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GLOBAL SOLVABILITY TO THE KIRCHHOFF EQUATION FOR A NEW CLASS OF INITIAL DATA

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Abstract: Introducing a new simple energy estimate, we prove the global solvability of the classical Kirchhoff equation for initial data $u(0,x)$, $u_t(0,x)$ in a suitable subset of the Sobolev class $H^2 \times H^1$.

Introduction

In this paper we investigate the question of the global solvability of the Kirchhoff equation

$$u_{tt} - m\left(\int_\Omega |\nabla u(x,t)|^2\,dx\right)\Delta u \,=\, 0 \quad \text{ in } \;\Omega\times[0,\infty)\;, \tag{1.1}$$

where $m(s)$ is a C^2-function such that $m(s) \geq \delta_0 > 0$, $\forall\, s \geq 0$. We shall consider the following cases:

(**1**) the Cauchy problem, when $\Omega = \mathbb{R}^n$;

(**2**) the periodic initial value problem, when $\Omega =]0, 2\pi[^n$;

(**3**) the initial-boundary value problem, if $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2 boundary.

Our results will be proved introducing a particular class of initial data in which we are able to show that the derivative of the nonlinear term $s(t) \stackrel{\text{def}}{=} \int_\Omega |\nabla u(x,t)|^2 dx$ is a *a-priori* bounded in every bounded interval $[0,T)$.

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1 – Formulation of the problem and main results

(1) In the case $\Omega = \mathbb{R}^n$, we consider the Cauchy problem

$$\begin{cases} u_{tt} - m\Big(\int_{\mathbb{R}^n} |\nabla u(x,t)|^2\,dx\Big)\,\Delta u \;=\; 0, \\ u(x,0) = u_0(x)\,, \qquad u_t(x,0) = u_1(x), \end{cases} \quad \text{in } \mathbb{R}^n\times[0,\infty)\,, \tag{1.2}$$

and we say that $u(x,t)$ is a *strong* solution of (1.2) in $\mathbb{R}^n \times [0,T)$ if

$$u(x,t) \in C^k([0,T); H^{2-k}(\mathbb{R}^n)) \quad \text{for } \ 0 \le k \le 2\,, \tag{1.3}$$

(analogously we define the *strong* periodic solutions if $\Omega = \,]0,2\pi[^n$). Besides, we introduce the following class of functions:

Definition 1.1. Let $\{\rho_j\}_{j\ge 0}$ be a given sequence of positive numbers such that $\rho_j \to +\infty$. We say that a L^2-function $f(x)$ belongs to $B^k_{\{\rho_j\}}(\mathbb{R}^n)$ if there exists $\eta > 0$ such that

$$\limsup_{j\to+\infty} \int_{|\xi|>\rho_j} |\xi|^{2k}\,|\hat f(\xi)|^2\,\exp\Big(\eta\,\rho_j^2/|\xi|\Big)\,d\xi \;<\; +\infty\,. \;\square \tag{1.4}$$

Then we have:

Theorem 1. *Let us suppose that the initial data satisfy*

$$u_0(x) \in B^2_{\{\rho_j\}}(\mathbb{R}^n)\,, \qquad u_1(x) \in B^1_{\{\rho_j\}}(\mathbb{R}^n)\,. \tag{1.5}$$

Then the Cauchy problem (1.2) has a unique global strong solution in $\mathbb{R}^n \times [0,\infty)$.

(2) By the same arguments, we may consider the periodic initial value problem. Namely, we assume that $\Omega = \,]0,2\pi[^n$ and that the unknown $u(x,t)$ is 2π-periodic in the space variables.

Denoting with $c_\xi(f)$, for $\xi \in \mathbb{Z}^n$, the Fourier coefficients of $f(x)$, with respect to the orthonormal system $e_\xi(x) = (2\pi)^{-n/2}\exp\{i\,\xi\,x\}$, we say that:

Definition 1.2. Let $\{\rho_j\}_{j\ge 0}$ be a given sequence of positive numbers such that $\rho_j \to +\infty$. A L^2 locally integrable 2π-periodic function $f(x)$ belongs to $B^k_{\{\rho_j\}}(]0,2\pi[^n)$ if there exists a constant $\eta > 0$ such that

$$\limsup_{j\to+\infty} \sum_{\xi\in\mathbb{Z}^n,\,|\xi|>\rho_j} |\xi|^{2k}\,|c_\xi(f)|^2\,\exp\Big(\eta\,\rho_j^2/|\xi|\Big) \;<\; +\infty\,. \;\square \tag{1.6}$$

Then, we have the following:

Corollary 2. *Let $\Omega =]0,2\pi[^n$ and suppose that the initial data are 2π-periodic functions such that $u_0(x) \in B^2_{\{\rho_j\}}(]0,2\pi[^n)$, $u_1(x) \in B^1_{\{\rho_j\}}(]0,2\pi[^n)$. Then the initial value problem*

$$\text{(1.7)} \qquad \begin{cases} u_{tt} - m\Big(\displaystyle\int_{]0,2\pi[^n} |\nabla u(x,t)|^2\,dx\Big)\Delta u \,=\, 0, \\ u(x,0) = u_0(x)\,, \qquad u_t(x,0) = u_1(x), \end{cases} \qquad \text{in } \mathbb{R}^n \times [0,\infty)\ ,$$

has a unique global strong periodic solution in $\mathbb{R}^n \times [0,\infty)$. ∎

(3) Finally, in the case of a bounded C^2 domain $\Omega \subset \mathbb{R}^n$, we consider the initial-boundary value problem:

$$\text{(1.8)} \qquad \begin{cases} u_{tt} - m\Big(\displaystyle\int_{\Omega} |\nabla u(x,t)|^2\,dx\Big)\,\Delta u \,=\, 0, \\ u(x,0) = u_0(x)\,, \qquad u_1(x,0) = u_1(x), \\ u(x,t) = 0 \quad \text{on } \ \partial\Omega \times [0,\infty), \end{cases} \qquad \text{in } \ \Omega \times [0,\infty)\ ,$$

and we say that $u(x,t)$ is a *strong* solution of (1.8) in $\Omega\times[0,T)$ if

$$\text{(1.9)} \qquad u(x,t) \,\in\, C^0([0,T); H^2(\Omega)) \cap C^1([0,T); H^1_0(\Omega)) \cap C^2([0,T); L^2(\Omega))\ .$$

As it is well known, we can find an orthonormal basis $\{v_i\}_{i\geq 0}$ of $L^2(\Omega)$ such that

$$-\Delta v_i = \lambda_i^2\, v_i\,, \qquad v_i \in H^1_0(\Omega)\ ,$$

where $\lambda_i > 0$, $\lambda_i \to +\infty$. Following the same lines, we say that $f(x) \in B^k_{\{\rho_j\}}(\Omega)$ if for some constant $\eta > 0$ we have

$$\text{(1.10)} \qquad \limsup_{j\to+\infty} \sum_{\lambda_i > \rho_j} |\lambda_i|^{2k}\, |c_i(f)|^2\, \exp\Big(\eta\, \rho_j^2/\lambda_i\Big) \,<\, +\infty\ ,$$

where $c_i(f)$ are the Fourier coefficients with respect to the basis $\{v_i\}_{i\geq 0}$. Then, we have:

Corollary 3. *Let $\Omega\subset\mathbb{R}^n$ be a bounded C^2 domain and assume that $u_0(x) \in B^2_{\{\rho_j\}}(\Omega)$ and $u_1(x) \in B^1_{\{\rho_j\}}(\Omega)$. Then the initial-boundary value problem (1.8) has a unique global strong solution $u(x,t)$ in $\Omega\times[0,\infty)$.* ∎

Remark 1.3. If the sequence $\{\rho_j\}_{j\geq 0}$ does not increase too fast, then $B^k_{\{\rho_j\}}(\mathbb{R}^n) = \mathcal{A}_{L^2}(\mathbb{R}^n)$ and $B^k_{\{\rho_j\}}(]0,2\pi[^n)$ coincide with the space $\mathcal{A}_{2\pi}(\mathbb{R}^n)$ of the analytic, 2π-periodic functions in $\mathbb{R}^n$. On the other hand, if $\rho_j \to +\infty$ and satisfies (for example) a condition like

$$\rho_{j+1} \geq 2\,\rho_j^2 \,, \tag{1.11}$$

it is easy to see that $\mathcal{A}_{L^2}(\mathbb{R}^n) \subsetneqq B^k_{\{\rho_j\}}(\mathbb{R}^n)$, $\mathcal{A}_{2\pi}(\mathbb{R}^n) \subsetneqq B^k_{\{\rho_j\}}(]0,2\pi[^n)$ and that the spaces $B^k_{\{\rho_j\}}(\mathbb{R}^n)$ and $B^k_{\{\rho_j\}}(]0,2\pi[^n)$ contain non-smooth functions. Unfortunately, it is also possible to prove that $B^k_{\{\rho_j\}}(\mathbb{R}^n)$ does not contain compactly supported functions (see the appendix). This means that the class $B^k_{\{\rho_j\}}(\mathbb{R}^n)$ is (in some sense) relatively small.

In the case of a bounded C^2-domain Ω, if the sequence $\{\rho_j\}_{j\geq 0}$ increase sufficiently fast, we have $\mathcal{A}_0(\Omega) \subsetneqq B^k_{\{\rho_j\}}(\Omega)$ where $\mathcal{A}_0(\Omega)$ denotes the space of all analytic functions in some neighborhood of $\overline{\Omega}$ such that $\Delta^l u = 0$ on $\partial\Omega$ for all $l \geq 0$.

In conclusion Theorem 1 and the Corollaries 1 and 2 extend the known global existence results in the space of real analytic functions. □

In this paper we shall give in details only the proof of the global solvability of the Cauchy problem (1.2), that is Theorem 1, because the Corollaries 2 and 3 can be proved following the same arguments.

Moreover, since the local solvability of the Kirchhoff equation (1.1) is well known (see [1], [5], [6], [12], [17], [18]), we shall only prove the basic *a-priori* estimates.

Finally we mention about the related works. The global solvability of the classical Kirchhoff equation in the space of analytic functions was originally proved by Bernstein [3], in one space dimension, and then extended by Pohožaev [15] to several space dimensions. Later, the result of [15] has been generalized in [2], [4], [9] considering the weakly hyperbolic case and more general second order elliptic operators. For quasi-analytic initial data the first result of global solvability was proved by Nishihara [10] and then generalized in [8], [20].

2 – The Linearized Equation

In this section we study the linearized equation derived from (1.1) setting $m(\int |\nabla u(x,t)|^2\,dx) = a(t)$ and $v(\xi,t) = \mathcal{F}_x\, u(\xi,t)$, i.e. applying the Fourier transform in the space variables. We use a technique introduced by Pohožaev [16] to obtain a second order conservation law for the Kirchhoff equation. See also [13], [14]. Let us consider the infinite system of linear oscillating equations of the form

$$v_{tt} + a(t)\,|\xi|^2\,v \;=\; 0 \quad \text{for } \; t\in[0,T), \;\; \xi\in\mathbb{R}^n \;, \tag{2.1}$$

where $0<T<\infty$, $v=v(\xi,t)$; $a(t)$ is a real valued function satisfying the conditions

$$a(t)\in C^2([0,T))\,, \qquad a(t)\geq\delta_0>0 \;\; \forall\, t\geq 0 \;. \tag{2.2}$$

Multiplying (2.1) by the factor $a_1(t)\,|\xi|^2\,\bar v_t$, we easily obtain that

$$\begin{aligned}\frac{d}{dt}\Big(a_1(t)\,|\xi|^2\,|v_t|^2 + a(t)\,a_1(t)\,|\xi|^4\,|v|^2\Big) \;&=\\ =\; a_1'(t)\,|\xi|^2\,|v_t|^2 + [a(t)\,a_1(t)]'\,|\xi|^4\,|v|^2 \;.\end{aligned} \tag{2.3}$$

While, multiplying by the term $a_2(t)\,|\xi|^2\,\bar v$ we find

$$\begin{aligned}\frac{d}{dt}\Big(a_2(t)\,|\xi|^2\,\Re\{\bar v\,v_t\}\Big) \;&=\\ =\; -a(t)\,a_2(t)\,|\xi|^4\,|v|^2 + a_2(t)\,|\xi|^2\,|v_t|^2 + a_2'(t)\,|\xi|^2\,\Re\{\bar v\,v_t\} \;,\end{aligned} \tag{2.4}$$

where $\Re\{z\}$ denotes the real part of $z\in\mathbb{C}$. Thus, introducing the quantity

$$\mathcal{E}(\xi,t) \stackrel{\text{def}}{=} \frac{1}{2}\,a_1(t)\,|\xi|^2\,|v_t|^2 + \frac{1}{2}\,a(t)\,a_1(t)\,|\xi|^4\,|v|^2 + a_2(t)\,|\xi|^2\,\Re\{\bar v\,v_t\} \;, \tag{2.5}$$

it follows that

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(\xi,t) \;&=\; \left[\frac{1}{2}(a(t)\,a_1(t))' - a(t)\,a_2(t)\right]|\xi|^4\,|v|^2\\ &\quad+\;\left[\frac{1}{2}\,a_1(t)' + a_2(t)\right]|\xi|^2\,|v_t|^2\\ &\quad+\; a_2(t)'\,|\xi|^2\,\Re\{\bar v\,v_t\} \;.\end{aligned} \tag{2.6}$$

Now, let us choose the coefficients $a_1(t)$, $a_2(t)$ such that

$$\begin{cases}\dfrac{1}{2}\,(a(t)\,a_1(t))' - a(t)\,a_2(t) \;=\; 0,\\[2mm] \dfrac{1}{2}\,a_1(t)' + a_2(t) \;=\; 0 \;.\end{cases} \tag{2.7}$$

A straightforward computation gives

$$a_1(t) = \frac{C}{\sqrt{a(t)}}, \quad a_2(t) = \frac{C}{4}\,\frac{a(t)'}{a(t)^{3/2}} \qquad \text{with } \; C\in\mathbb{R} \;. \tag{2.8}$$

In the following, we fix $C=1$. Then, taking $a_1(t)$, $a_2(t)$ as in (2.8) we finally have

$$\frac{d}{dt}\mathcal{E}(\xi,t) \;=\; a_2(t)'\,|\xi|^2\,\Re\{\bar v\,v_t\}\ . \tag{2.9}$$

Defining the energy

$$E(\xi,t) \stackrel{\text{def}}{=} \frac{1}{2}\,a_1(t)\,|\xi|^2\,|v_t|^2 + \frac{1}{2}\,a(t)\,a_1(t)\,|\xi|^4\,|v|^2\ , \tag{2.10}$$

we can state the following lemma:

Lemma 2.1. *Let us suppose that* $0<T<\infty$. *Then, for every* $\varepsilon>0$ *there exists* $\rho_\varepsilon>0$ *such that*

$$E(\xi,t) \le 4\,E(\xi,0) \quad \textit{in}\ \ [0,T-\varepsilon]\ , \tag{2.11}$$

for every $|\xi|\ge\rho_\varepsilon$.

Proof: Let us fix $K>0$ such that

$$|a_2(t)|,\ |a_2(t)'| \;\le\; K \quad \text{in}\ \ [0,T-\varepsilon]\ . \tag{2.12}$$

Then, having

$$|\xi|^3\,|v|\,|v_t| \;\le\; \frac{1}{2\sqrt{a(t)}}\ |\xi|^2\,|v_t|^2 + \frac{\sqrt{a(t)}}{2}\ |\xi|^4\,|v|^2 \;\equiv\; E(\xi,t)\ ,$$

for $|\xi|>0$ the equations (2.5)–(2.9) give

$$\frac{d}{dt}\Big[E(\xi,t) + a_2(t)\,|\xi|^2\,\Re\{\bar v\,v_t\}\Big] \;\le\; K\,|\xi|^2\,|\bar v\,v_t| \;\le\; K\,\frac{E(\xi,t)}{|\xi|}\ . \tag{2.13}$$

Now, we can choose $\rho_1>0$ such that

$$|\xi|\ge\rho_1 \quad\Longrightarrow\quad \Big|a_2(t)\,|\xi|^2\,\Re\{\bar v\,v_t\}\Big| \le \frac{E(\xi,t)}{2} \quad \text{for}\ \ 0\le t\le T-\varepsilon\ . \tag{2.14}$$

Namely, we set $\rho_1=2K$. Then, for $|\xi|\ge\rho_1$, we have

$$\begin{aligned} E(\xi,t) \;&\le\; E(\xi,0) - \Big[a_2(\tau)\,|\xi|^2\,\Re\{\bar v\,v_t\}\Big]_0^t + \frac{K}{|\xi|}\int_0^t E(\xi,\tau)\,d\tau \\ &\le\; \frac{3}{2}\,E(\xi,0) + \frac{1}{2}\,E(\xi,t) + \frac{K}{|\xi|}\int_0^t E(\xi,\tau)\,d\tau\ . \end{aligned} \tag{2.15}$$

Hence, by Gronwall's Lemma, it follows that

$$E(\xi,t) \;\le\; 3\,E(\xi,0)\,\exp\!\left(\frac{2K}{|\xi|}\,t\right)\ . \tag{2.16}$$

Thus, it is sufficient to take $\rho_\varepsilon \geq \rho_1$ such that

$$|\xi| \geq \rho_\varepsilon \implies \frac{2\,K(T-\varepsilon)}{|\xi|} \leq \ln\!\left(\frac{4}{3}\right) . \blacksquare \tag{2.17}$$

Remark 2.2. A similar conclusion holds true if $T = +\infty$. Precisely, the estimate (2.11) holds in the interval $[0, 1/\varepsilon]$, provided $|\xi| \geq \rho_\varepsilon$ for a suitable $\rho_\varepsilon > 0$. □

Moreover, we easily have:

Corollary 2.3. *Let $T < \infty$. Then for every $\varepsilon > 0$ there exists $\rho_\varepsilon > 0$ such that*

$$\frac{E(\xi,0)}{4} \leq E(\xi,t) \leq 4\,E(\xi,0) \quad \text{in } [0, T-\varepsilon] \tag{2.18}$$

for all $|\xi| \geq \rho_\varepsilon$. ■

3 – A-priori estimates for $0 \leq t \leq T-\varepsilon$

Let us consider the infinite system of nonlinear oscillating equations

$$\begin{cases} v_{tt} + m\Big(\displaystyle\int |\xi|^2\,|v(\xi,t)|^2\,d\xi\Big)\,|\xi|^2\,v = 0, \\ v(\xi,0) = v_0(\xi)\,, \quad v_t(\xi,0) = v_1(\xi), \end{cases} \quad \text{in } \mathbb{R}^n_\xi \times [0,\infty)\ , \tag{3.1}$$

where $m(s) \in C^2([0,\infty))$ with $m(s) \geq \delta_0 > 0$. From now on we assume that the initial data $v_0(\xi)$, $v_1(\xi)$ are L^2 functions such that

$$\int \Big(1+|\xi|^4\Big)\,|v_0(\xi)|^2\,d\xi < \infty\,, \qquad \int \Big(1+|\xi|^2\Big)\,|v_1(\xi)|^2\,d\xi < \infty \tag{3.2}$$

and that $v(\xi,t)$ is the (unique) local *strong* solution of (3.1) in $\mathbb{R}^n_\xi \times [0,T)$. Namely:

Definition 3.1. We say that $v(\xi,t)$ is a strong solution of (3.1) in $\mathbb{R}^n_\xi \times [0,T)$ if

$$\Big(1+|\xi|^{2-j}\Big)\,\partial_t^j v(\xi,t) \in C^0([0,T); L^2(\mathbb{R}^n_\xi))\ , \tag{3.3}$$

for $0 \leq j \leq 2$ and $t \mapsto v(\cdot,t)$ satisfies (3.1) in $[0,T)$. □

Remark 3.2. Condition (3.3) implies that

$$t \longmapsto s(t) \stackrel{\text{def}}{=} \int |\xi|^2 \, |v(\xi,t)|^2 \, d\xi \tag{3.4}$$

is a well defined C^2-function on $[0,T)$. Setting then

$$a(t) = m(s(t)) \ , \tag{3.5}$$

it follows that $a(t) \in C^2([0,T))$ with $a(t) \geq \delta_0$ and defining (with $C{=}1$)

$$\begin{aligned} a_1(t) &= \frac{1}{\sqrt{m(s(t))}} \stackrel{\text{def}}{=} m_1(s(t)) \ , \\ a_2(t) &= \frac{1}{4} \frac{m'(s(t))}{\sqrt{m(s(t))^3}} \, s'(t) \stackrel{\text{def}}{=} m_2(s(t)) \, s'(t) \ , \end{aligned} \tag{3.6}$$

we can apply the energy estimates of the previous section. □

To continue, let us recall that:

First order conservation law. Assuming $v(\xi,t)$ a local strong solution of (3.1) and multiplying by the factor $\bar{v}_t$, we find the well known identity:

$$\mathcal{H}(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n_\xi} |v_t(\xi,t)|^2 \, d\xi \, + \, \Phi\Big(\int_{\mathbb{R}^n_\xi} |\xi|^2 \, |v(\xi,t)|^2 \, d\xi \Big) \equiv \mathcal{H}(0) \stackrel{\text{def}}{=} \mathcal{H}_0 \ , \tag{3.7}$$

where $\Phi(s) \stackrel{\text{def}}{=} \int_0^s m(z) \, dz$ satisfies $\Phi(s) \geq \delta_0 \, s$ for all $s \geq 0$. □

Thus, we have the a-priori estimate

$$\int |v_t(\xi,t)|^2 \, d\xi \, + \, \delta_0 \int |\xi|^2 \, |v(\xi,t)|^2 \, d\xi \, \leq \, \mathcal{H}_0 \ , \tag{3.8}$$

uniformly for $0 \leq t < T$. In particular, we have $0 \leq s(t) \leq \mathcal{H}_0/\delta_0$ and

$$\int_{|\xi| \leq \rho} |\xi|^2 \, |v| \, |v_t| \, d\xi \, \leq \, \rho \, \frac{\mathcal{H}_0}{2\sqrt{\delta_0}} \ , \tag{3.9}$$

for all $\rho \geq 0$. Note also that

$$\delta_0 \leq m(s(t)) \leq M \quad \text{and} \quad |m_2(s(t))|, \ |m_2'(s(t))| \, \leq \, M/2 \ ,$$

for a suitable constant $M > 0$ because $m(s) \in C^2([0,\infty))$.

Applying Lemma 2.1 we can state the following:

Corollary 3.3. *For every $\varepsilon > 0$ there exists $\rho_\varepsilon > 0$ such that*

$$|s'(t)| \leq \rho \frac{\mathcal{H}_0}{\sqrt{\delta_0}} + 8 \int_{|\xi|>\rho} \frac{E(0,\xi)}{|\xi|} \, d\xi \tag{3.10}$$

for all $\rho \geq \rho_\varepsilon$ and $t \in [0, T-\varepsilon]$.

Proof: Deriving with respect to t, we have

$$\begin{aligned} |s'(t)| &\leq 2 \int_{|\xi|\leq\rho} |\xi|^2 \, |v| \, |v_t| \, d\xi + 2 \int_{|\xi|>\rho} |\xi|^2 \, |v| \, |v_t| \, d\xi \\ &\leq \frac{\rho}{\sqrt{\delta_0}} \int_{|\xi|\leq\rho} \Big[|v_t|^2 + \delta_0 \, |\xi|^2 \, |v|^2 \Big] \, d\xi + 2 \int_{|\xi|>\rho} |\xi|^2 \, |v| \, |v_t| \, d\xi \\ &\leq \rho \frac{\mathcal{H}_0}{\sqrt{\delta_0}} + 2 \int_{|\xi|>\rho} \frac{E(\xi,t)}{|\xi|} \, d\xi \, . \end{aligned} \tag{3.11}$$

Now, taking ρ_ε according to Lemma 2.1, namely

$$\rho_\varepsilon = 2 \, K \max\left\{1, \frac{T-\varepsilon}{\ln(4/3)}\right\} \tag{3.12}$$

where $K > 0$ satisfies

$$|m_2 \, s'(t)| \, , \; |m_2' \, s'(t)^2 + m_2 \, s''(t)| \leq K \quad \text{in} \;\; [0, T-\varepsilon] \, , \tag{3.13}$$

we can apply the inequality (2.11), that is $E(\xi,t) \leq 4 \, E(\xi,0)$ for $|\xi| \geq \rho_\varepsilon$. Clearly, this immediately implies the estimate (3.10). ∎

4 – A-priori estimates for $T - \varepsilon \leq t < T$

Now, assuming suitable conditions on the initial data $v_0(\xi)$ and $v_1(\xi)$ of problem (3.1), we shall prove that the energy $E(\xi,t)$ is uniformly bounded in the interval $[0,T)$, if $T < +\infty$.

To begin with, for $\rho \geq 1$, let us set

$$\mathcal{E}_\rho(t) \stackrel{\text{def}}{=} \int_{|\xi|>\rho} \frac{E(\xi,t)}{|\xi|} \, d\xi \, . \tag{4.1}$$

Rewriting the expression of the energy $E(\xi,t)$ with the positions (3.5)–(3.6), we have

$$E(\xi,t) = \frac{|\xi|^2 \, |v_t|^2}{2 \sqrt{m(s(t))}} + \frac{1}{2} \sqrt{m(s(t))} \; |\xi|^4 \, |v|^2 \, . \tag{4.2}$$

Besides, taking into account of (3.8) and (3.11), it follows that

$$|s'(t)| \le \rho\,\frac{\mathcal{H}_0}{\sqrt{\delta_0}} + 2\,\mathcal{E}_\rho(t)\ , \tag{4.3a}$$

$$\begin{aligned} |s''(t)| &\le 2\int |\xi|^2\,|v_t|^2\,d\xi \,+\, 2\,m(s(t))\int |\xi|^4\,|v|^2\,d\xi \\ &\le 2\,\rho^2\,[1+M/\delta_0]\,\mathcal{H}_0 + 4\sqrt{M}\int_{|\xi|>\rho} E(\xi,t)\,d\xi\ . \end{aligned} \tag{4.3b}$$

Hence, from (2.4)–(2.9), we have the following estimate

$$\begin{aligned} E'(\xi,t) + \frac{d}{dt}\Big(m_2(s(t))\,s'(t)\,|\xi|^2\,\Re\{\bar v\,v_t\}\Big) &\le \\ &\le \Big|m_2'(s(t))\,s'(t)^2 + m_2(s(t))\,s''(t)\Big|\,|\xi|^2\,|v|\,|v_t| \\ &\le C_1\Big[(\mathcal{H}_0+\mathcal{H}_0^2)\,\rho^2 + \mathcal{E}_\rho(t)^2 + \int_{|\xi|>\rho} E(\xi,t)\,d\xi\Big]\,\frac{E(\xi,t)}{|\xi|}\ , \end{aligned} \tag{4.4}$$

for a suitable constant $C_1 > 0$ depending only on M, δ_0. Now, let $K_1 \ge 1$ be a given constant (we will fix K_1 and ε in the following). Then, as long as

$$\mathcal{E}_\rho(t)^2 + \int_{|\xi|>\rho} E(\xi,t)\,d\xi \;\le\; K_1\ , \tag{4.5}$$

we have

$$\begin{aligned} E(\xi,t) \;\le\; E(\xi,T-\varepsilon) &- \Big[m_2(s(\tau))\,s'(\tau)\,|\xi|^2\,\Re\{\bar v\,v_t\}\Big]_{T-\varepsilon}^{t} \\ &+\, C_1\,\frac{(\mathcal{H}_0+\mathcal{H}_0^2)\,\rho^2 + K_1}{|\xi|}\int_{T-\varepsilon}^{t} E(\xi,\tau)\,d\tau \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} \Big|m_2(s(t))\,s'(t)\,|\xi|^2\,\Re\{\bar v\,v_t\}\Big| &\le |m_2|\,\Big[\rho\,\frac{\mathcal{H}_0}{\sqrt{\delta_0}} + 2\,\mathcal{E}_\rho(t)\Big]\,\frac{E(\xi,t)}{|\xi|} \\ &\le \frac{M}{|\xi|}\,\Big[\rho\,\frac{\mathcal{H}_0}{\sqrt{\delta_0}} + \sqrt{K_1}\Big]\,E(\xi,t)\ . \end{aligned} \tag{4.7}$$

Now, if we assume that

$$\mathcal{H}_0 \;\le\; \frac{\sqrt{\delta_0}}{3\,M}\ , \tag{4.8}$$

then

$$\Big|m_2(s(t))\,s'(t)\,|\xi|^2\,\Re\{\bar v\,v_t\}\Big| \;\le\; \frac{E(\xi,t)}{2} \qquad \text{provided}\ \ |\xi| \ge \max\{\rho,\, 6\,M\sqrt{K_1}\}\ .$$

This implies that, as long as (4.5) holds and $|\xi| \geq \max\{\rho,\, 6\,M\sqrt{K_1}\}$, we have

$$E(\xi,t) \;\leq\; 3\,E(\xi,T-\varepsilon) + 2\,C_1\,\frac{(\mathcal{H}_0+\mathcal{H}_0^2)\,\rho^2+K_1}{|\xi|}\int_{T-\varepsilon}^{t} E(\xi,\tau)\,d\tau\ . \tag{4.9}$$

Hence, applying the classical Gronwall's lemma,

$$E(\xi,t) \;\leq\; 3\,E(\xi,T-\varepsilon)\,\exp\Big\{2\,C_1\,\frac{(\mathcal{H}_0+\mathcal{H}_0^2)\,\rho^2+K_1}{|\xi|}\,(t-T+\varepsilon)\Big\}\ . \tag{4.10}$$

Finally, having $\rho,\ K_1 \geq 1$, in order to verify the condition (4.5), it will be sufficient to require that

$$\int_{|\xi|>\rho} E(\xi,t)\,d\xi \;\leq\; \frac{\sqrt{K_1}}{2}\ . \tag{4.11}$$

To this end, recalling Lemma 2.1 and (3.12), (3.13) we assume that

$$\rho \;\geq\; \max\Big\{1,\ \rho_\varepsilon,\ 6\,M\sqrt{K_1}\Big\}\ , \tag{4.12}$$

thus, by the estimate (2.11), we know that $E(\xi,T-\varepsilon) \leq 4\,E(\xi,0)$ in (4.10). Then, condition (4.5) will be verified as long as

$$\begin{aligned} I(t) &\stackrel{\text{def}}{=} \int_{|\xi|>\rho} E(\xi,0)\,\exp\Big\{2\,C_1\frac{(\mathcal{H}_0+\mathcal{H}_0^2)\,\rho^2+K_1}{|\xi|}\,(t-T+\varepsilon)\Big\}\,d\xi \\ &\leq \frac{\sqrt{K_1}}{24}\ . \end{aligned} \tag{4.13}$$

In order to extend the solution $v(\xi,t)$, we require now that (4.13) holds for $T-\varepsilon \leq t < T$. Clearly, this leads to a sufficient condition for the global solvability of the Cauchy problem (3.1). More precisely, we assume that the initial data $v_0(\xi)$, $v_1(\xi)$ satisfy the following conditions:

Definition 4.1. Let $\{\rho_j\}_{j\geq 0}$ be a given sequence of positive numbers such that $\rho_j \to +\infty$. We say that a L^2-function $f(\xi)$ belong to $\tilde{B}^k_{\{\rho_j\}}(\mathbb{R}^n)$ if there exist $\eta > 0$ and $\mathcal{C} \geq 0$ such that

$$\int_{|\xi|>\rho_j} |\xi|^{2k}\,|f(\xi)|^2\,\exp\{\eta\,\rho_j^2/|\xi|\}\,d\xi \;\leq\; \mathcal{C} \qquad \forall\, j\geq 0\ . \ \square \tag{4.14}$$

Then, if $v_0(\xi) \in \tilde{B}^2_{\{\rho_j\}}\{\mathbb{R}^n\}$, $v_1(\xi) \in \tilde{B}^1_{\{\rho_j\}}\{\mathbb{R}^n\}$, it is easy to see that (4.13) holds true in $[T-\varepsilon, T)$ provided we choose $\varepsilon > 0$ sufficiently small, $\rho = \rho_j \geq 1$ and

$K_1 \geq 1$ large enough. In fact, since $|\xi|^2 v_0(\xi)$, $|\xi|\, v_1(\xi) \in L^2$, we have $\mathcal{H}_0 < \infty$ and

$$E(\xi,0) \,\leq\, C_2 \left(|\xi|^2 \, |v_1(\xi)|^2 + |\xi|^4 \, |v_0(\xi)|^2 \right) , \tag{4.15}$$

with $C_2 = \frac{1}{2\sqrt{\delta_0}} + \frac{\sqrt{M}}{2}$. Hence, for $T-\varepsilon \leq t < T$, we have

$$\begin{aligned} I(t) \,\leq\, & C_2 \int_{\rho^2 \geq |\xi| > \rho} \left(|\xi|^2 \, |v_1(\xi)|^2 + |\xi|^4 \, |v_0(\xi)|^2 \right) \exp\left\{ \frac{C_3\, \rho^2 + 2\, C_1\, K_1}{|\xi|} \, \varepsilon \right\} d\xi \\ & + C_2 \int_{|\xi| > \rho^2} \left(|\xi|^2 \, |v_1(\xi)|^2 + |\xi|^4 \, |v_0(\xi)|^2 \right) \exp\left\{ \frac{C_3\, \rho^2 + 2\, C_1\, K_1}{|\xi|} \, \varepsilon \right\} d\xi \end{aligned} \tag{4.16}$$

where $C_3 = 2\, C_1\, (\mathcal{H}_0 + \mathcal{H}_0^2)$. Then, since $v_0(\xi)$, $v_1(\xi)$ satisfy the inequality (4.14) (with $k=2$ and $k=1$ respectively) for suitable constants $\eta_i > 0$ and $\mathcal{C}_i \geq 0$, we choose K_1 large such that

$$\frac{\sqrt{K_1}}{48} - 1 \,\geq\, C_2\, \mathcal{C}_i \quad \text{ for } \; i = 0,\, 1 \tag{4.17}$$

and $\varepsilon > 0$ small such that, for $\rho \geq 1$,

$$\frac{C_3\, \rho^2 + 2\, C_1\, K_1}{|\xi|} \, \varepsilon \,<\, \eta_i \, \frac{\rho^2}{|\xi|} \quad \text{ in } \; \rho \leq |\xi| \leq \rho^2 \; \text{ for } i = 0,\, 1 \; . \tag{4.18}$$

In this way, for all $\rho = \rho_j$ the value of the first integral in (4.16) is less than $\dfrac{\sqrt{K_1}}{24} - 2$.

Finally, noting that

$$\lim_{\rho \to +\infty} \int_{|\xi| > \rho^2} \left(|\xi|^2 \, |v_1(\xi)|^2 + |\xi|^4 \, |v_0(\xi)|^2 \right) \exp\left\{ \frac{C_3\, \rho^2 + 2\, C_1\, K_1}{|\xi|} \, \varepsilon \right\} d\xi \,=\, 0 \tag{4.19}$$

we take $j \geq 0$ large, such that the value of the second integral in (4.16) is less than 2.

Summarizing up, we have proved the following:

Lemma 4.2. *Let $v(\xi,t)$ be a local strong solution of the problem (3.1) in the stripe $\mathbb{R}^n_\xi \times [0,T)$ with $0 < T < +\infty$. Assume that (4.8) holds and that*

$$v_0(\xi) \in \tilde{B}^2_{\{\rho_j\}}(\mathbb{R}^n_\xi) \, , \qquad v_1(\xi) \in \tilde{B}^1_{\{\rho_j\}}(\mathbb{R}^n_\xi) \; .$$

Then the integral $\displaystyle\int_{\mathbb{R}^n} E(\xi,t)\, d\xi$ is uniformly bounded in $[0,T)$.

Proof: By the previous arguments, see (4.11), we know that

$$\int_{|\xi|>\rho_j} E(\xi,t)\,d\xi \;\leq\; \frac{\sqrt{K_1}}{2}\,, \qquad \forall\, t\in[0,T)\;, \tag{4.20}$$

for suitable constants K_1 satisfying (4.17) and $\rho_j \geq \max\{1,\, \rho_\varepsilon,\, 6\,M\sqrt{K_1}\}$ such that the value of the second integral in (4.16) is less than 2. On other hand for $|\xi| \leq \rho_j$, by (3.8), we have

$$\int_{|\xi|\leq\rho_j} E(\xi,t)\,d\xi \;\leq\; \frac{1}{2\sqrt{\delta_0}}\left(1+\frac{\sqrt{M}}{\sqrt{\delta_0}}\right)\rho_j^2\,\mathcal{H}_0\;, \tag{4.21}$$

for all $t\in[0,T)$. ∎

5 – Global solvability for small data

From the estimates of the previous sections, it is now straightforward to prove the global solvability provided the initial data $(u_0(x),u_1(x)) \in B^2_{\{\rho_j\}}(\mathbb{R}^n)\times B^1_{\{\rho_j\}}(\mathbb{R}^n)$ is sufficiently small. Namely, we assume that (4.8) holds true with

$$\mathcal{H}_0 \;=\; \int_{\mathbb{R}^n} |u_1(x)|^2\,dx \;+\; \Phi\left(\int_{\mathbb{R}^n} |\nabla u_0(x)|^2\,dx\right)\,.$$

In fact, let us consider again problem (1.2)

$$\begin{cases} u_{tt} - m\left(\displaystyle\int_{\mathbb{R}^n} |\nabla u(x,t)|^2\,dx\right)\Delta u \;=\; 0, \\ u(0,x) = u_0(x)\,, \qquad u_t(0,x) = u_1(x), \end{cases} \quad \text{in } \;\mathbb{R}^n\times[0,\infty)\;, \tag{1.2}$$

with $u_0(x)\in B^2_{\{q_j\}}(\mathbb{R}^n)$, $u_1(x)\in B^1_{\{q_j\}}(\mathbb{R}^n)$. Since we know that the Cauchy problem for the Kirchhoff equation (in the space of strong solutions, with $u_0(x)\in H^2(\mathbb{R}^n)$ and $u_1(x)\in H^1(\mathbb{R}^n)$, see [12], [17], [18]) is well posed, we can define

$$\mathcal{T} \overset{\text{def}}{=} \sup\Big\{T>0 \;|\; \exists!\;\; u(x,t)\;\text{ strong solution }\;\text{in }\;\mathbb{R}^n\times[0,T)\Big\}\;. \tag{5.1}$$

Besides, having $u_0(x)\in B^2_{\{\rho_j\}}(\mathbb{R}^n)$ and $u_1(x)\in B^1_{\{\rho_j\}}(\mathbb{R}^n)$, it easily follows that $v_0(\xi)=\hat{u}_0(\xi)$ and $v_1(\xi)=\hat{u}_1(\xi)$ satisfy the conditions of Definition 4.1, for a suitable subsequence $\{\tilde\rho_j\}_{j\geq 0}$. More precisely, we have

$$v_0(\xi)\in \tilde{B}^2_{\{\tilde\rho_j\}}(\mathbb{R}^n_\xi)\,, \qquad v_1(\xi)\in \tilde{B}^1_{\{\tilde\rho_j\}}(\mathbb{R}^n_\xi)\;.$$

Assume, by contradiction, that $\mathcal{T} < +\infty$. From Lemma 4.2, if (4.8) holds, it follows that the norms

$$\|u(\cdot,t)\|_{H^2}\,, \qquad \|u_t(\cdot,t)\|_{H^1}$$

are uniformly bounded in the interval $[0,\mathcal{T})$. Thus, by (4.3), the coefficient $m(s(t))$ is uniformly bounded in the C^2-norm and there exists $a(t) \in C^1([0,\mathcal{T}])$ such that $a(t) = m(s(t))$ in $[0,\mathcal{T})$. Hence, we may consider the linear problem

$$w_{tt} - a(t)\,\Delta w \,=\, 0 \quad \text{with} \;\; w(x,0) = u_0(x), \;\; w_t(x,0) = u_1(x)\;. \tag{5.2}$$

Clearly, this problem has a global strong solution $w(x,t)$ in $[0,\mathcal{T}]\times\mathbb{R}^n$ and by the uniqueness property we have $u(x,t) = w(x,t)$ in $[0,\mathcal{T})\times\mathbb{R}^n$. This means that there exist the limits

$$\begin{cases} \lim\limits_{t\to\mathcal{T}^-} u(x,t) \,=\, w(x,\mathcal{T}) & \text{in } H^2, \\ \lim\limits_{t\to\mathcal{T}^-} u_t(x,t) \,=\, w_t(x,\mathcal{T}) & \text{in } H^1\;. \end{cases} \tag{5.3}$$

Now, using again the local existence theorem for the Kirchhoff equation, with initial data for $t = \mathcal{T}$ given by $u(x,\mathcal{T}) = w(x,\mathcal{T})$ and $u_t(x,\mathcal{T}) = w_t(x,\mathcal{T})$, we can extend $u(x,t)$ (as strong solution of (1.2)) to a larger stripe $\mathbb{R}^n\times[0,\mathcal{T}_1)$ with $\mathcal{T}_1 > \mathcal{T}$. Clearly, this contradicts the definition of $\mathcal{T}$ and concludes the proof of Theorem 1 in the case of small initial data. ∎

Remark 5.1. Observe that for all $t \geq 0$ we have

$$u(t,\cdot) \in B^2_{\{\rho_j\}}(\mathbb{R}^n)\,, \qquad \partial_t u(t,\cdot) \in B^1_{\{\rho_j\}}(\mathbb{R}^n)\;. \;\square \tag{5.4}$$

6 – Proof of Theorem 1

We shall only see how to perform the energy estimates of Section 4, for $t \in [T-\varepsilon, T)$, in the case the condition (4.8) does not hold. We use the fact that the initial data $(u_0(x), u_1(x))$ belongs to $B^2_{\{\rho_j\}}(\mathbb{R}^n) \times B^1_{\{\rho_j\}}(\mathbb{R}^n)$, but we don't require that $u_0(x)$, $u_1(x)$ are small. First of all, setting

$$\lambda \,=\, 3\,\frac{M\,\mathcal{H}_0}{\sqrt{\delta_0}} + 1\;, \tag{6.1}$$

for every $\rho \geq 1$ we can write:

$$
\begin{aligned}
|s'(t)| \ &\leq \ 2\int_{|\xi|\leq\rho} |\xi|^2\,|v|\,|v_t|\,d\xi \ + \ 2\int_{\rho<|\xi|\leq\lambda\,\rho} |\xi|^2\,|v|\,|v_t|\,d\xi \\
&\quad + \ 2\int_{|\xi|<\lambda\,\rho} |\xi|^2\,|v|\,|v_t|\,d\xi \\
&\leq \ \rho\,\frac{\mathcal{H}_0}{\sqrt{\delta_0}} \ + \ 2\int_{\rho<|\xi|\leq\lambda\,\rho} |\xi|^2\,|v|\,|v_t|\,d\xi \ + \ 2\,\mathcal{E}_{\lambda\,\rho}(t)\ .
\end{aligned}
\tag{6.2}
$$

Now, we use the trivial inequality $E(\xi,t)' \leq M^{3/2}\,|s'(t)|\,E(\xi,t)$ (with $M > 0$ the constant introduced after (3.9)) to estimate the second term in right hand side of (6.2). Namely, by (2.11), for $T-\varepsilon \leq t < T$ and $\rho \geq \rho_\varepsilon$ we have

$$
E(\xi,t) \ \leq \ 4\,E(\xi,0)\,\exp\Big\{M^{3/2}\int_{T-\varepsilon}^{t} |s'(\tau)|\,d\tau\Big\}\ . \tag{6.3}
$$

To continue, let $\tilde{K}_1,\,\tilde{K}_2 \geq 1$ and let us suppose that $\rho \geq \max\{1,\,\rho_\varepsilon\}$. Then, following the argument developed in (4.5)–(4.19) of Section 4, as long as

$$
\mathcal{E}_{\lambda\,\rho}(t)^2 + \int_{|\xi|>\lambda\,\rho} E(\xi,t)\,d\xi \ \leq \ \tilde{K}_1\ , \tag{6.4a}
$$

$$
\int_{\rho<|\xi|\leq\lambda\,\rho} E(\xi,t)\,d\xi \ \leq \ \tilde{K}_2\ , \tag{6.4b}
$$

in the interval $[T-\varepsilon, T)$, we have

$$
\int_{T-\varepsilon}^{t} |s'(\tau)|\,d\tau \ \leq \ \varepsilon\bigg(\rho\,\frac{\mathcal{H}_0}{\sqrt{\delta_0}} + \frac{2\,\tilde{K}_2}{\rho} + 2\sqrt{\tilde{K}_1}\bigg)\ . \tag{6.5}
$$

By (6.3), this means that the condition (6.4b) holds true if

$$
\begin{aligned}
J(\rho,\varepsilon) \ &\stackrel{\text{def}}{=} \int_{\rho<|\xi|\leq\lambda\,\rho} E(\xi,0)\,\exp\bigg\{\varepsilon\,M^{3/2}\bigg[\rho\,\frac{\mathcal{H}_0}{\sqrt{\delta_0}} + \frac{2\,\tilde{K}_2}{\rho} + 2\sqrt{\tilde{K}_1}\bigg]\bigg\}\,d\xi \\
&\leq \ \frac{\tilde{K}_2}{4}\ .
\end{aligned}
\tag{6.6}
$$

To verify (6.6) we use the fact that $u_0(x) \in B^2_{\{\rho_j\}}(\mathbb{R}^n)$, $u_1(x) \in B^1_{\{\rho_j\}}(\mathbb{R}^n)$. More precisely, we know that there exist a sequence $\{\rho_j\}_{j\geq 0}$, $\rho_j \to +\infty$, and two constants $\eta > 0$, $C \geq 0$ such that

$$
\int_{|\xi|>\rho_j} E(\xi,0)\,\exp\Big\{\eta\,\rho_j^2/|\xi|\Big\}\,d\xi \ \leq \ C \tag{6.7}
$$

for $j \geq 0$ sufficiently large. Then, for $\varepsilon > 0$ small enough, namely

$$0 < \varepsilon \leq \frac{\eta\, \delta_0^{1/2}}{3\,\lambda\, M^{3/2}\, \mathcal{H}_0}\ ,$$

and $\rho = \rho_j \geq \max\{1,\, \rho_\varepsilon\}$ sufficiently large, we have

$$\varepsilon\, M^{3/2} \left[\rho_j\, \frac{\mathcal{H}_0}{\sqrt{\delta_0}} + \frac{2\,\tilde{K}_2}{\rho_j} + 2\sqrt{\tilde{K}_1} \right] \leq \frac{\eta\, \rho_j}{2\,\lambda} \leq \frac{\eta\, \rho_j^2}{2\,|\xi|} \tag{6.8}$$

for all $\xi \in \mathbb{R}^n_\xi$ such that $\rho_j \leq |\xi| \leq \lambda\, \rho_j$. By (6.7) this implies that

$$J(\rho_j, \varepsilon) \leq C\, \exp\Big\{ -\eta\, \rho_j / 2\, \lambda \Big\} \leq \frac{\tilde{K}_2}{4} \tag{6.9}$$

provided $j \geq 0$ is sufficiently large. Summarizing up, for $\varepsilon > 0$ small and $\rho = \rho_j$ with $j \geq 0$ large enough, the condition (6.4b) is verified and the estimate (6.5) holds true in the interval $[T - \varepsilon, T)$ as long as (6.4a) holds. Thus, taking in the following $\rho = \rho_j$, we can substitute (4.3a), (4.3b) with

$$|s'(t)| \leq \rho_j\, \frac{\mathcal{H}_0}{\sqrt{\delta_0}} + \frac{2\,\tilde{K}_2}{\rho_j} + 2\,\mathcal{E}_{\lambda\, \rho_j}(t)\ , \tag{6.10a}$$

$$|s''(t)| \leq 2\,\lambda^2\, \rho_j^2\, [1 + M/\delta_0]\, \mathcal{H}_0 + 4\sqrt{M} \int_{|\xi| > \lambda\, \rho_j} E(\xi, t)\, d\xi\ . \tag{6.10b}$$

Then, as long as (6.4a) holds in $[T - \varepsilon, T)$, we can estimate $E(\xi, t)$, for $|\xi| > \lambda\, \rho_j$, using exactly the same arguments of Section 4. In fact, for $\rho_j \geq \max\{1, \rho_\varepsilon\}$ large enough, instead of (4.7) we now have

$$\begin{aligned} \Big| m_2(s(t))\, s'(t)\, |\xi|^2\, \Re\{\bar{v}\, v_t\} \Big| &\leq \frac{M}{|\xi|} \left[\rho_j\, \frac{\mathcal{H}_0}{\sqrt{\delta_0}} + \frac{\tilde{K}_2}{\rho_j} + \sqrt{\tilde{K}_1} \right] E(\xi, t) \\ &\leq \frac{E(\xi, t)}{2}\ , \end{aligned} \tag{6.11}$$

because $|\xi| > \lambda\, \rho_j$ and λ satisfies (6.1). This completes the proof of Theorem 1. ∎

Appendix

Here we sketch the proof of the fact that $B^k_{\{\rho_j\}}(\mathbb{R})$ does not contain nontrivial compactly supported functions. To begin with, let us recall the following result (see [11], Theorem XII):

Theorem (Paley–Wiener). *Let $\phi(\xi) \in L^2(\mathbb{R})$ be a real non-negative function not equivalent to zero. A necessary and sufficient condition that there should exist a complex-valued function $f(x)$ vanishing for $x \geq x_o$ for some $x_o \in \mathbb{R}$, and such that $|\hat{f}(\xi)| = \phi(\xi)$, is that*

$$\int_{-\infty}^{+\infty} \frac{|\ln \phi(\xi)|}{1+\xi^2}\, d\xi \; < \; \infty \, . \; \blacksquare \tag{a.1}$$

We shall see that, if $f(x) \in B^k_{\{\rho_j\}}(\mathbb{R}) \setminus \{0\}$ has compact support, then the above integral cannot converge.

First of all, observe that $\hat{f}(\xi)$ is a bounded continuous function such that $|\hat{f}(\xi)| \to 0$ when $\xi \to \pm\infty$. Thus $|\hat{f}(\xi)| \leq 1$ for ξ large. Then, it is easy to see that, for $\rho = \rho_j$ large enough,

$$\begin{aligned} \int_{\rho}^{\rho^2} \frac{|\ln |\hat{f}(\xi)||}{1+|\xi|^2}\, d\xi \; &\geq \\ \geq \; \inf &\left\{ J(y) \stackrel{\text{def}}{=} -\int_{\rho}^{\rho^2} \frac{\ln y(t)}{1+t^2}\, dt \;\middle|\; 0 < y(t) \leq 1, \; \int_{\rho}^{\rho^2} t^{2k}\, y(t)\, e^{\eta\, \rho^2/t}\, dt \; = \; \mathcal{C} \right\} \end{aligned} \tag{a.2}$$

where $\eta > 0$ is the constant in Definition 1.1 and $\mathcal{C} > 0$ satisfy

$$\limsup_{j \to +\infty} \int_{|\xi| > \rho_j} |\xi|^{2k}\, |\hat{f}(\xi)|^2\, \exp\Big(\eta\, \rho_j^2/|\xi|\Big)\, d\xi \; < \; \mathcal{C} \, . \tag{a.3}$$

Let us evaluate the infimum of the functional $J(y)$ for $y(t) \in \mathcal{D} \stackrel{\text{def}}{=} C^0([\rho, \rho^2]) \cap \{0 < y(t) \leq 1\}$ and constrained by $\int_{\rho}^{\rho^2} t^{2k}\, y(t)\, e^{\eta\, \rho^2/t}\, dt = \mathcal{C}$.

Since $J(y)$ is strictly convex in $\mathcal{D}$ and the constrain is a linear functional, by elementary variational calculus (see [19]) we know that $J(y)$ has an absolute minimum if there exists $y_0(t) \in \mathcal{D}$ such that

$$\delta\, \tilde{J}(y_0) = 0 \, , \qquad \int_{\rho}^{\rho^2} t^{2k}\, y_0(t)\, e^{\eta\, \rho^2/t}\, dt \; = \; \mathcal{C} \; , \tag{a.4}$$

where $\tilde{J}(y) \stackrel{\text{def}}{=} J(y) + \lambda \int_{\rho}^{\rho^2} t^{2k}\, y(t)\, e^{\eta\, \rho^2/t}\, dt$ with $\lambda \in \mathbb{R}$. But the conditions in (a.4) imply that

$$y_0(t) = \frac{1}{\lambda}\, \frac{e^{-\eta\, \rho^2/t}}{t^{2k}\, (1+t^2)} \qquad \text{with} \;\; \lambda \approx \frac{\mathcal{C}}{\rho} \;\; \text{as } \rho \to +\infty \, . \tag{a.5}$$

Thus, at least for $k > -1/2$ (if $k \leq -1/2$ we must change a little our argument), $y_0(t) \in \mathcal{D}$ and satisfies $-\ln\, y_0(t) \geq \eta\, \rho^2/t + 2(k+1) \ln t - \ln \rho + O(1)$, provided

$t \in [\rho, \rho^2]$ and $\rho > 0$ is large enough. Finally, the last inequality gives

$$(a.6) \qquad -\int_\rho^{\rho^2} \frac{\ln y_0(t)}{1+t^2}\, dt \;\geq\; \frac{\eta}{4} \quad \text{as} \quad \rho \to +\infty \;,$$

and (a.6) implies that the integral (a.1) cannot converge if $f(x) \in B^k_{\{\rho_j\}}(\mathbb{R})\backslash\{0\}$.

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