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POSITIVE PERIODIC SOLUTIONS OF IMPULSIVE DELAY DIFFERENTIAL EQUATIONS WITH SIGN-CHANGING COEFFICIENT *

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Abstract: Consider the existence and nonexistence of positive periodic solutions of the non-autonomous delay differential equation

$$(*) \qquad x'(t) = -a(t)\,x(t) + \lambda\,h(t)\,f\big(x(t-\tau(t))\big)\,, \qquad t \neq t_k, \;\; k \in Z\;,$$

where a and h may change sign, with the impulses as follows

$$(**) \qquad x(t_k) = (1+b_k)\,x(t_k^-)\,, \qquad k \in Z\;.$$

It is shown that the system $(*)$–$(**)$ has positive periodic solutions under certain reasonable conditions and no positive periodic solutions under some other conditions. Some applications and examples are given to illustrate our Theorems.

1 – Introduction and definitions

Consider the existence and nonexistence of positive periodic solutions of the following system consisting of the non-autonomous delay differential equation

$$(1) \qquad x'(t) = -a(t)\,x(t) + \lambda\,h(t)\,f\Big(x(t-\tau(t))\Big)\,, \qquad t \neq t_k, \;\; k \in Z$$

and the impulses as follows

$$(2) \qquad x(t_k) = (1+b_k)\,x(t_k^-)\,, \qquad k \in Z\;,$$

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where Z is the set of all integers, $\{t_k\}$ an increasing sequence with $\lim_{k\to+\infty} t_k = +\infty$ and $\lim_{k\to-\infty} t_k = -\infty$, $\lambda > 0$ a parameter. $a\colon R \to R$, $h\colon R \to R$, $\tau\colon R \to R$ are continuous and T-periodic functions with $T > 0$, and $f\colon R^+ \to R^+$ is continuous, $R = (-\infty, +\infty)$, $R^+ = [0, +\infty)$.

Kuang and Smith in [18] studied the existence of periodic solutions of the delay differential equation

$$x'(t) = -f\Big(x(t), x(t-\tau(t))\Big) . \tag{3}$$

As is pointed in [18,19], there has not many studies for the existence of periodic solutions of delay differential equations. For the more references, we refer to papers [1–9,11] or the monograph [10,12]. Cheng and Zhang [6] studied the existence of positive periodic solutions of the delay differential equation

$$x'(t) = -a(t)\, x(t) + \lambda\, h(t)\, g\Big(x(t-\tau(t))\Big) . \tag{4}$$

They proved that (4) has positive periodic solutions for some $\lambda > 0$ under the following assumptions

($\mathbf{H}_1$) a, h and g are non-negative continuous functions and a, h are T-periodic with $\int_0^T a(s)\, ds > 0$.

($\mathbf{H}_2$) $\lim_{x\to 0} g(x)/x = l \in [0, +\infty]$ and $\lim_{x\to+\infty} g(x)/x = L \in [0, +\infty]$.

However, the existence and nonexistence problems of positive T-periodic solutions of (4) have not been studied if a and h change sign.

On the other hand, the differential equations with impulses are a basic tool to describe the evolution processes which are subjected to abrupt changes in their state. Many biological, physical and engineering applications exhibit impulsive effects. We refer to [10,11,12] and the references cited therein. The simple case of impulsive differential equations is that in which impulsive effects occur at periodic fixed moment of the time and the impulsive functions that regulate the process are linear. For example, we see Eq. (1) with impulsive conditions (2) which was posed in [10], used and studied in [7,8,9]. (1) is the one dimension version of many interesting ecological models. In particular, (1) can be interpreted as the Malthus Population Model subjected to perturbation with periodic delay. In [7], the oscillatory property of solutions of (1)–(2) was studied, however, the existence of positive solutions is not investigated.

Very recently, Nieto in [17] used a complicated method to study the existence of non-positive solutions of the following periodic boundary value problem

$$\begin{cases} u'(t) + F(t, u(t)) = 0, & \text{a.e. } t \in J'\,, \\ u(t_j^+) = u(t_j^-) + I_j(u(t_j^-)), & j = 1, 2, ..., p\,, \\ u(0) = u(T)\ . \end{cases}$$

We will present a different way. To the best of our knowledge, the existence and nonexistence of positive periodic solutions of impulsive delay differential equation (1)–(2) have not been studied.

In this paper, we give the assumptions as follows

($\mathbf{A}_1$) $f\colon R^+ \to R^+$ is continuous and $f(0) > 0$.

($\mathbf{A}_2$) $h : R \to R$ is continuous T-periodic function and there is $k > 1$ such that

$$\int_t^{t+T} G(t,s)\, h^+(s)\, ds \ \ge\ k \int_t^{t+T} G(t,s)\, h^-(s)\, ds \qquad \text{for all } \ t\ ,$$

where

$$G(t,s) \ =\ \frac{\exp\Bigl(\int_t^s a(u)\, du\Bigr) \displaystyle\prod_{s<t_k\le t+T} (1+b_k)}{\exp\Bigl(\int_0^T a(u)\, du\Bigr) - \displaystyle\prod_{0<t_k\le T} (1+b_k)}\ ,$$

$h^+(t) = \max\{h(t),\ 0\}$ and $h^-(t) = \max\{-h(t),\ 0\}$.

($\mathbf{A}_3$) $a\colon R \to R$ is a continuous T-periodic function and satisfies $\exp(\int_0^T a(u)du) > \prod_{0\le t_k<T}(1+b_k)$ and $\prod_{t\le t_k<t+T}(1+b_k) =$ constant and $b_k > -1$ for all $k \in Z$.

Our aim in this paper is to establish the existence and nonexistence criteria of positive T-periodic solutions of (1) with impulses (2) (system (1)–(2) for short) and to apply the main theorems to the following equations

$$\text{(5)} \qquad \begin{cases} N'(t) = -\mu(t)\, N(t) + \lambda\, p(t)\, e^{-rN(t-\tau(t))}, & t \ne t_k\,, \\ N(t_k) = (1+b_k)\, N(t_k^-), & k \in Z\ , \end{cases}$$

and

$$\text{(6)} \qquad \begin{cases} N'(t) = -\mu(t)\, N(t) + \lambda\, p(t)\, \dfrac{1}{1+N^n(t-\tau(t))}, & t \ne t_k\,, \\ N(t_k) = (1+b_k)\, N(t_k^-), & k \in Z\ , \end{cases}$$

where $\mu(t)$, $p(t)$ and $\tau(t)$ are positive continuous functions, $n > 0$ is real number and λ a parameter. Eq.(5) was called hematopoiesis model (Weng and Liang [13]), where $N(t)$ is the number of red blood cells at time t. This is a generalized model of red blood cell system introduced by Wazewska–Czyzewska and Lasota[14]

$$N'(t) \,=\, \mu\, N(t) + p\, e^{-rN(t-\tau)} \ ,$$

where μ, r, p and τ are positive constants $\mu \in (0,1)$. Chow in [15] studied the existence of periodic solution of a class of differential equations similar to (5). Certain equations similar to Eq. (6) was posed and studied in [16]. However, the existence of positive periodic solution of (5) or (6) and has not been studied for the case where the coefficients p and μ change sign.

This paper is organized as follows. In section 2, we give our main results. The applications to (5) and (6) and two examples will be given to illustrate our theorems in section 3.

2 – Main results and proofs

Suppose that $y\colon R \to [0,+\infty)$ is continuous function and consider the solution of the equation

$$x'(t) = -a(t)\, x(t) + y(t)\,, \qquad t \neq t_k, \;\; k \in Z \ , \tag{7}$$

with the impulsive effects (2). We find that

$$\Big(x(t)\, e^{\int_0^t a(s)ds}\Big)' \,=\, e^{\int_0^t a(s)ds} y(t)\,, \qquad t \neq t_k, \;\; k \in Z \ .$$

Suppose that $t \le t_k < t_{k+1} < \cdots < t_{k+l} \le t+T$. After integration from t to $t+T$, we get

$$\begin{cases} x(t_k^-)\, e^{\int_0^{t_k} a(u)du} - x(t)\, e^{\int_0^t a(u)du} = \displaystyle\int_t^{t_k} y(s)\, e^{\int_0^s a(u)du}\, ds, & t < t_k\,, \\ x(t_k^-) = x(t_k), & t = t_k\ , \end{cases}$$

$$x(t_{k+1}^-)\, e^{\int_0^{t_{k+1}} a(u)du} - x(t_k)\, e^{\int_0^{t_k} a(u)du} \,=\, \int_{t_k}^{t_{k+1}} y(s)\, e^{\int_0^s a(u)du}\, ds \ ,$$

$$\cdots\cdots$$

$$x(t_{k+l}^-)\, e^{\int_0^{t_{k+l}} a(u)du} - x(t_{k+l-1})\, e^{\int_0^{t_{k+l-1}} a(u)du} \,=\, \int_{t_{k+l-1}}^{t_{k+l}} y(s)\, e^{\int_0^s a(u)du}\, ds \ ,$$

$$\begin{cases} x(t+T)\, e^{\int_0^{t+T} a(u)du} - x(t_{k+l})\, e^{\int_0^{t_{k+l}} a(u)du} = \displaystyle\int_{t_{k+l}}^{t+T} y(s)\, e^{\int_0^s a(u)du}\, ds, & t_{k+l} < t+T\,, \\ x(t_{k+l}) = (1+b_k)\, x(t_{k+l}^-), & t_{k+l} = t+T\,. \end{cases}$$

Combining (2), it follows that

$$x(t) = \int_t^{t+T} G(t,s)\, y(s)\, ds\,, \qquad t \in R\,, \tag{8}$$

where

$$G(t,s) \;=\; \frac{\exp\Bigl(\displaystyle\int_t^s a(u)\,du\Bigr) \displaystyle\prod_{s<t_k\le t+T} (1+b_k)}{\exp\Bigl(\displaystyle\int_0^T a(u)\,du\Bigr) - \displaystyle\prod_{0<t_k\le T} (1+b_k)}\;. \tag{9}$$

Let $I^t = [t,t+T]$, $b_k^+ = \max\{0,b_k\}$, $b_k^- = \max\{0,-b_k\}$,

$$I_1^t = \Bigl\{t \in I^t\colon\; a(t) \ge 0\Bigr\}\,, \qquad I_2^t = \Bigl\{t \in I^t\colon\; a(t) < 0\Bigr\}$$

and

$$a^+(t) = \max\{0,a(t)\}\,, \qquad a^-(t) = \min\{0,a(t)\}\;.$$

Since

$$\begin{aligned}
G(t,s) \;&=\; \frac{\exp\Bigl(\displaystyle\int_t^s a(u)\,du\Bigr) \displaystyle\prod_{s<t_k\le t+T} (1+b_k)}{\exp\Bigl(\displaystyle\int_0^T a(u)\,du\Bigr) - \displaystyle\prod_{0<t_k\le T} (1+b_k)} \\
&\le\; \frac{\exp\Bigl(\Bigl[\displaystyle\int_{[t,s]\,\cap\, I_1^t} a(\tau)\,d\tau \;+ \int_{[t,s]\,\cap\, I_2^t} a(\tau)\,d\tau\Bigr]\Bigr) \displaystyle\prod_{t<t_k\le t+T} (1+b_k^+)}{\exp\Bigl(\displaystyle\int_0^T a(\tau)\,d\tau\Bigr) - \displaystyle\prod_{0<t_k\le T} (1+b_k)} \\
&\le\; \frac{\exp\Bigl(\displaystyle\int_{[t,t+T]\,\cap\, I_1^t} a(\tau)\,d\tau\Bigr) \displaystyle\prod_{0<t_k\le T} (1+b_k^+)}{\exp\Bigl(\displaystyle\int_0^T a(\tau)\,d\tau\Bigr) - \displaystyle\prod_{0<t_k\le T} (1+b_k)} \\
&\le\; \frac{\exp\Bigl(\displaystyle\int_0^T a^+(\tau)\,d\tau\Bigr) \displaystyle\prod_{0<t_k\le T} (1+b_k^+)}{\exp\Bigl(\displaystyle\int_0^T a(\tau)\,d\tau\Bigr) - \displaystyle\prod_{0<t_k\le T} (1+b_k)} \;=\; M\,,
\end{aligned}$$

and

$$G(t,s) = \frac{\exp\left(\int_t^s a(u)\,du\right) \prod_{s<t_k\le t+T}(1+b_k)}{\exp\left(\int_0^T a(u)\,du\right) - \prod_{0<t_k\le T}(1+b_k)}$$

$$\ge \frac{\exp\left(\int_{[t,t+T]\cap I_2^t} a(\tau)\,d\tau\right) \prod_{0<t_k\le T}(1-b_k^-)}{\exp\left(\int_0^T a(\tau)\,d\tau\right) - \prod_{0<t_k\le T}(1+b_k)}$$

$$\ge \frac{\exp\left(\int_0^T a^-(\tau)\,d\tau\right) \prod_{0<t_k\le T}(1-b_k^-)}{\exp\left(\int_0^T a(\tau)\,d\tau\right) - \prod_{0<t_k\le T}(1+b_k)} = N\ ,$$

we have the following Lemma.

Lemma 1. *Suppose that $y(t)$ is a non-negative T-periodic solution and $x(t)$ is a solution of system* (2) *and* (7)*. Then $x(t)\ge\sigma\|x\|$, for $t\in R$, where $\sigma = N/M$ and $\|x\| = \sup\{|x(t)|\colon t\in[0,T]\}$.*

Proof: It is easy to see that

$$x(t) = \int_t^{t+T} G(t,s)\,y(s)\,ds \ \ge\ N\int_t^{t+T} y(s)\,ds = N\int_0^T y(s)\,ds\ ,$$

and

$$M\int_0^T y(s)\,ds = M\int_t^{t+T} y(s)\,ds \ \ge\ \int_t^{t+T} G(t,s)\,y(s)\,ds\ .$$

The proof is obvious and is omitted. ∎

Now, let X be the set of all real piecewise continuous T-periodic functions with finite discontinuities of the first type on [0,T] and endowed with the usual linear structure as well as the norm $\|x\| = \sup_{t\in[0,T]}|x(t)|$. i.e.

$$X = \Big\{x\colon R\to R \text{ is } T\text{-periodic and continuous in } (t_k,t_{k+1}) \text{ with } x(t_k^-) = x(t_k),\ x(t_{k+1}^+) \text{ exist for all } k\Big\}\ .$$

Then X is a Banach space. Let

$$K = \Big\{x\in X\colon\ x(t)\ge\sigma\|x\|,\ \ t\in[0,T]\Big\}\ .$$

Then K is a cone of space X. The main results of this paper are as follows.

Theorem 1. *Suppose that* (A_1)–(A_3) *hold. Then there is a positive number* λ^* *such that the system* (1)–(2) *has at least one positive* T*-periodic solution for* $\lambda \in (0, \lambda^*)$.

Theorem 2. *Suppose that* (A_1)–(A_3) *hold and for any* $a > 0$*, there is* $\mu > 0$ *such that*

$$k\,\mu \;\geq\; f(x) - \sigma\,a \;\geq\; \mu \qquad \text{for} \;\; x \in [\sigma a, a] \;.$$

Then (1)–(2) *has no positive* T*-periodic solution for*

$$\lambda \;>\; \left[\sigma\Big(1-\frac{1}{k}\Big)\,\frac{\exp\Big(\int_0^T a^-(u)\,du\Big)\prod\limits_{0<t_k\leq T}(1-b_k)}{\exp\Big(\int_0^T a(u)\,du\Big)-\prod\limits_{0<t_k\leq T}(1+b_k)}\int_0^T h^+(s)\,ds\right]^{-1} .$$

In order prove Theorems 1 and 2, we need the following Lemma.

Lemma 2. *Suppose* (A_1)–(A_3) *hold. Then for every* $0 < \delta < 1$*, there exists a positive number* $\overline{\lambda}$ *such that, for* $\lambda \in (0, \overline{\lambda})$*, the equation*

$$(10)\qquad \begin{cases} x'(t) = -a(t)\,x(t) + \lambda\,h^+(t)\,f\Big(x(t-\tau(t))\Big), & t \neq t_k\,, \\ x(t_k) = (1+b_k)\,x(t_k^-), & k \in Z\,, \end{cases}$$

has a positive T*-periodic solution* $\overline{x}_\lambda$ *with* $\|\overline{x}_\lambda\| \to 0$ *as* $\lambda \to 0$ *and*

$$(11)\qquad \overline{x}_\lambda \;\geq\; \lambda\,\delta\,f(0)\,\|p(t)\| \;,$$

where

$$p(t) \;=\; \int_t^{t+T} G(t,s)\,h^+(s)\,ds \;.$$

Proof: We know that $p(t) \geq 0$ for $t \in R$ and (10) is equivalent to the integral equation

$$(12)\qquad x(t) \;=\; \lambda \int_t^{t+T} G(t,s)\,h^+(s)\,f\Big(x(s-\tau(s))\Big)\,ds \;:=\; T\,x(t) \;,$$

where $x \in X$. It is easy to prove that T is completely continuous, $TK \subset K$ and the fixed point of T is solution of (1)–(2). We shall apply the Leray–Schauder degree theory to prove T has at least one fixed point for small λ.

Let $\epsilon > 0$ be such that

$$f(t) \geq \delta f(0) \quad \text{for } \ 0 \leq t \leq \epsilon \ . \tag{13}$$

Suppose that

$$0 < \lambda < \frac{\epsilon}{2\,\|p\|\,\overline{f}(\epsilon)} := \overline{\lambda}\ ,$$

where $\overline{f}(t) = \max_{0 \leq s \leq t} f(s)$. Since

$$\lim_{t \to 0^+} \frac{\overline{f}(t)}{t} = +\infty \quad \text{ and } \quad \overline{f}(\epsilon)/\epsilon < 1/(2\,\|p\|\,\lambda)\ ,$$

there is $r_\lambda \in (0, \epsilon)$ such that

$$\frac{\overline{f}(r_\lambda)}{r_\lambda} = \frac{1}{2\,\lambda\,\|p\|}\ ,$$

which implies $r_\lambda \to 0$ as $\lambda \to 0$.

Now, consider the homotopy equation

$$u = \theta\,Tu\,, \qquad \theta \in (0,1)\ .$$

Let $u \in X$ and $\theta \in (0,1)$ be such that $u = \theta\,Tu$. We claim that $\|u\| \neq r_\lambda$. In fact, if

$$u(t) \;=\; \theta\lambda \int_t^{t+T} G(t,s)\,h^+(s)\,f\Big(u(s - \tau(s))\Big)\,ds\ ,$$

set

$$w(t) \;=\; \theta\lambda \int_t^{t+T} G(t,s)\,h^+(s)\,\overline{f}(\|u\|)\,ds \;\leq\; \theta\,\lambda\,\overline{f}(\|u\|)\,p(t)\ ,$$

then by $f(u) \leq \overline{f}(\|u\|)$, we know that $u(t) \leq w(t)$ for all $t \in R$. Moreover, we have

$$\|u\| \leq \lambda\,\|p\|\,\overline{f}(\|u\|)\ ,$$

i.e.

$$\frac{\overline{f}(\|u\|)}{\|u\|} \geq \frac{1}{\lambda\,\|p\|}\ ,$$

which implies that $\|u\| \neq r_\lambda$. Thus by Leray–Schauder degree theory, T has a fixed point $\overline{x}_\lambda$ with

$$\|\overline{x}_\lambda\| \leq r_\lambda < \epsilon\ .$$

Moreover, combining (12) and (13), we get

$$\overline{x}_\lambda \geq \lambda\,\delta f(0)\,p(t)\,, \qquad t \in R\ . \tag{14}$$

This completes the proof. ∎

Proof of Theorem 1: Let

$$q(t) = \int_t^{t+T} G(t,s)\, h^-(s)\, ds \;, \tag{15}$$

then $q(t) \geq 0$. Since $p(t)/q(t) \geq k > 1$, we can choose $d \in (0,1)$ such that $k\,d > 1$. There is $c > 0$ such that $|f(y)| \leq k\,d\,f(0)$ for $y \in [0,c]$, then

$$q(t)\,|f(y)| \leq d\,p(t)\,f(0)\,, \qquad t \in R, \;\; y \in [0,c] \;.$$

Fix $\delta \in (d,1)$ and let $\lambda^* > 0$ be such that

$$\|\overline{x}_\lambda\| + \lambda\,\delta f(0)\,\|p\| \leq c\,, \qquad \lambda \in (0,\lambda^*) \;, \tag{16}$$

where $\overline{x}_\lambda$ is given in Lemma 1, we have

$$|f(x) - f(y)| \;\leq\; f(0)\,\frac{\delta - d}{2} \tag{17}$$

for $x,y \in [-c,c]$ with $|x-y| \leq \lambda^* \delta f(0)\|p\|$.

Let $\lambda \in (0,\lambda^*)$, we look for a solution x_λ of the form $\overline{x}_\lambda + y_\lambda$ such that y_λ solves the following equation

$$\begin{cases} y'(t) = -a(t)\,y(t) + \lambda\,h^+(t)\Big[f\Big(\overline{x}_\lambda(t-\tau(t)) + y(t-\tau(t))\Big) - f\Big(\overline{x}_\lambda(t-\tau(t))\Big)\Big] \\ \qquad\qquad - \lambda\,h^-(t)\,f\Big(\overline{x}_\lambda(t-\tau(t)) + y(t-\tau(t))\Big)\,, \qquad t \neq t_k \;, \\ \\ y(t_k) = (1+b_k)\,y(t_k^-)\,, \qquad k \in Z \;. \end{cases} \tag{18}$$

For each $w \in X$, let $y = Tw$ be the T-periodic solution of (18). Then T is completely continuous. Let $y \in X$ and $\theta \in (0,1)$ be such that $y = \theta\,Ty$, then we have

$$\begin{cases} y'(t) = -a(t)\,y(t) + \theta\,\lambda\,h^+(t)\Big[f\Big(\overline{x}_\lambda(t-\tau(t)) + y(t-\tau(t))\Big) - f\Big(\overline{x}_\lambda(t-\tau(t))\Big)\Big] \\ \qquad\qquad - \theta\,\lambda\,h^-(t)\,f\Big(\overline{x}_\lambda(t-\tau(t)) + y(t-\tau(t))\Big)\,, \qquad t \neq t_k \;, \\ \\ y(t_k) = (1+b_k)\,y(t_k^-)\,, \qquad k \in Z \;. \end{cases}$$

We claim that $\|y\| \neq \lambda\,\delta f(0)\|p\|$. Suppose to the contrary that $\|y\| = \lambda\,\delta f(0)\|p\|$, then by (16) and (17), we get

$$\|\overline{x}_\lambda + y\| \;\leq\; \|\overline{x}_\lambda\| + \|y\| \;\leq\; c \tag{19}$$

and

$$\left|f(\overline{x}_\lambda + y) - f(\overline{x}_\lambda)\right| \;\leq\; f(0)\,\frac{\delta-d}{2}\;. \tag{20}$$

By using (12) and $q(t)|f(y)| \leq d\,p(t)f(0)$, there holds

$$\begin{aligned}
|y(t)| \;&=\; \lambda\,\left|\int_t^{t+T} G(t,s)\,h^+(s)\Big[f\Big(\overline{x}_\lambda(s-\tau(s)) + y(s-\tau(s))\Big) - f\Big(\overline{x}_\lambda(s-\tau(s))\Big)\Big]\,ds\right.\\
&\qquad\left. +\,\lambda\int_t^{t+T} G(t,s)\,h^-(s)\,f\Big(\overline{x}_\lambda(s-\tau(s)) + y(s-\tau(s))\Big)\,ds\right|\\
&\leq\; \lambda\int_t^{t+T} G(t,s)\,h^+(s)\,f(0)\,\frac{\delta-d}{2}\;ds \;+\; \lambda\int_t^{t+T} G(t,s)\,h^-(s)\,\frac{p(t)}{q(t)}\;d\,f(0)\,ds\\
&\leq\; \lambda\,\frac{\delta-d}{2}\,p(t) + \lambda\,d\,f(0)\,p(t)\\
&=\; \lambda\,\frac{\delta+d}{2}\,f(0)\,p(t)\;.
\end{aligned}$$

In particular,

$$\|y\| \;\leq\; \lambda\,\frac{\delta+d}{2}\,f(0)\,\|p\| \;<\; \lambda\,\delta f(0)\,\|p\|\;, \tag{21}$$

this is a contradiction and the claim is true. Thus by Leray–Schauder degree theory, T has a fixed point y_λ with

$$\|y_\lambda\| \leq \lambda\,\delta f(0)\,\|p\|\;.$$

Using Lemma 1 and (21), we obtain

$$\begin{aligned}
x_\lambda(t) \;&\geq\; \overline{x}_\lambda - \|y_\lambda\|\\
&\geq\; \lambda\,\delta f(0)\,p(t) - \lambda\,\frac{\delta+d}{2}\,f(0)\,p(t)\\
&=\; \lambda\,\frac{\delta-d}{2}\,f(0)\,p(t) \;>\; 0\;,
\end{aligned}$$

i.e., x_λ is a positive T-periodic solution. The proof is complete. ∎

Proof of Theorem 2: Suppose that (1)–(2) has a positive T-periodic solution $x(t)$. Assume $\|x\|=a$. From the condition in Theorem 2, there is $\mu>0$ such that

$$k\,\mu \;\geq\; f(x) - \sigma\,a \;\geq\; \mu \quad \text{ for }\; x\in[\sigma a, a]\;.$$

So

$$\int_t^{t+T} G(t,s)\,h(s)\Big[f\Big(x(s-\tau(s))\Big) - \sigma a\Big]\,ds \;=\;$$

$$
\begin{aligned}
&= \int_t^{t+T} G(t,s)\, h^+(s) \Big[f\Big(x(s-\tau(s))\Big) - \sigma a \Big]\, ds \\
&\quad - \int_t^{t+T} G(t,s)\, h^-(s) \Big[f\Big(x(s-\tau(s))\Big) - \sigma a \Big]\, ds \\
&\ge \int_t^{t+T} G(t,s)\, h^+(s)\, \mu\, ds \,-\, \int_t^{t+T} G(t,s)\, h^-(s)\, k\, \mu\, ds \\
&= \mu \left[\int_t^{t+T} G(t,s)\, h^+(s)\, ds \,-\, k \int_t^{t+T} G(t,s)\, h^-(s)\, ds \right] \\
&\ge 0.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
x(t) &= \lambda \int_t^{t+T} G(t,s)\, h(s)\, f\Big(x(s-\tau(s))\Big)\, ds \\
&\ge \lambda \int_t^{t+T} G(t,s)\, h(s)\, \sigma a\, ds \\
&= \lambda\, \sigma a \left(\int_0^T G(t,s)\, h^+(s)\, ds \,-\, \int_0^T G(t,s)\, h^-(s)\, ds \right) \\
&\ge \lambda\, \sigma a \left(1 - \frac{1}{k}\right) \int_0^T G(t,s)\, h^+(s)\, ds \\
&\ge \lambda\, \sigma a \left(1 - \frac{1}{k}\right) \left(\frac{\exp\Big(\int_0^T a^-(\tau)\, d\tau\Big) \prod_{0<t_k\le T} (1-b_k^-)}{\exp\Big(\int_0^T a(\tau)\, d\tau\Big) - \prod_{0<t_k\le T} (1+b_k)} \int_0^T h^+(s)\, ds \right) \\
&> a \,=\, \|x\| \,,
\end{aligned}
$$

which is a contradiction. Thus (1)–(2) has no positive T-periodic solution. ∎

3 – Applications and examples

In this section, we apply Theorem 1 to system (5) and (6).

Corollary 3. *Suppose that μ, p, τ, are piecewise continuous T-periodic functions, $p(t) \ge 0$ with $\int_0^T p(s) > 0$ and $\exp(\int_0^T \mu(u)\, du) > \prod_{0\le t_k<T}(1+b_k)$, $b_k > -1$ for all k, $\prod_{t<t_k\le t+T}(1+b_k) =$ constant. Then system (5) has at least one positive T-periodic solution for every $\lambda > 0$.* ∎

Corollary 4. *Under the assumptions in Corollary 3, the system (6) has at least one positive T-periodic solution for every $\lambda > 0$.* ∎

Let $f(t,x) = e^{-rx}$ or $f(t,x) = 1/(1+x^n)$, then corollaries 3 and 4 are directly acquired from Theorem 1.

Example 1. Consider the equation

$$x'(t) \;=\; -x(t) + \lambda\, h(t)\, f\Big(x(t-\sin t)\Big)\;, \tag{22}$$

where

$$h(t) = \begin{cases} 80 - \dfrac{81}{\pi}(t-2\,k\pi), & 2\,k\pi \le t < 2\,k\pi + \pi\;, \\[2ex] 80 + \dfrac{81}{\pi}(t-2(k+1)\pi), & 2\,k\pi + \pi \le t < 2(k+1)\pi\;, \end{cases}$$

with $k \in Z$, $\tau(t) = \sin t$, $T = 2\pi$, $a(t) \equiv 1$ and $f\colon R \to R$ is continuous with $f(0) > 0$. □

It is easy to know that

$$\begin{aligned}
\inf_{t\in R} \frac{\displaystyle\int_t^{t+T} G(t,s) h^+(s)\,ds}{\displaystyle\int_t^{t+T} G(t,s)\, h^-(s)\,ds} &= \inf_{t\in R} \frac{\displaystyle\int_t^{t+T} e^{s-t}\, h^+(s)\,ds}{\displaystyle\int_t^{t+T} e^{s-t}\, h^-(s)\,ds} \\
&\ge \inf_{t\in R} \frac{\displaystyle\int_t^{t+T} e^{t}\, h^+(s)\,ds}{\displaystyle\int_t^{t+T} e^{t+2\pi}\, h^-(s)\,ds} \\
&= e^{-2\pi} \inf_{t\in R} \frac{\displaystyle\int_t^{t+T} h^+(s)\,ds}{\displaystyle\int_t^{t+T} h^-(s)\,ds} \\
&= 640\, e^{-2\pi} \;>\; 1\;.
\end{aligned}$$

By Theorem 1, there is $\lambda^* > 0$ such that (22) has at least one positive T-periodic solution.

Example 2. Consider the following problem

$$\begin{cases} x'(t) + 2\,x(t) = \lambda\, h(t)\,\Big(2\,x(t-\sin t) + \alpha\Big)\,, & t \ne t_k\,, \\ x(t_k) = 6\,x(t_k^-)\,, \quad k \in Z\;, \end{cases} \tag{23}$$

where $\alpha > 0$, $t_k = 2\,k\pi$ for $k \in Z$,

$$h(t) = \begin{cases} e^{4\pi} - x\,\dfrac{72\,e^{8\pi}+1}{72\,\pi\,e^{4\pi}}, & 0 \le t \le \dfrac{72\,\pi\,e^{8\pi}}{72\,e^{8\pi}+1}\,, \\ -\dfrac{e^{4\pi}(72\,e^{8\pi}+1)}{\pi}\left(x - \dfrac{72\,\pi\,e^{8\pi}}{72\,e^{8\pi}+1}\right), & \dfrac{72\,\pi\,e^{8\pi}}{72\,e^{8\pi}+1} \le t \le \pi\,, \\ h(2\,\pi - t), & \pi + \pi \le t \le 2\,\pi\ , \end{cases}$$

and $h(t+2\pi) = h(t)$ for every t. By Theorem 2, (23) has no positive 2π-periodic solution if

$$\lambda > \frac{(72\,e^{8\pi}+1)\,(e^{4\pi}-6)}{36\,\pi\,e^{8\pi}\,(12\,e^{4\pi}-1)}\ . \ \square$$

In fact, it is easy to check that $\prod_{t\le t_k \le t+2\pi}(1+b_k) \equiv 6$ and (A_3) holds. Let $f(x) = 2\,x + \alpha$. Then f is continuous and $f(0) = \alpha > 0$, so (A_1) holds.

$$G(t,s) \ = \ \frac{e^{2(s-t)} \displaystyle\prod_{s<t_k\le t+2\pi} (1+b_k)}{e^{4\pi}-6}\ .$$

$$\int_t^{t+T} G(t,s)\,h^+(s)\,ds \ \ge \ \int_t^{t+T} \frac{1}{e^{4\pi}-6}\,h^+(s)\,ds \ = \ \frac{1}{e^{4\pi}-6}\int_0^T h^+(s)\,ds$$
$$= \ \frac{72\,\pi\,e^{8\pi}}{2(e^{4\pi}-6)\,(72\,e^{8\pi}+1)}\ .$$

$$\int_t^{t+T} G(t,s)\,h^-(s)\,ds \ \le \ \frac{6\,e^{4\pi}}{e^{4\pi}-6}\int_0^T h^-(s)\,ds \ \le \ \frac{3\,\pi\,e^{4\pi}}{(e^{4\pi}-6)\,(72\,e^{8\pi}+1)}\ .$$

$$k \ge 12\,e^{4\pi} > 1\,, \qquad \sigma = \frac{1}{6\,e^{4\pi}}\ .$$

On the other hand, for any $a > 0$, choose $\mu = \sigma a + \alpha$, we have $f(x) - \sigma a = 2\,x + \alpha - \sigma a \in [\sigma a + \alpha, 2a + \alpha - \sigma a]$ for $x \in [\sigma a, a]$.

Since $k\mu = k\,\sigma a + \alpha k \ge 2a + \alpha - \sigma a$, we get

$$\mu \ \le \ f(x) - \sigma a \ \le \ k\mu \quad \text{ for } \ x \in [\sigma a, a]\ .$$

By Theorem 2, (23) has no positive 2π-periodic solution if

$$\lambda \ > \ \frac{(72\,e^{8\pi}+1)\,(e^{4\pi}-6)}{36\,\pi\,e^{8\pi}\,(12\,e^{4\pi}-1)}\ .$$

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