p_m+1})...(\gamma_{m_1}\gamma_i^{-1})(\gamma_i\gamma_{m_1+1}\gamma_i^{-1}) \\ &\quad ...(\gamma_i\gamma_{m_2})...(\gamma_{m_s}\gamma_i^{-1})(\gamma_i\gamma_{m_s+1}\gamma_i^{-1})...(\gamma_i\gamma_{2g+2d-2}\gamma_i^{-1}) \\ &= \psi_1\psi_2...\psi_{t-1}(\psi_t\psi_{t+1}...\psi_{t+m-1})\psi_{t+m}\psi_{t+m+1}...\psi_{t+m+s-1} . \end{aligned}$$

But from this second relation we get

$$\psi_t\psi_{t+1}...\psi_{t+m-1} = \psi_1...\psi_{t-1}\psi_{t+m}\psi_{t+m+1}...\psi_{t+m+s-1} = 1$$

by the first relation.

So we have $\sigma_*(\gamma_i\gamma_{m_1}) = \psi_{t+m} = Id_*(\gamma_i\gamma_{m_1})$. In the same way we see that $\sigma_*(\gamma_i\gamma_{m_k}) = \psi_{t+m+k-1} = Id_*(\gamma_i\gamma_{m_k}),\ \forall\, k = 1, ..., s$.

Therefore we have proven that in case 2.(b)(i)

$$\sigma_* = Id_*\colon\ H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z}) \to H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})\ .$$

It remains case 2.(b)(ii).

Here we assume that $\tau_i = \tau_j = (a, b)$, $\forall\, h > j$ such that $b \in \operatorname{supp}(\tau_h)$, $\tau_h = \tau_j$, $a \in \operatorname{supp}(\tau_h)$, $h < i$, so $\tau_h \neq \tau_i$ and $\sigma(b) = b$.

Take b as base point, $h = \min\{r|\ a \in \operatorname{supp}(\tau_r)\}$, so γ_i and γ_h are two edges of the maximal tree.

The generators and relations of $\pi_1(\mathcal{C}, b)$ are described above, whence we have to calculate $\sigma_*(g_j)\ \forall\, j$.

Since $\mathcal{G}$ doesn't contain any polygon and $\tau_{2g+2d-2}\tau_{2d+2g-1}...\tau_1(a) = a$, there must exist at least an index p_1 such that $\tau_{p_1} = \tau_h = (a, c),\ p_1 > h$.

Now by 1.7, we know that if $h < p_1 < p_2 < ... < p_m < i = r_1 < j = r_2 < ... < r_t < m_1 < m_2 < ... < m_s$ is a saturated angle in a with $\tau_h = \tau_{p_1} = ... = \tau_{p_m} = (a, c)$, and $\tau_i = \tau_j = \tau_{r_3} = ... = \tau_{r_t} = (b, a)$, then t is even.

Now since $\tau_{2g+2d-2}\tau_{2d+2g-1}...\tau_1(a) = a$, and there aren't polygons, then m is even and s is odd, or m is odd and s is even. But we know that $\sigma(a) = c$, so m cannot be odd, otherwise $\{\sigma(b), \sigma(a)\} = \{\rho_{i-1}(b), \rho_{i-1}(a)\} = \{b, a\}$, and this is a contradiction.

Therefore we have m even and s odd (see Fig. 5).

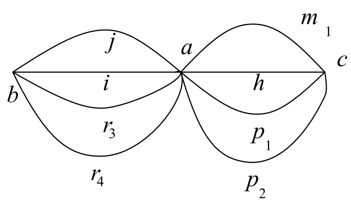

Fig. 5

We calculate at first

$$\begin{aligned}\sigma_*(g_1) &= \sigma_*(\gamma_i\gamma_j)\\ &= \big(\gamma_1^{-1}\gamma_2^{-1}...\gamma_{h-1}^{-1}\gamma_h\gamma_{h+1}^{-1}...\gamma_{p_1-1}^{-1}\gamma_{p_1}^{-1}\gamma_{p_1+1}^{-1}...\gamma_{p_2-1}^{-1}\gamma_{p_2}^{-1}\gamma_{p_2+1}^{-1}\\ &\quad ...\gamma_{p_m-1}^{-1}\gamma_{p_m}^{-1}\gamma_{p_m+1}^{-1}...\gamma_{i-1}^{-1}\gamma_i^{-1}\gamma_{i-1}...\gamma_{p_m+1}...\gamma_{p_2}....\gamma_{p_1}...\gamma_h...\gamma_1\big)\,.\end{aligned}$$

$$\begin{aligned}
&\cdot(\gamma_1^{-1}\gamma_2^{-1}...\gamma_{h-1}^{-1}\gamma_h\gamma_{h+1}^{-1}...\gamma_{p_1-1}^{-1}\gamma_{p_1}^{-1}\gamma_{p_1+1}^{-1}...\gamma_{p_2-1}^{-1}\gamma_{p_2}^{-1}\gamma_{p_2+1}^{-1}\\
&\quad ...\,\gamma_{p_m-1}^{-1}\gamma_{p_m}^{-1}\gamma_{p_m+1}^{-1}...\gamma_{i-1}^{-1}\gamma_i^{-1}\gamma_{i+1}^{-1}...\gamma_{j-1}^{-1}\gamma_j^{-1}\gamma_{j-1}...\gamma_i\gamma_{i-1}\\
&\quad ...\,\gamma_{p_m+1}...\gamma_{p_2}...\gamma_{p_1}...\gamma_h...\gamma_1)\\
=\;&(\gamma_1^{-1})...(\gamma_{h-1}^{-1})(\gamma_h^{-1})...(\gamma_{p_1}^{-1})...(\gamma_{p_2}^{-1})...(\gamma_{p_m}^{-1})...(\gamma_{i-1}^{-1})(\gamma_i^{-2})(\gamma_{i+1}^{-1})\\
&\quad ...\,(\gamma_{j-1}^{-1})(\gamma_j^{-2})(\gamma_j\gamma_i^{-1})(\gamma_i\gamma_{j-1}\gamma_i^{-1})...(\gamma_i\gamma_{i+1}\gamma_i^{-1})(\gamma_i^2)(\gamma_{i-1})\\
&\quad ...\,(\gamma_{p_m})...(\gamma_{p_1})...(\gamma_h)...(\gamma_1)\\
=\;&(\gamma_j\gamma_i^{-1}) \;=\; g_1 \;.
\end{aligned}$$

Analogously we obtain: $\sigma_*(g_k)=\sigma_*(\gamma_i\gamma_{r_k+1})=g_k\ \ \forall\, k=1,...,t-1.$

With a similar computation we get: $\sigma_*(g_t)=g_t$ and

$$\sigma_*(g_{t+l})=\sigma_*(\gamma_i\gamma_{p_{l+1}}\gamma_h^{-1}\gamma_i^{-1})=g_{t+l}\,,\qquad \forall\, l=1,...,m-1\;.$$

Finally we have to calculate $\sigma_*(g_{t+m})$ and we obtain:

$$\sigma_*(g_{t+m})=g_1g_2...g_{t-1}g_{t+m}\;.$$

Then we see that from the relations we get $g_1g_2...g_{t-1}=1$. In fact, we have $\gamma_1\gamma_2...\gamma_{2d+2g-2}=1$ and

$$\begin{aligned}
1 \;=\;&\gamma_1\gamma_2...\gamma_{2d+2g-2}\\
=\;&(\gamma_1)...(\gamma_{h-1})(\gamma_h)...(\gamma_{p_1})...(\gamma_{p_m})...(\gamma_{i-1})(\gamma_i\gamma_{i+1}\gamma_i^{-1})...(\gamma_i\gamma_{j-1}\gamma_i^{-1})(\gamma_i\gamma_j)\\
&\quad ...\,(\gamma_{r_3}\gamma_i^{-1})...(\gamma_i\gamma_{r_t})(\gamma_{r_t+1})...(\gamma_{m_1})...(\gamma_{m_s})...(\gamma_{2g+2d-2})\\
=\;&g_1...g_{t-1}\;.
\end{aligned}$$

So $\sigma_*(g_{t+m})=g_{t+m}$.

Similarly one can verify that $\sigma_*(g_{t+m+r})=\sigma_*(\gamma_i\gamma_h\gamma_{m_{r+1}}\gamma_i^{-1})=g_{t+m+r}$, $\forall\, r=1,...,s-1$.

Thus we have proven that $\sigma_*:\ H_1(\mathcal{C},\mathbf{Z}/2\mathbf{Z})\to H_1(\mathcal{C},\mathbf{Z}/2\mathbf{Z})$ is the identity in all the cases of 1.13.

Now, in order to conclude that $a(\sigma)=0$ and $\nu(\sigma)=g+1$ we use the fact that for a real curve $(\mathcal{C},\sigma)$, $\nu(\sigma)=g+1-\dim(\sigma+id)_*(H_1(\mathcal{C},\mathbf{Z}/2\mathbf{Z}))$ (see e.g. [Ci-Pe], Corollary 4.1.8).

We have just proven that $\sigma_*=id$, so $\nu(\sigma)=g+1$ and therefore $a(\sigma)=0$. ∎

Let us now define the reduced graph $\mathcal{G}_{red}$ of the monodromy graph $\mathcal{G}$ of f to be the graph obtained from $\mathcal{G}$ in this way: for every two vertices we remove some edges of $\mathcal{G}$ in such a way that there remains only one edge connecting the two vertices.

A polygon in $\mathcal{G}$ is said to be odd if it has an odd number of edges.

We prove the following

Theorem 2.2. *Let $\mathcal{C}$ be a real smooth algebraic curve of genus g, $\sigma\colon \mathcal{C} \to \mathcal{C}$ be the antiholomorphic involution that gives the real structure, assume that $\nu(\sigma) \neq 0$, let $f : \mathcal{C} \to \mathbf{P}^1_{\mathbf{C}}$ be a generic real algebraic function of degree $d \geq 2$ ($f \circ \sigma(x) = \overline{f(x)}$, $\forall\, x \in \mathcal{C}$).*

Assume that all the critical values of f are real and positive (if they are not positive it suffices to perform a base point change).

Let $\mathcal{G}$ be the monodromy graph of f. Assume furthermore that any two polygons contained in $\mathcal{G}_{red}$ have no common edges. Set $\sigma_* : H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z}) \to H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})$. Then

$$\dim(\sigma + identity)_*(H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})) \;=\; \#\{odd\ polygons\ in\ \mathcal{G}_{red}\}\ .$$

So

$$\nu(\sigma) \;=\; g + 1 - \#\{odd\ polygons\ in\ \mathcal{G}_{red}\}\ .$$

Proof: Assume first of all that the graph $\mathcal{G}$ has only 3 vertices and that $\mathcal{G}_{red}$ is a triangle. Then there exists a vertex which is fixed by σ.

Let us now take as base point one vertex of the triangle which is fixed by σ, let us call it b, then since $\sigma(b) = b$, and case 2.(b) of 1.13 cannot occur (since b is a vertex of a triangle), b must satisfy the hypothesis of 2.(a) of 1.13.

So we take as base point a vertex b such that $\sigma(b) = b$ and b satisfies the hypothesis of case 2.(a) of 1.13.

Assume that $\tau_i = \tau_j = \tau_{r_1} = ... = \tau_{r_t} = (a, b)$, $\tau_h = \tau_{m_1} = ... = \tau_{m_r} = (b, c)$, $\tau_k = (c, a)$, $i < j < r_1 < ... < r_t < h < m_1 < ... < m_r$ is a saturated angle in b. k can either be smaller or bigger than i.

Assume first of all that $k < i$. Let $k_n < k_{n-1} < ... < k_1 < k < i$ be a saturated angle in a, $\tau_{k_j} = \tau_k = (c, a)$, $\forall\, j = 1, ..., n$.

As in Theorem 2.1 we have generators for $H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})$:

$$g_1 = \gamma_i\gamma_j,\quad g_2 = \gamma_i\gamma_{r_1},\quad ...,\quad g_{t-1} = \gamma_i\gamma_{r_t},\quad g_t = \gamma_h\gamma_{m_1},\quad ...,\quad g_{t+r-1} = \gamma_h\gamma_{m_r}\ ,$$
$$g_1' = \gamma_i\gamma_k\gamma_h^{-1},\quad g_2' = \gamma_i\gamma_{k_1}\gamma_h^{-1},\quad ...,\quad g_{n+1}' = \gamma_i\gamma_{k_n}\gamma_h^{-1}\ .$$

The relations are analogue to the relations in Theorem 2.1, so we have:

$$S_1 \;=\; \gamma_1 ... \gamma_{2g+2d-2} \;=\; g_1 g_2 ... g_{t-1} g_t g_{t+1} ... g_{t+r+1}\ ,$$

therefore we get $g_1 g_2 ... g_{t-1} = g_t g_{t+1} ... g_{t+r+1}$.

$$S_2 = \gamma_i \gamma_1 \gamma_2 ... \gamma_{2g+2d-2} \gamma_i^{-1} = g'_1 ... g'_{n+1} g_1 ... g_{t-1} ,$$

whence we have $g_1 g_2 ... g_{t-1} = g'_1 g'_2 ... g'_{n+1}$.

$$\begin{aligned}
\sigma_*(g_1) &= \sigma_*(\gamma_i \gamma_j) \\
&= (\gamma_1^{-1} \gamma_2^{-1} ... \gamma_{i-1}^{-1} \gamma_i^{-1} \gamma_{i-1} ... \gamma_1) \\
&\quad . (\gamma_1^{-1} \gamma_2^{-1} ... \gamma_{i-1}^{-1} \gamma_i^{-1} \gamma_{i+1}^{-1} ... \gamma_{j-1}^{-1} \gamma_j^{-1} \gamma_{j-1} ... \gamma_i \gamma_{i-1} ... \gamma_1) \\
&= (\gamma_1)^{-1} ... (\gamma_{i-1})^{-1} (\gamma_i)^{-2} (\gamma_{i+1})^{-1} ... (\gamma_{j-1})^{-1} (\gamma_j)^{-1} (\gamma_j \gamma_i^{-1}) (\gamma_i \gamma_{j-1} \gamma_i^{-1}) \\
&\quad ... (\gamma_i \gamma_{i-1} \gamma_i^{-1}) (\gamma_i^2) (\gamma_{i-1}) ... (\gamma_1) \\
&= g_1 \ .
\end{aligned}$$

Analogously one sees that $\sigma_*(g_j) = g_j$, $\forall\, j = 1, ..., t-1$, and $\sigma_*(g_{t+j}) = g_{t+j}$, $\forall\, j = 0, ..., r+1$.

$$\begin{aligned}
\sigma_*(g'_1) &= \sigma_*(\gamma_i \gamma_k \gamma_h^{-1}) \\
&= (\gamma_1^{-1} ... \gamma_{k_n}^{-1} ... \gamma_{k_2}^{-1} ... \gamma_{k_1}^{-1} ... \gamma_{k-1}^{-1} \gamma_k^{-1} ... \gamma_{i-1}^{-1} \gamma_i^{-1} \gamma_{i-1} ... \gamma_{k+1} \gamma_k ... \gamma_{k_1} ... \gamma_{k_n} ... \gamma_1) \\
&\quad . (\gamma_1^{-1} ... \gamma_{k_n}^{-1} ... \gamma_{k_1}^{-1} ... \gamma_{k-1}^{-1} \gamma_k^{-1} \gamma_{k-1} ... \gamma_{k_1} ... \gamma_{k_n} ... \gamma_1) \\
&\quad . (\gamma_1^{-1} ... \gamma_{k_n}^{-1} ... \gamma_{k_1}^{-1} ... \gamma_{k-1}^{-1} \gamma_k^{-1} ... \gamma_{i-1}^{-1} \gamma_i^{-1} \gamma_{i+1}^{-1} ... \gamma_j^{-1} ... \gamma_{r_1}^{-1} ... \gamma_{r_t}^{-1} \\
&\quad ... \gamma_{h-1}^{-1} \gamma_h \gamma_{h-1} ... \gamma_{r_t} ... \gamma_{r_1} ... \gamma_j ... \gamma_i ... \gamma_k ... \gamma_{k_1} ... \gamma_{k_n} ... \gamma_1) \\
&= (\gamma_1^{-1}) ... (\gamma_{k_n}^{-1}) ... (\gamma_{k_1}^{-1}) ... (\gamma_{k-1}^{-1}) (\gamma_k^{-1}) ... (\gamma_{i-1}^{-1}) (\gamma_i^{-2}) (\gamma_i \gamma_{i-1} \gamma_i^{-1}) \\
&\quad ... (\gamma_i \gamma_{k+1} \gamma_i^{-1}) (\gamma_i \gamma_k^{-2} \gamma_i^{-1}) (\gamma_i \gamma_k \gamma_h^{-1}) (\gamma_h \gamma_{k+1} \gamma_h^{-1}) \\
&\quad ... (\gamma_h \gamma_{i-1}^{-1} \gamma_h^{-1}) (\gamma_h \gamma_i^{-1} \gamma_h^{-1}) ... (\gamma_h \gamma_j^{-1} \gamma_h^{-1}) ... (\gamma_h \gamma_{r_1}^{-1} \gamma_h^{-1}) ... (\gamma_h \gamma_{r_t}^{-1} \gamma_h^{-1}) \\
&\quad ... (\gamma_h \gamma_{h-1}^{-1} \gamma_h^{-1}) (\gamma_h^2) (\gamma_{h-1}) ... (\gamma_{r_t+1}) (\gamma_{r_t} \gamma_i^{-1}) (\gamma_i \gamma_{r_t-1} \gamma_i^{-1}) \\
&\quad ... (\gamma_{r_2} \gamma_i^{-1}) (\gamma_i \gamma_{r_2-1} \gamma_i^{-1}) ... (\gamma_i \gamma_{r_1}) ... (\gamma_{j+1}) (\gamma_j \gamma_i^{-1}) (\gamma_i \gamma_{j-1} \gamma_i^{-1}) \\
&\quad ... (\gamma_i \gamma_{i+1} \gamma_i^{-1}) (\gamma_i^2) (\gamma_{i-1}) ... (\gamma_k) ... (\gamma_{k_1}) ... (\gamma_{k_n}) ... (\gamma_1) \\
&= g'_1\, g_{t-1}\, g_{t-2} ... g_1 \ .
\end{aligned}$$

$$\begin{aligned}
\sigma_*(g'_2) &= \sigma_*(\gamma_i \gamma_{k_1} \gamma_h^{-1}) \\
&= (\gamma_1^{-1} ... \gamma_{k_n} ... \gamma_{k_{n-1}}^{-1} ... \gamma_2^{-1} ... \gamma_{k_1-1}^{-1} \gamma_{k_1}^{-1} \gamma_{k_1+1}^{-1} ... \gamma_{k-1}^{-1} \gamma_k^{-1} \gamma_{k+1}^{-1} \\
&\quad ... \gamma_{i-1}^{-1} \gamma_i^{-1} \gamma_{i-1} ... \gamma_{k+1} \gamma_k \gamma_{k-1} ... \gamma_{k_1+1} \gamma_{k_1} \gamma_{k_1-1} ... \gamma_{k_2} ... \gamma_{k_n} ... \gamma_1)
\end{aligned}$$

$$
\begin{aligned}
&\cdot(\gamma_1^{-1}...\gamma_{k_n}^{-1}...\gamma_{k_2}^{-1}...\gamma_{k_1-1}^{-1}\gamma_{k_1}^{-1}\gamma_{k_1-1}...\gamma_{k_2}...\gamma_{k_n}...\gamma_1)\\
&\cdot(\gamma_1^{-1}...\gamma_{k_n}...\gamma_{k_2}...\gamma_{k_1-1}^{-1}\gamma_{k_1}^{-1}...\gamma_k^{-1}...\gamma_i^{-1}...\gamma_j^{-1}...\gamma_{r_1}^{-1}...\gamma_{r_t}^{-1}\\
&...\gamma_{h-1}^{-1}\gamma_h\gamma_{h-1}...\gamma_{r_t}...\gamma_{r_2}...\gamma_{r_1}...\gamma_j...\gamma_i...\gamma_k...\gamma_{k_1}...\gamma_{k_n}...\gamma_1)\\
= \ &(\gamma_1^{-1})...(\gamma_{k_n}^{-1})...(\gamma_{k_2}^{-1})...(\gamma_{k_1}^{-1})...(\gamma_k^{-1})...(\gamma_{i-1}^{-1})(\gamma_i^{-2})(\gamma_i\gamma_{i-1}\gamma_i^{-1})\\
&...(\gamma_i\gamma_{k+1}\gamma_i^{-1})(\gamma_i\gamma_k\gamma_h^{-1})(\gamma_h\gamma_{k-1}\gamma_h^{-1})\\
&...(\gamma_h\gamma_{k_1+1}\gamma_h^{-1})(\gamma_h\gamma_{k_1}^{-1}\gamma_i^{-1})(\gamma_i\gamma_{k_1+1}\gamma_i^{-1})...(\gamma_i\gamma_k^{-1}\gamma_h^{-1})(\gamma_h\gamma_{k+1}^{-1}\gamma_h^{-1})\\
&...(\gamma_h\gamma_i^{-1}\gamma_h^{-1})...(\gamma_h\gamma_j^{-1}\gamma_h^{-1})...(\gamma_h\gamma_{r_1}^{-1}\gamma_h^{-1})...(\gamma_h\gamma_{r_t}^{-1}\gamma_h^{-1})\\
&...(\gamma_h\gamma_{h-1}\gamma_h^{-1})(\gamma_h^2)(\gamma_{h-1})...(\gamma_{r_t+1})(\gamma_{r_t}\gamma_i^{-1})...(\gamma_{r_2}\gamma_i^{-1})...(\gamma_i\gamma_{r_1})(\gamma_{r_1-1})\\
&...(\gamma_j\gamma_i^{-1})(\gamma_i\gamma_{j-1}\gamma_i^{-1})...(\gamma_i\gamma_{i+1}\gamma_i^{-1})(\gamma_i^2)(\gamma_{i-1})...(\gamma_k)...(\gamma_{k_1})...(\gamma_{k_n})...(\gamma_1)\\
= \ &g_1'\,g_2'\,g_1'\,g_{t-1}\,g_{t-2}\,...\,g_1\\
= \ &g_2'\,g_{t-1}\,g_{t-2}\,...\,g_1\ .
\end{aligned}
$$

In the same way one can see that $\sigma_*(g_d') = g_d'g_1g_2...g_{t-1}\ \ \forall\, d = 2, ..., n.$

We claim now that $\operatorname{rank}(\sigma + Id)_* = 1.$

In fact let us first of all consider the following matrix whose columns are the images through $(\sigma + Id)_*$ of the generators $g_1, ..., g_{t-1}, g_t, ..., g_{t+r-1}, g_1', ..., g_{n+1}'$.

$$
\begin{pmatrix}
0 & \cdots & 0 & 1 & 1 & \cdots & 1\\
0 & \cdots & 0 & 1 & 1 & \cdots & 1\\
\vdots & & \vdots & \vdots & \vdots & & \vdots\\
0 & \cdots & 0 & 1 & 1 & \cdots & 1\\
0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\vdots & & \vdots & \vdots & \vdots & & \vdots\\
0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
$$

We have:

$$Im(\sigma + Id)_* \, = \, \Big\langle (\sigma + Id)_*(g_1') \Big\rangle \ .$$

Therefore we only have to check that $(\sigma+Id)_*(g_1') = g_1...g_{t-1}$ is not the identity. But this is true, since the only relations between the g_i's and the g_j''s are S_1 and S_2. Thus we have proven that $\operatorname{rank}(\sigma + Id)_* = 1.$

Assume now that $k > i$, more precisely suppose that $\tau_i = \tau_j = \tau_{r_1} = ... = \tau_{r_t} = (a, b)$, $\tau_k = \tau_{k_1} = ... = \tau_{k_s} = (a, c)$, $\tau_h = \tau_{m_1} = ... = \tau_{m_r} = (b, c)$, with t even and s odd, where $i < j < r_1 < ... < r_t < k < k_1 < ... < k_s < h < m_1 < ... < m_r$ are the labels of a saturated subgraph.

We compute the action of σ only on the set of generators:

$$\begin{aligned} g' &= \gamma_i\, \gamma_k\, \gamma_h^{-1}\ , \\ g_1' &= \gamma_i\, \gamma_{k_1}\, \gamma_h^{-1}\ , \\ g_2' &= \gamma_i\, \gamma_{k_2}\, \gamma_h^{-1}\ , \\ &\ \ \vdots \\ g_s' &= \gamma_i\, \gamma_{k_s}\, \gamma_h^{-1}\ , \end{aligned}$$

since on the other generators we have already seen that σ acts as the identity.

We have the relations

$$S_1 = \gamma_1\, \gamma_2 \dots \gamma_{2g+2d-2} = g_1\, g_2 \dots g_{t-1}\, g_t\, g_{t+1} \dots g_{t+r+1}\ ,$$

so we get $g_1\, g_2 \dots g_{t-1} = g_t\, g_{t+1} \dots g_{t+r+1}$.

$$S_2 = \gamma_i\, \gamma_1\, \gamma_2 \dots \gamma_{2g+2d-2}\, \gamma_i^{-1} = g_1\, g_2 \dots g_{t-1}\, g'\, g_1' \dots g_s'\ ,$$

therefore we obtain $g_1\, g_2 \dots g_{t-1} = g'\, g_1'\, g_2' \dots g_s'$.

$$\begin{aligned} \sigma_*(g') &= \sigma_*(\gamma_i\gamma_k\gamma_h^{-1}) \\ &= (\gamma_1^{-1} \dots \gamma_{i-1}^{-1}\gamma_i^{-1}\gamma_{i-1} \dots \gamma_1) \\ &\quad .\,(\gamma_1^{-1} \dots \gamma_{i-1}^{-1}\gamma_i^{-1} \dots \gamma_j^{-1} \dots \gamma_{r_1}^{-1} \dots . \gamma_{r_t}^{-1} \dots \gamma_{k-1}^{-1}\gamma_k^{-1}\gamma_{k-1} \dots \gamma_{r_t} \dots . \gamma_{r_1} \dots \gamma_j \dots \gamma_i \dots \gamma_1) \\ &\quad .\,(\gamma_1^{-1} \dots \gamma_i^{-1} \dots \gamma_j^{-1} \dots \gamma_{r_1}^{-1} \dots \gamma_{r_t}^{-1} \dots \gamma_{k-1}^{-1}\gamma_k^{-1} \dots \gamma_{k_1}^{-1} \dots \gamma_{k_s}^{-1} \dots \gamma_{h-1}^{-1}\gamma_h\gamma_{h-1} \\ &\quad \dots \gamma_{k_s} \dots \gamma_{k_1} \dots \gamma_k \dots \gamma_{r_t} \dots \gamma_{r_1} \dots \gamma_j \dots \gamma_i \dots \gamma_1) \\ &= (\gamma_1^{-1}) \dots (\gamma_{i-1}^{-1})(\gamma_i^{-2})(\gamma_{i+1}^{-1}) \dots (\gamma_{j-1}^{-1})(\gamma_j^{-2})(\gamma_j\gamma_i^{-1})(\gamma_i\gamma_{j+1}\gamma_i^{-1}) \\ &\quad \dots (\gamma_i\gamma_{r_1-1}\gamma_i^{-1})(\gamma_i\gamma_{r_1}) \dots (\gamma_{r_t}\gamma_i^{-1})(\gamma_i\gamma_{r_t+1}\gamma_i^{-1}) \\ &\quad .\,(\gamma_i\gamma_{k-1}^{-1}\gamma_i^{-1})(\gamma_i\gamma_k^{-2}\gamma_i^{-1})(\gamma_i\gamma_{k+1}^{-1}\gamma_i^{-1}) \\ &\quad \dots (\gamma_i\gamma_{k_1-1}^{-1}\gamma_i^{-1})(\gamma_i\gamma_{k_1}^{-1}\gamma_h^{-1})(\gamma_h\gamma_{k_1+1}^{-1}\gamma_h^{-1}) \dots (\gamma_h\gamma_{k_2-1}^{-1}\gamma_h^{-1})(\gamma_h\gamma_{k_2}^{-1}\gamma_i^{-1}) \\ &\quad \dots (\gamma_i\gamma_{k_s}^{-1}\gamma_h^{-1})(\gamma_h\gamma_{k_s+1}^{-1}\gamma_h^{-1}) \dots (\gamma_h\gamma_{h-1}\gamma_h^{-1})(\gamma_h^2)(\gamma_{h-1}) \\ &\quad \dots (\gamma_{k_s}) \dots (\gamma_{k_2})(\gamma_{k_2-1}) \dots (\gamma_{k_1+1})(\gamma_{k_1}) \dots (\gamma_{r_t+1})(\gamma_{r_t}\gamma_i^{-1})(\gamma_i\gamma_{r_t-1}\gamma_i^{-1}) \\ &\quad \dots (\gamma_{r_2}\gamma_i^{-1})(\gamma_i\gamma_{r_1})(\gamma_{r_1-1}) \dots (\gamma_j\gamma_i^{-1})(\gamma_i\gamma_{j-1}\gamma_i^{-1}) \dots (\gamma_i^2)(\gamma_{i-1}) \dots (\gamma_1) \\ &= g_1' \dots g_s'\ . \end{aligned}$$

In the same way we have $\sigma_*(g_1') = g'\, g_2'\, g_3' \dots g_s'$.
Analogously $\sigma_*(g_i') = g'\, g_1' \dots g_{i-1}'\, g_{i+1}' \dots g_s'$, $\forall\, i$.

We claim now that $\operatorname{rank}(\sigma + Id)_* = 1$.

In fact, as above, let us first of all consider the following matrix whose columns are the images through $(\sigma + Id)_*$ of the generators g_i's, $g', g'_1, ..., g'_s$.

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

We notice, as before, that we have:

$$Im(\sigma + Id)_* \;=\; \Big\langle (\sigma + Id)_*(g') \Big\rangle \;.$$

Therefore we only have to check that

$$(\sigma + Id)_*(g') \;=\; g' g'_1 ... g'_s \;=\; g_1 ... g_{t-1}$$

is not the identity. But this is true, since the only relations are S_1 and S_2. Thus we have proven that $\operatorname{rank}(\sigma + Id)_* = 1$.

Assume that $\mathcal{G}$ has d vertices and $\mathcal{G}_{red}$ is a polygon with d edges.

We want to show that, as in the case of triangles, in order to compute the rank of $(\sigma + Id)_*$ it suffices to restrict ourself to the case in which every two vertices of $\mathcal{G}$ are connected by exactly 2 edges. In fact, when we have polygons in the reduced graph, we have generators of the type of the g'_i's that we have obtained in the case of triangles, as it is illustrated in Figure 6, where we have

$$\begin{aligned} g'_1 &= \gamma_{k_1}\gamma_{k_2} \cdots \gamma_{k_r}\gamma_{i_1}\gamma_{h_s}^{-1} \cdots \gamma_{h_2}^{-1}\gamma_{h_1}^{-1} \;, \\ g'_2 &= \gamma_{k_1}\gamma_{k_2} \cdots \gamma_{k_r}\gamma_{i_2}\gamma_{h_s}^{-1} \cdots \gamma_{h_2}^{-1}\gamma_{h_1}^{-1} \;, \\ &\;\vdots \\ g'_t &= \gamma_{k_1}\gamma_{k_2} \cdots \gamma_{k_r}\gamma_{i_t}\gamma_{h_s}^{-1} \cdots \gamma_{h_2}^{-1}\gamma_{h_1}^{-1} \;, \end{aligned}$$

Notice that in the figure we only drew the subgraph of $\mathcal{G}$ we are interested in.

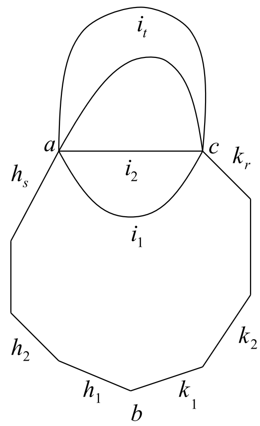

Fig. 6

Assume that $i_1 < i_2 < ... < i_t$ and that $\{i_1, i_2, ..., i_t\}$ is a saturated angle both in a and in c.

Then one can easily prove, as in the case of triangles, that if $\sigma_*(g'_{i_j}) = h$, then $\sigma_*(g'_{i_{j+1}}) = hg'_{i_{j+1}}g'_{i_j}$, so that

$$(\sigma_* + Id)(g'_{i_j}) \;=\; h\,g'_{i_j}\ ,$$

$$(\sigma_* + Id)(g'_{i_{j+1}}) \;=\; h\,g'_{i_{j+1}}\,g'_{i_j}\,g'_{i_{j+1}} \;=\; h\,g'_{i_j}\ .$$

Therefore

$$(\sigma_* + Id)(g'_{i_1}) \;=\; (\sigma_* + Id)(g'_{i_2}) \;=\; ... \;=\; (\sigma_* + Id)(g'_{i_t})\ .$$

This allows us to reduce ourselves to the case in which every two vertices are connected by exactly two edges. In fact if in Figure 6 t is even, we can delete all the edges connecting a and c with labels i_j with $j \geq 3$. If t is odd, we know from the properties of monodromy graphs of real generic algebraic functions described in section 1, that there must exist at least another edge connecting a and c with a label m either bigger then i_t or smaller then i_1 (of course then the set of indices $\{m, i_1, ..., i_t\}$ need not be anymore a saturated angle in a and c, as Figure 7 shows).

If t is odd then we delete all the edges with labels $i_2, ..., i_t$ and we are left with the two edges with labels i_1 and m. The procedure that we have just described is illustrated in the two examples given in Figure 7.

Observe that in Figure 7, when we remove some edges, we relabel the edges of the new graph mantaining the previous order, only in order to have the labels in the right set of indices.

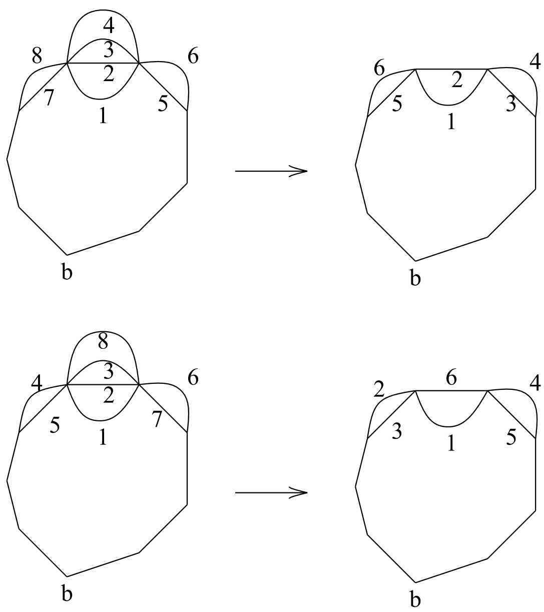

Fig. 7

First of all we treat the case in which the polygon given by $\mathcal{G}_{red}$ is odd, i.e. it has an odd number of vertices, and every two vertices are connected by exactly 2 edges.

To understand better the situation, we describe precisely the case of pentagons and then we explain how to get the general case of an odd polygon.

For pentagons we have the different graphs of Figure 8.

Notice that every other graph of this type can be treated analogously. In all the examples in Figure 8 we take as base point the vertex b.

In Figure 8 1) a Schreier system of representatives for $\pi_1(\mathbf{P}^1 - B, 0)$ modulo $f_*\pi_1(\mathcal{C}-f^{-1}(B), b)$ is the following: $L_0=1$, $L_1=\gamma_1$, $L_2=\gamma_3$, $L_3=\gamma_1\gamma_9$, $L_4=\gamma_3\gamma_5$.

A list of generators is the following: $g_1 = \gamma_1\gamma_2,\ g_2 = \gamma_3\gamma_4,\ g_3 = \gamma_1\gamma_9\gamma_{10}\gamma_1^{-1}$, $g_4 = \gamma_3\gamma_5\gamma_6\gamma_3^{-1},\ g_1' = \gamma_1\gamma_9\gamma_7\gamma_5^{-1}\gamma_3^{-1},\ g_2' = \gamma_1\gamma_9\gamma_8\gamma_5^{-1}\gamma_3^{-1}$.

We have seen in 2.1 that $\sigma_*(g_i) = g_i$ for all i, and as in the case of triangles one easily sees that $(\sigma + Id)_*(g_1') = (\sigma + Id)_*(g_2')$.

Therefore the image of $(\sigma_* + Id)$ is generated by $(\sigma_* + Id)(g_1')$ and the only thing that we have to prove is that $(\sigma_* + Id)(g_1') \neq 0$. We have:

$$\sigma_*(g_1') = g_2\, g_4\, g_2'$$

and the relations $S_1 = \gamma_1\gamma_2...\gamma_8$, $S_2 = \gamma_1\gamma_1\gamma_2...\gamma_8\gamma_1^{-1}$, $S_3 = \gamma_3\gamma_1\gamma_2...\gamma_8\gamma_3^{-1}$, $S_4 = \gamma_1\gamma_9\gamma_1\gamma_2...\gamma_8\gamma_9^{-1}\gamma_1^{-1}$, give: $g_1 = g_2 = g_3 = g_4$, $g_2' = g_1 g_1'$.

Thus we get: $H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z}) = \langle g_1, g_1' \rangle$,

$$\sigma_*(g_1') = g_2\, g_4\, g_2' = g_1\, g_2\, g_4 = g_1^2\, g_1\, g_1' = g_1\, g_1' \ .$$

Then $(\sigma_* + Id)(g_1') = g_1$ and the rank of $(\sigma_* + Id)$ is one.

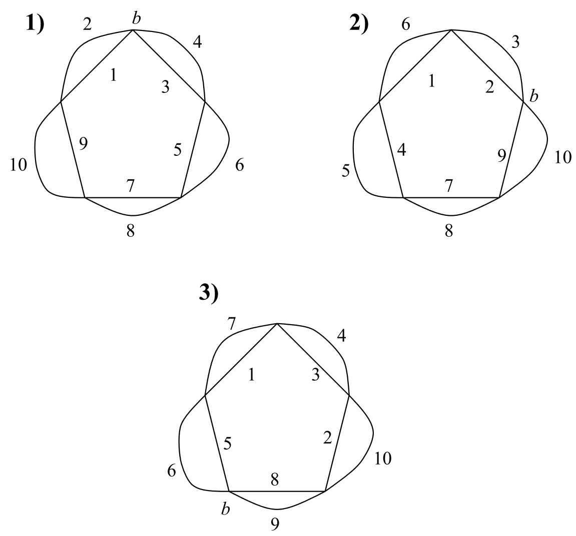

Fig. 8

Now we treat case 2) in Figure 8. Here we choose $L_0 = 1$, $L_1 = \gamma_2$, $L_2 = \gamma_9$, $L_3 = \gamma_2\gamma_1$, $L_4 = \gamma_9\gamma_7$.

We have a list of generators: $g_1 = \gamma_2\gamma_3$, $g_2 = \gamma_9\gamma_{10}$, $g_3 = \gamma_2\gamma_1\gamma_6\gamma_2^{-1}$, $g_4 = \gamma_9\gamma_7\gamma_8\gamma_9^{-1}$, $g_1' = \gamma_2\gamma_1\gamma_4\gamma_7^{-1}\gamma_9^{-1}$, $g_2' = \gamma_2\gamma_1\gamma_5\gamma_7^{-1}\gamma_9^{-1}$. Also here we only have to determine the image of g_1' by $(\sigma_* + Id)$. We obtain as above $\sigma_*(g_1') = g_1 g_4 g_2' = g_1^2 g_1 g_1' = g_1 g_1'$, since the relations S_i's give: $g_i = g_j \ \forall\, i, j$, $g_2' = g_1 g_1'$.

So it remains to consider case 3) in Figure 8. Here we have: $L_0 = 1$, $L_1 = \gamma_5$, $L_2 = \gamma_8$, $L_3 = \gamma_5\gamma_1$, $L_4 = \gamma_8\gamma_2$. We have a list of generators: $g_1 = \gamma_5\gamma_6$, $g_2 = \gamma_8\gamma_9$, $g_3 = \gamma_5\gamma_1\gamma_7\gamma_5^{-1}$, $g_4 = \gamma_8\gamma_2\gamma_{10}\gamma_8^{-1}$, $g_1' = \gamma_5\gamma_1\gamma_3\gamma_2^{-1}\gamma_8^{-1}$, $g_2' = \gamma_5\gamma_1\gamma_4\gamma_2^{-1}\gamma_8^{-1}$.

One computes: $\sigma_*(g'_1) = g_1 g'_1$, and the relations still give: $g_i = g_j \ \forall i, j$, $g'_2 = g_1 g'_1$. Thus the statement is proven in the case of pentagons.

In general, for an odd polygon with d edges and such that every two vertices are connected by 2 edges (i.e. the genus of $\mathcal{C}$ is 1), we obtain a similar list of generators: $g_1, ..., g_{d-1}$, g'_1, g'_2 such that $\sigma_*(g_i) = g_i \ \forall i$, $(\sigma_* + Id)(g'_1) = (\sigma_* + Id)(g'_2) = g_1$. Therefore the rank of $(\sigma_* + Id)$ is one.

Assume now that $\mathcal{G}$ is a graph made of d edges with d even, such that the reduced graph $\mathcal{G}_{red}$ is a polygon with d edges.

We have already remarked that we can restrict out attention to the case in which every two vertices of $\mathcal{G}$ are connected by exactly 2 edges. First of all we consider the case in which $d = 4$. We will prove that the action of σ_* on the generators obtained from the edges of the square is the identity. Following 1.13 we see that for a square in $\mathcal{G}$ we have basically the three following possibilities illustrated in Figure 9:

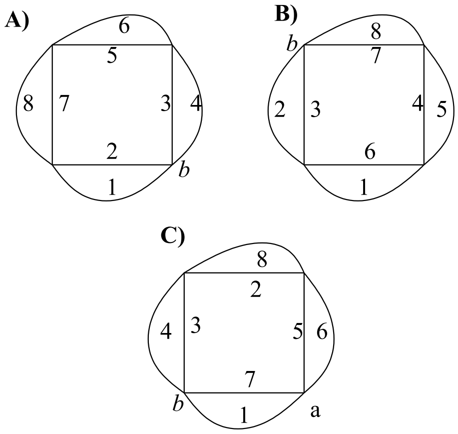

Fig. 9

So we treat at first the simplified case A).

We take b as base point. We choose: $L_0 = 1$, $L_1 = \gamma_1$, $L_2 = \gamma_3$, $L_3 = \gamma_1\gamma_7$.

A list of generators is: $g_1 = \gamma_1\gamma_2$, $g_2 = \gamma_3\gamma_4$, $g_3 = \gamma_1\gamma_7\gamma_8\gamma_1^{-1}$, $g'_1 = \gamma_1\gamma_7\gamma_5\gamma_3^{-1}$, $g'_2 = \gamma_1\gamma_7\gamma_6\gamma_3^{-1}$.

We have relations:

$$S_1 = \gamma_1\gamma_2 \dots \gamma_8 = (\gamma_1\gamma_2)(\gamma_3\gamma_4)(\gamma_5)(\gamma_6)(\gamma_7)(\gamma_8) = g_1 \, g_2, \quad \text{so} \quad g_1 = g_2 \ ,$$
$$S_2 = \gamma_1\gamma_1\gamma_2 \dots \gamma_8\gamma_1^{-1} = g_1 \, g_3, \quad \text{so} \quad g_3 = g_1 \ ,$$
$$S_3 = \gamma_3\gamma_1\gamma_2 \dots \gamma_8\gamma_3^{-1} = g_2 \, g'_1 \, g'_2 \ .$$

So we have $g_2' = g_1 g_1'$ and $H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z}) = \langle g_1, g_1' \rangle$.

$\sigma_*(g_1) = g_1$, $\sigma_*(g_1') = g_2 g_2' = g_1 g_1 g_1' = g_1'$. Therefore we have seen that in case A) $\sigma_* = Id$.

Now we consider case B).

We have: $L_0 = 1$, $L_1 = \gamma_2$, $L_2 = \gamma_7$, $L_3 = \gamma_2\gamma_1$.

A list of generators is the following: $g_1 = \gamma_2\gamma_3$, $g_2 = \gamma_7\gamma_8$, $g_3 = \gamma_2\gamma_1\gamma_6\gamma_2^{-1}$, $g_1' = \gamma_2\gamma_1\gamma_4\gamma_7^{-1}$, $g_2' = \gamma_2\gamma_1\gamma_5\gamma_7^{-1}$. The relations give:

$$\begin{aligned}
S_1 &= \gamma_1\gamma_2 \dots \gamma_8 = g_1\, g_2, \quad \text{so we have } \ g_1 = g_2\ , \\
S_2 &= \gamma_2\gamma_1\gamma_2 \dots \gamma_8\gamma_2^{-1} = g_1'\, g_2'\, g_3\ , \\
S_3 &= \gamma_7\gamma_1\gamma_1\gamma_2 \dots \gamma_8\gamma_1^{-1}\gamma_7^{-1} = g_1'\, g_2'\, g_2\ ,
\end{aligned}$$

therefore we have $g_1 = g_2 = g_3$, $g_2' = g_1 g_1'$, and $H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z}) = \langle g_1, g_1' \rangle$.

$\sigma_*(g_1) = g_1$, $\sigma_*(g_1') = g_1 g_2' = g_1^2 g_1' = g_1'$. Thus we have seen that also in case B) $\sigma_* = Id$.

Now we treat case C). We observe that here all the vertices are moved by σ. We take at first b as base point, $\sigma(b) = a$. A list of generators is: $g_1 = \gamma_1\gamma_7$, $g_2 = \gamma_3\gamma_4$, $g_3 = \gamma_1\gamma_5\gamma_6\gamma_1^{-1}$, $g_1' = \gamma_1\gamma_5\gamma_2\gamma_3^{-1}$, $g_2' = \gamma_1\gamma_5\gamma_8\gamma_3^{-1}$. The relations are:

$$\begin{aligned}
S_1 &= \gamma_1\gamma_2 \dots \gamma_8 = g_3\, g_1, \quad \text{so we have } \ g_3 = g_1\ , \\
S_2 &= \gamma_1\gamma_1\gamma_2 \dots \gamma_8\gamma_1^{-1} = g_2\, g_1, \quad \text{so we obtain } \ g_2 = g_1\ , \\
S_3 &= \gamma_1\gamma_5\gamma_1\gamma_2 \dots \gamma_8\gamma_5^{-1}\gamma_1^{-1} = g_1'\, g_2\, g_2'\ ,
\end{aligned}$$

therefore we get $g_2' = g_1 g_1'$, and $H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z}) = \langle g_1, g_1' \rangle$. Now $\sigma(b) = a$ so we take a as base point in the target. A list of generators is the following: $\psi_1 = \gamma_1\gamma_7$, $\psi_2 = \gamma_5\gamma_6$, $\psi_3 = \gamma_1\gamma_3\gamma_4\gamma_1^{-1}$, $\psi_1' = \gamma_1\gamma_3\gamma_2\gamma_5^{-1}$, $\psi_2' = \gamma_1\gamma_3\gamma_8\gamma_5^{-1}$.

The relations are:

$$\begin{aligned}
S_1' &= \gamma_1\gamma_2 \dots \gamma_8 = \psi_3\, \psi_1, \quad \text{so we have } \ \psi_1 = \psi_3\ , \\
S_2' &= \gamma_1\gamma_1\gamma_2 \dots \gamma_8\gamma_1^{-1} = \psi_2\, \psi_1, \quad \text{therefore we get } \ \psi_2 = \psi_1\ , \\
S_3' &= \gamma_1\gamma_3\gamma_1\gamma_2 \dots \gamma_8\gamma_3^{-1}\gamma_1^{-1} = \psi_1'\, \psi_2\, \psi_2'\ ,
\end{aligned}$$

so we have $\psi_2' = \psi_1\psi_1'$ and $H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z}) = \langle \psi_1, \psi_1' \rangle$.

$\sigma_*(g_1) = \sigma_*(\gamma_1\gamma_7) = \psi_2\psi_1\psi_3 = \psi_1$, $\sigma_*(g_1') = \sigma_*(\gamma_1\gamma_5\gamma_2\gamma_3^{-1}) = \psi_1'$.

On the other hand $Id_*(g_1) = \psi_1$, $Id_*(g_1') = \psi_1'$.

Therefore also in case C) $\sigma_* = Id_*$.

Now in the same way one can see that if $\mathcal{G}_{red}$ is an even polygon, then $\sigma_* = Id_*$.

In fact we restrict ourselves to the case in which every two vertices are connected by exactly two edges and we find as above a list of generators $g_1, ..., g_{d-1}$, g_1', g_2' and relations S_i such that $H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z}) = \langle g_1, g_1' \rangle$ and $\sigma_*(g_1) = g_1$, $\sigma_*(g_1') = g_1'$, if the base point b is fixed; otherwise, if $\mathcal{G}$ has no vertex which is fixed by σ, we have $\sigma_*(g_1) = \psi_1$, $\sigma_*(g_1') = \psi_1'$, where ψ_1 and ψ_1' are the corresponding generators that we have obtained by changing base point to $\sigma(b)$.

Therefore the theorem is proven in the case in which $\mathcal{G}_{red}$ is a polygon. Now one has only to use Theorem 2.1 to obtain the statement under the assumption that every two polygons in $\mathcal{G}_{red}$ do not have any common edge. ∎

Now using the proof of 2.2 we obtain the following

Theorem 2.3. *Let $\mathcal{C}$ be a real smooth algebraic curve of genus g, $\sigma\colon \mathcal{C} \to \mathcal{C}$ be the antiholomorphic involution that gives the real structure, assume that $\nu(\sigma) \neq 0$, let $f : \mathcal{C} \to \mathbf{P}^1_{\mathbf{C}}$ be a generic real algebraic function of degree $d \geq 2$ ($f \circ \sigma(x) = \overline{f(x)}$, $\forall x \in \mathcal{C}$).*

Assume that all the critical values of f are real and positive (if they are not positive it suffices to perform a base point change).

Let $\mathcal{G}$ be the monodromy graph of f. Set $\sigma_*\colon H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z}) \to H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})$. Then

$$\dim(\sigma + identity)_*(H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})) \leq \rho(\mathcal{G}_{red}) ,$$

where $\rho(\mathcal{G}_{red})$ is the minimal number of polygons of $\mathcal{G}_{red}$ whose union is the union of all the polygons of $\mathcal{G}_{red}$. So

$$\nu(\sigma) \geq g + 1 - \rho(\mathcal{G}_{red}) .$$

Proof: The inequality obviously holds if the reduced graph $\mathcal{G}_{red}$ does not contain any polygon (see 2.1) or if every pair of polygons in $\mathcal{G}_{red}$ have no common edges (see 2.2).

So we must understand what happens when we have two polygons in $\mathcal{G}_{red}$ that have at least a common edge. Assume that the reduced graph is the union of two polygons as in Figure 10.

In Figure 10 we have chosen labels of the edges of the reduced graph as follows: for any two vertices a and a' of $\mathcal{G}_{red}$ we have taken the smallest label of the edges of $\mathcal{G}$ connecting a and a'.

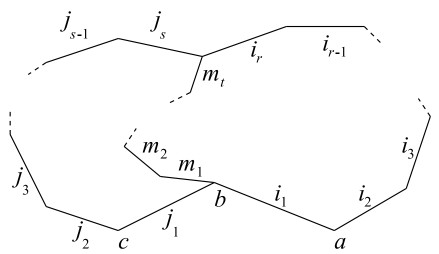

Fig. 10

Then we choose the following set of Schreier representatives for $\pi_1(\mathbf{P}^1 - B, 0)$ modulo $f_*\pi_1(\mathcal{C} - f^{-1}(B), b)$:

$$\begin{aligned}
L_0 &= 1 \ , \\
L_1 &= \gamma_{i_1} \ , \\
L_2 &= \gamma_{i_1}\gamma_{i_2} \ , \\
&\vdots \\
L_{r-1} &= \gamma_{i_1}\gamma_{i_2} \cdots \gamma_{i_{r-1}} \ , \\
L'_1 &= \gamma_{j_1} \ , \\
L'_2 &= \gamma_{j_1}\gamma_{j_2} \ , \\
&\vdots \\
L'_{s-1} &= \gamma_{j_1}\gamma_{j_2} \cdots \gamma_{j_{s-1}} \ , \\
L''_1 &= \gamma_{m_1} \ , \\
L''_2 &= \gamma_{m_1}\gamma_{m_2} \ , \\
&\vdots \\
L''_t &= \gamma_{m_1}\gamma_{m_2} \cdots \gamma_{m_t} \ .
\end{aligned}$$

As in the proof of 2.2 we can assume that every two vertices are connected by exactly two edges. Then if the reduced graph is as in the figure, we have that the genus of $\mathcal{C}$ is 2 and $H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z})$ is generated by g_1, h_1, g'_1 h'_1, where $g_1 = \gamma_{i_1}\gamma_n$, n is the label of the other edge connecting b with a, $h_1 = \gamma_{j_1}\gamma_p$, p is the label of the other edge connecting b with c,

$$\begin{aligned}
g'_1 &= L_{r-1}\,\gamma_{i_r}(L''_t)^{-1} \ , \\
h'_1 &= L'_{s-1}\,\gamma_{j_s}(L''_t)^{-1} \ .
\end{aligned}$$

Then, since $\sigma_*(g_1) = g_1$, $\sigma_*(h_1) = h_1$, the image of $(\sigma_* + Id)$ is generated by the elements: $(\sigma_* + Id)(g_1')$, $(\sigma_* + Id)(h_1')$.

Therefore the rank of $(\sigma_* + Id)$ is smaller or equal to 2.

In conclusion, to determine the rank of $(\sigma_* + Id)$ we can assume that the monodromy graph $\mathcal{G}$ is such that every two vertices are connected by exactly two edges. Under this assumption, every polygon in a tessellation of $\mathcal{G}_{red}$ gives rise to two generators g_i, g_i' with $\sigma_*(g_i) = g_i$, so that only g_i' possibly contributes to increase the rank of $(\sigma_* + Id)$. But then $\mathrm{rank}(\sigma_* + Id) \leq \rho(\mathcal{G}_{red})$, where $\rho(\mathcal{G}_{red})$ is the minimal number of polygons in $\mathcal{G}_{red}$ whose union is the union of all the polygons in $\mathcal{G}_{red}$. ∎

Remark 2.4. Notice that in the proof of 2.3, if $\mathcal{G}_{red}$ is the union of two polygons with common edges, we cannot conclude that $\mathrm{rank}(\sigma_* + Id) = 2$, as it is shown in the following example (Fig. 11). □

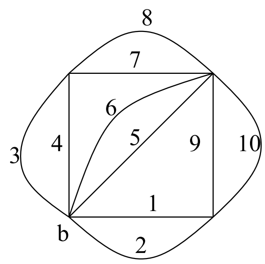

Fig. 11

We have $L_0 = 1$, $L_1 = \gamma_1$, $L_2 = \gamma_5$, $L_3 = \gamma_4$.

The generators are $g_1 = \gamma_1\gamma_2$, $g_2 = \gamma_5\gamma_6$, $g_3 = \gamma_3\gamma_4$, $g_1' = \gamma_1\gamma_9\gamma_5^{-1}$, $g_2' = \gamma_1\gamma_{10}\gamma_5^{-1}$, $h_1' = \gamma_5\gamma_7\gamma_3^{-1}$, $h_2' = \gamma_5\gamma_8\gamma_3^{-1}$.

The relations are $S_1 = \gamma_1\gamma_2...\gamma_{10} = g_1g_2g_3$, $S_2 = \gamma_1\gamma_1\gamma_2...\gamma_{10}\gamma_1^{-1} = g_1g_1'g_2'$, $S_3 = \gamma_5\gamma_1\gamma_2...\gamma_{10}\gamma_5^{-1} = g_2h_1'h_2'g_1'g_2'$, therefore $g_2 = g_1g_3$, $g_2' = g_1g_1'$, $h_2' = g_3h_1'$, $H_1(\mathcal{C}, \mathbf{Z}/2\mathbf{Z}) = \langle g_1, g_3, g_1', h_1'\rangle$ and $\sigma_*(g_i) = g_i$, $i = 1, 3$, $\sigma_*(g_1') = g_1g_3g_1'$, $\sigma_*(h_1') = g_1g_3h_1'$,

$$(\sigma_* + Id) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $(\sigma_* + Id)(g_1') = (\sigma_* + Id)(h_1')$.

Remark 2.5. Let $\mathcal{T}$ be a graph that is the union of two polygons that have some edges in common. Then there exists a monodromy graph $\mathcal{G}$ of a real generic algebraic function with all real critical values with $\mathcal{G}_{red} = \mathcal{T}$, such that

1. If the two polygons are both odd, then $\operatorname{rank}(\sigma_* + Id) = 2$.

2. If one polygon is odd and the other is even, then $\operatorname{rank}(\sigma_* + Id) = 1$.

3. If both polygons are even, then $\sigma_* = Id$. □

Proof: One can easily compute that it suffices to consider the graph $\mathcal{G}$ in Figure 12. ∎

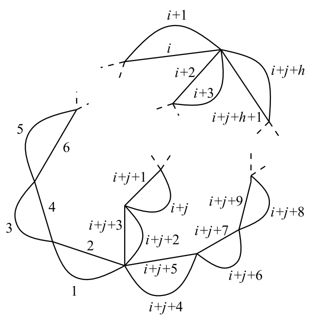

Fig. 12

Remark 2.6. Notice that for the graph in Figure 13 we have

$$\operatorname{rank}(\sigma_* + Id) = 2 \ ,$$

although there is a polygon with an even number of edges. □

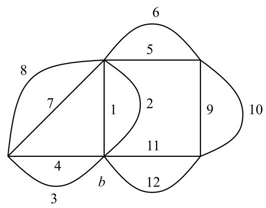

Fig. 13

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