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A NEW PROOF OF THE EXISTENCE OF HIERARCHIES OF POISSON–NIJENHUIS STRUCTURES *

J. Monterde

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Abstract: Given a Poisson–Nijenhuis manifold, a two-parameter family of Poisson–Nijenhuis structures can be defined. As a consequence we obtain a new and noninductive proof of the existence of hierarchies of Poisson–Nijenhuis structures.

1 – Introduction

One of the main characteristics of the theory of Poisson–Nijenhuis structures is the possibility of constructing from a Poisson–Nijenhuis structure, a hierarchy of new ones. The different proofs of the existence of such a hierarchy that can be found in the literature all used proof by induction ([3], [9]).

The aim of this note is to obtain, from a single Poisson–Nijenhuis structure, (P,N), a two-parameter family of Poisson–Nijenhuis structures (P_t, N_s), $t,s\in\mathbb{R}$. Such a family provides a noninductive way of proving the existence of the well known hierarchy of associated Poisson–Nijenhuis structures. In fact, we can say that the two-parameter family is a kind of integration of the hierarchy: all the structures of the hierarchy can be obtained as successive partial derivatives evaluated at $(0,0)$ of the two-parameter structures (P_t, N_s).

In Section 2.2, we prove a consequence of this approach related to generating operators of Gerstenhaber brackets. Let $(A, [\![\ ,\ ]\!], \wedge)$ be a Gerstenhaber alge-

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bra. If δ is a generating operator of a Gerstenhaber bracket and N is a degree 0 Nijenhuis endomorphism of the associative algebra $(A, \wedge)$, then $[\delta, i_N]$ is a generating operator of the deformation by N of the Gerstenhaber bracket, where i_N denotes the extension, as a degree 0 derivation, of N to the whole algebra. This is Theorem 1. In Corollary 5 we apply the results of Section 2 to the different Gerstenhaber brackets which can be associated to a Poisson–Nijenhuis structure.

2 – Nijenhuis tensors and their integral flow

Let $(E, [\ ,\ ])$ be a graded Lie algebra. In the applications that we shall give here, $(E, [\ ,\ ])$ will be the vector space of smooth vector fields over a manifold M, $\mathfrak{X}(M)$, together with the usual Lie bracket of vector fields, or the vector space of differential 1-forms, $\Omega^1(M)$, with the Lie bracket of 1-forms associated to a Poisson structure, or that of differential forms, $\Omega(M)$, together with the Koszul–Schouten bracket of differential forms on a Poisson manifold.

We can define the Frölicher–Nijenhuis bracket, $[\ ,\ ]_{FN}$ of two degree 0 endomorphisms of E, N, L, as

$$\begin{aligned}[N,L]_{FN}(X,Y) := \ & [NX,LY]+[LX,NY]-N\Big([LX,Y]+[X,LY]\Big)\\ & -L\Big([NX,Y]+[X,NY]\Big)+(NL+LN)[X,Y]\ ,\end{aligned} \tag{2.1}$$

for all $X, Y \in E$.

The Frölicher–Nijenhuis bracket of N with itself is called the Nijenhuis torsion of N, and N is said to be Nijenhuis if its Nijenhuis torsion vanishes.

Definition 1. Let $(E, [\ ,\ ]_\nu)$ be a graded Lie algebra. Given a degree 0 endomorphism, N, we can define the deformation of the Lie bracket, $[\ ,\ ]_\nu$, by means of N as

$$[X,Y]_{N.\nu} = [NX,Y]_\nu + [X,NY]_\nu - N[X,Y]_\nu\ ,$$

for all $X, Y \in E$. □

If the Nijenhuis torsion of N vanishes, then $[\ ,\ ]_{N.\nu}$ is a Lie bracket.

Occasionally, the deformed bracket will be simply denoted by $[\ ,\ ]_N$.

Let Φ_t be a one-parameter group of graded automorphisms of degree 0 of the vector space E and let N be its infinitesimal generator, $N = \frac{d}{dt}|_{t=0}\Phi_t$.

Then
$$\frac{d}{dt}[X,Y]_{\Phi_t} = [\Phi_t NX, Y] + [X, \Phi_t NY] - \Phi_t N[X,Y]$$
and, in particular,
$$\frac{d}{dt}|_{t=0}[X,Y]_{\Phi_t} = [X,Y]_N .$$
So, we can think of the deformed bracket as the first derivative evaluated at $t = 0$ of the one-parameter family of deformed brackets $[\ ,\ ]_{\Phi_t}$.

2.1. The integral flow of a $(1,1)$-tensor field

As examples and because we will need them in the applications that we shall give later, let us recall how to construct the one-parameter groups of graded endomorphisms from their infinitesimal generators in some cases.

Let M be a manifold and let N be a $(1,1)$ tensor on M, i.e., N is a bundle map $N : TM \to TM$. We shall denote its transpose by $N^* : T^*M \to T^*M$.

Let us consider the $(1,1)$-tensor field defined by the formal series $exp(tN)$. It has been previously used, for example in [3], page 41, as a way of justifying why the deformed bracket is called a deformed bracket. Such an expression, $exp(tN) = \sum_{i=0}^{\infty} \frac{1}{i!} t^i N^i$, is in principle just a formal expression. But for each point $m \in M$, N_m is an endomorphism of T_mM, and then, as is well known, the series $exp(tN_m)$ is always convergent. Therefore, $\Phi_t = exp(tN)$ is a well-defined automorphism of the vector bundle TM for all $t \in \mathbb{R}$.

Associated to the tensor field N we can define a zero-degree derivation of the algebra of differential forms on M, $\Omega(M)$. This derivation is denoted by i_N, and it is defined as the extension as a derivation of the map, $f \mapsto i_N f := 0$ for any smooth function f and for any differential 1-form α, $\alpha \mapsto i_N\alpha := N^*\alpha$.

The transpose of Φ_t is $\Phi_t^* = exp(tN^*)$, and it can be extended as an automorphism of $\Omega(M)$ which we shall also denote by Φ_t, in an abuse of notation. Note that this automorphism is the identity on $\Omega^0(M) = C^\infty(M)$. The derivative with respect to t of the automorphism Φ_t^* gives rise to the derivation i_N. Note that the following conditions are satisfied
$$\Phi_t^{*\prime} = \Phi_t^* \circ i_N = i_N \circ \Phi_t^* ,$$
$$\Phi_0^* = Id , \qquad \Phi_{t+s}^* = \Phi_t^* \circ \Phi_s^* .$$

It is in this sense that we can think of Φ_t^* as the integral flow of the zero-degree derivation i_N. Analogous relations are valid for Φ_t. Now we have all the ingredients to study what happens when N is a Nijenhuis tensor.

It is well known that if N is Nijenhuis, then all the powers of N are also Nijenhuis. Moreover, the Frölicher–Nijenhuis brackets $[N^k, N^\ell]_{FN}$ vanish for all $k, l \in \mathbb{N}$. This is the so-called hierarchy of Nijenhuis operators.

Since N^k is the k^{th}-derivative at $t = 0$ of the one-parameter group Φ_t, it is natural to ask whether Φ_t is also Nijenhuis. Below, we shall give a noninductive proof of the existence of the hierarchy of Nijenhuis operators. We shall obtain a direct proof of the following statement: N is Nijenhuis if and only if Φ_t is Nijenhuis.

Proposition 1. *Let $(E, [\ ,\ ])$ be a graded Lie algebra. Let Φ_t be a one-parameter group of graded automorphisms of degree 0 of the vector space E, and let N its infinitesimal generator. Then N is Nijenhuis if and only if Φ_t is Nijenhuis. In other words,*

$$[\Phi_t X, \Phi_t Y] = \Phi_t [X, Y]_{\Phi_t}\ , \tag{2.2}$$

for all $t \in \mathbb{R}$ if and only if the torsion of N vanishes.

Proof: First note that the second derivative of the Nijenhuis torsion of Φ_t evaluated at $t = 0$ is exactly the Nijenhuis torsion of N, up to a constant factor. Indeed,

$$\begin{aligned} \frac{d^2}{dt^2}|_{t=0}[\Phi_t, \Phi_t]_{FN} \ &= \ 2\, \frac{d}{dt}|_{t=0}[\Phi_t, N \circ \Phi_t]_{FN} \\ &= \ 2\Big([N, N]_{FN} + [Id, N^2]_{FN}\Big) \ = \ 2\,[N, N]_{FN}\ . \end{aligned}$$

Therefore, if we suppose first that Φ_t is Nijenhuis, then N also is Nijenhuis.

Reciprocally, let us now suppose that N is Nijenhuis. The converse needs a kind of double integration process. We shall show as a first step that the Frölicher–Nijenhuis bracket of N with Φ_t vanishes.

The first derivative of $[N, \Phi_t]_{FN}$ is $[N, N \circ \Phi_t]_{FN}$. An easy computation using Eq. (2.1) shows that, for any $X, Y \in E$,

$$\begin{aligned} &[N, N \circ \Phi_t]_{FN}(X, Y) \ = \\ &\qquad = \ N \circ [N, \Phi_t]_{FN}(X, Y) + [N, N]_{FN}(\Phi_t X, Y) + [N, N]_{FN}(X, \Phi_t Y)\ . \end{aligned}$$

Moreover, $[N, \Phi_0]_{FN} = [N, Id]_{FN} = 0$. Therefore, if N is Nijenhuis, we find that $[N, \Phi_t]_{FN}$ is a solution of the first-order differential equation, $\Psi_t' = N \circ \Psi_t$, with the initial condition $\Psi_0 = 0$. But the trivial solution, $\Psi_t = 0$, is a solution of the same differential equation with the same initial condition, so, by uniqueness of solutions of first-order differential equations with identical initial conditions, $[N, \Phi_t]_{FN} = 0$.

Now, let us show that Φ_t is Nijenhuis. We shall prove in fact that, for any $t, s \in \mathbb{R}$, $[\Phi_t, \Phi_s]_{FN} = 0$. The first derivative of $[\Phi_t, \Phi_s]_{FN}$ with respect to t is $2[N \circ \Phi_t, \Phi_s]_{FN}$. Once again, a simple computation using Eq. (2.1) shows that

$$\begin{aligned} [N \circ \Phi_t, \Phi_s]_{FN}(X, Y) \; = \; & N \circ [\Phi_t, \Phi_s]_{FN}(X, Y) \\ & + [N, \Phi_s]_{FN}(\Phi_t X, Y) + [N, \Phi_s]_{FN}(X, \Phi_t Y) \\ & - \Phi_s \circ [N, \Phi_t]_{FN}(X, Y) - [\Phi_{t+s}, N]_{FN}(X, Y) \; , \end{aligned}$$

where we have applied $\Phi_t \circ \Phi_s = \Phi_{t+s}$. Therefore, since $[N, \Phi_t]_{FN} = 0$ for all $t \in \mathbb{R}$, we find that $[\Phi_t, \Phi_s]_{FN}$ is a solution of $\Psi_t' = N \circ \Psi_t$. Moreover it satisfies the initial condition, $[\Phi_0, \Phi_s]_{FN} = [Id, \Phi_s]_{FN} = 0$. Using the same arguments as before, we obtain $[\Phi_t, \Phi_s]_{FN} = 0$. In particular, $[\Phi_t, \Phi_t]_{FN} = 0$. ∎

Remark 1. Note that we have shown that N is Nijenhuis if and only if

$$\Phi_{-t}[\Phi_t X, \Phi_t Y] = [X, Y]_{\Phi_t} \; .$$

In other words, the conjugation of the old Lie bracket by Φ_t is precisely its deformation by Φ_t. □

Corollary 1 (The hierarchy of Nijenhuis tensors)**.** *If* N *is Nijenhuis, then* $[N^k, N^\ell]_{FN} = 0$ *for any* $k, \ell \in \mathbb{N}$.

Proof: Let us recall that in the proof of Proposition 1 we proved that if N is Nijenhuis then $[\Phi_t, \Phi_s]_{FN} = 0$ for any $t, s \in \mathbb{R}$. Now taking successive partial derivatives with respect to t and s and evaluating them at $t = 0$ and $s = 0$, we deduce that $[N^k, N^\ell]_{FN} = 0$. ∎

2.2. Relationship with Gerstenhaber brackets

We will show an application to the computation of a generating operator of a Gerstenhaber bracket.

If $\mathbf{A}$ is a $\mathbb{Z}_2$-graded commutative, associative algebra, then an *odd Poisson bracket* or a $\mathbb{Z}_2$-*Gerstenhaber bracket* on $\mathbf{A}$ is, by definition, a bilinear map, $[\![\ ,\ ]\!]$, from $\mathbf{A} \times \mathbf{A}$ to $\mathbf{A}$, satisfying, for any $f, g, h \in \mathbf{A}$,

- $[\![f, g]\!] = -(-1)^{(|f|-1)(|g|-1)}[\![g, f]\!]$, (skew-symmetry)

- $[\![f,[\![g,h]\!]]\!]=[\![[\![f,g]\!],h]\!]+(-1)^{(|f|-1)(|g|-1)}[\![g,[\![f,h]\!]]\!]$, (graded Jacobi identity)
- $[\![f,gh]\!]=[\![f,g]\!]h+(-1)^{(|f|-1)|g|}g[\![f,h]\!]$, (Leibniz rule)
- $|[\![f,g]\!]|=|f|+|g|-1 \quad (\text{mod. } 2)$.

An algebra $\mathbf{A}$ together with a bracket satisfying the above conditions is called an *odd Poisson algebra* or a *$\mathbb{Z}_2$-Gerstenhaber algebra*.

A linear map of odd degree, $\Delta:\mathbf{A}\to\mathbf{A}$, such that, for all $a,b\in\mathbf{A}$,

$$[\![f,g]\!] \;=\; (-1)^{|f|}\Big(\Delta(fg)-(\Delta f)g-(-1)^{|f|}f(\Delta g)\Big)\,, \tag{2.3}$$

is called a *generator* or a *generating operator* of this bracket.

Lemma 1. *Assume that $(A,[\![\ ,\ ]\!],\wedge)$ is a Gerstenhaber algebra. Let δ be a generator of the bracket. Let Φ be an automorphism of the associative algebra $(A,\wedge)$.*

Then, a generating operator of the conjugation of the bracket by the automorphism Φ is the conjugation of the generating operator, $\Phi^{-1}\circ\delta\circ\Phi$.

Let us suppose now that Φ_t is a one-parameter group of automorphisms of the associative algebra $(A,\wedge)$. It is easy to check that now, the infinitesimal generator, $N=\frac{d}{dt}|_{t=0}\Phi_t$, is a derivation of $(A,\wedge)$.

Theorem 1. *Let $(A,[\![\ ,\ ]\!],\wedge)$ be a Gerstenhaber algebra. Let δ be a generator of the Gerstenhaber bracket $[\![\ ,\ ]\!]$. Let Φ_t be a one-parameter group of automorphisms of the associative algebra $(A,\wedge)$, and let N be its infinitesimal generator. If N is Nijenhuis, then the deformed Gerstenhaber bracket, $[\![\ ,\ ]\!]_N$, is generated by $[\delta,N]$.*

Proof: By Remark 1 we know that the deformed bracket $[\![\ ,\ ]\!]_{\Phi_t}$, agrees with the conjugation by Φ_t of the bracket $[\![\ ,\ ]\!]$. Now, by lemma 1, $\Phi_{-t}\circ\delta\circ\Phi_t$ is a generating operator of $[\![\ ,\ ]\!]_{\Phi_t}$. By taking derivatives with respect to t at $t=0$ we find that $[\delta,N]$ is a generating operator of $[\![\ ,\ ]\!]_N$. ∎

Let us apply this result to a particular case: the deformation by a Nijenhuis tensor of the Schouten–Nijenhuis bracket of multivector fields.

Let M be a manifold and let us consider the Gerstenhaber algebra $(\Gamma(\Lambda TM),[\ ,\ ]_{SN},\wedge)$, where $[\ ,\ ]_{SN}$ denotes the Schouten–Nijenhuis bracket. Let δ be a generating operator of this bracket (see [5]).

Let N be a Nijenhuis tensor with respect to the usual Lie bracket of vector fields. We know that the deformed bracket, $[\ ,\ ]_N$, is also a Lie bracket on $\mathfrak{X}(M)$. It is then possible to define a Gerstenhaber bracket on the algebra of multivector fields, extending the deformed bracket, $[\ ,\ ]_N$, as a biderivation on the algebra of multivector fields. We shall denote the resulting Gerstenhaber bracket by $[\ ,\ ]^N_{SN}$.

Corollary 2. *If δ is a generator of the Schouten–Nijenhuis bracket, then the Gerstenhaber bracket, $[\ ,\ ]^N_{SN}$, is generated by $[\delta, i_N]$.*

Proof: Let Φ_t be the one-parameter group of automorphism of TM having N as infinitesimal generator. Let us, by an abuse of language, also denote by Φ_t the extension of $\Phi_t : TM \to TM$ as an automorphism of the whole algebra of multivector fields. The one-parameter group of automorphisms of the algebra of multivector fields Φ_t has the derivation i_N as infinitesimal generator (see subsection 2.1).

By Theorem 1, the deformation by i_N of the Schouten–Nijenhuis bracket is a Gerstenhaber bracket with $[\delta, i_N]$ as a generating operator. Finally, it is easy to check that the deformation by i_N of the Schouten–Nijenhuis bracket agrees with the Gerstenhaber bracket, $[\ ,\ ]^N_{SN}$. Indeed, one can check that they agree when acting on a pair of smooth functions and/or vector fields. ∎

3 – Poisson–Nijenhuis structures

Let us first recall the definition of Poisson–Nijenhuis structures. Among all the equivalent definitions we prefer the one from [3].

Definition 2. Given a Poisson bivector, P, on a differentiable manifold, M, we can define a Lie algebra bracket on $\Omega(M)$ by

$$\begin{aligned} [\![\alpha,\beta]\!]_{\nu(P)} &= \mathcal{L}_{\#_P\alpha}\beta - \mathcal{L}_{\#_P\beta}\alpha - dP(\alpha,\beta)\ , \\ [\![\alpha,f]\!]_{\nu(P)} &= P(\alpha, df)\ , \\ [\![f,g]\!]_{\nu(P)} &= 0\ , \end{aligned}$$

for all $\alpha,\beta \in \Omega^1(M)$ and $f,g \in C^\infty(M)$, where $\#_P\alpha$ denotes the vector field defined by $(\#_P\alpha)(f) = P(\alpha, df)$ for any $f \in C^\infty(M)$, and extending the Lie algebra bracket to the whole $\Omega(M)$ by the Leibniz rule. This bracket is known as the Koszul–Schouten bracket associated to the Poisson bivector P. □

The adjoint operator N^* can be seen as a $C^\infty(M)$-linear map $N^* : \Omega^1(M) \to \Omega^1(M)$ as usual.

Definition 3. A Nijenhuis tensor N and a Poisson tensor P on a manifold M are called compatible, that is, the pair (P, N) is called a Poisson–Nijenhuis structure, if

$$i) \quad N \circ \#_P = \#_P \circ N^*$$

and if

$$ii) \quad [\![\alpha, \beta]\!]_{\nu(NP)} = [\![\alpha, \beta]\!]_{N^*.\nu(P)} \ ,$$

for all $\alpha, \beta \in \Omega(M)$. ▫

Note that the compositions $N \circ \#_P$ and $\#_P \circ N^*$ define two not necessarily skewsymmetric $(2,0)$-tensor fields, denoted by NP and PN^*, such that $N \circ \#_P = \#_{NP}$ and $\#_P \circ N^* = \#_{PN^*}$. The tensor fields are then

$$(NP)(\alpha, \beta) = P(\alpha, N^*\beta)\,, \qquad (PN^*)(\alpha, \beta) = P(N^*\alpha, \beta)\ .$$

Thus, the first condition in the definition of a Poisson–Nijenhuis manifold can be written as $NP = PN^*$. This condition guarantees that

$$N \circ \#_P \ = \ \#_{NP} \ = \ \#_{PN^*} \ = \ \#_P \circ N^*\ .$$

In addition we can deduce that $NP = PN^*$ is skewsymmetric.

The second condition can be expressed in another way. Let us define the concomitant $C(P, N)$ by

$$C(P,N)(\alpha,\beta) \ = \ [\![\alpha, \beta]\!]_{\nu(NP)} - [\![\alpha, \beta]\!]_{N^*.\nu(P)} \ ,$$

for all $\alpha, \beta \in \Omega^1(M)$. Because $N \circ \#_P = \#_P \circ N^*$, $C(P, N)$ is a tensor field of type $(3, 0)$. Thus the second condition is just the vanishing of $C(P, N)$.

The concomitant $C(P, N)$ can be also written as

$$\begin{aligned} C(P,N)(\alpha,\beta) \ &= \ \mathcal{L}_{P\alpha}(N^*\beta) - N^*\mathcal{L}_{P\alpha}\beta - \mathcal{L}_{P\beta}(N^*\alpha) + N^*\mathcal{L}_{P\beta}\alpha \\ &\qquad + \ dNP(\alpha,\beta) - N^*dP(\alpha,\beta) \\ &= (\mathcal{L}_{P\alpha}N^*)\beta - (\mathcal{L}_{P\beta}N^*)\alpha + dNP(\alpha,\beta) - N^*dP(\alpha,\beta)\ . \end{aligned} \tag{3.1}$$

Let us recall the definition of compatibility of Poisson structures.

Definition 4. Poisson structures P_0 and P_1 on the same manifold M are compatible if the sum $P_0 + P_1$, is also a Poisson structure. ▫

Remark 2. Let us recall that this is equivalent to

$$\#_{P_0}[\![\alpha,\beta]\!]_{\nu(P_1)} + \#_{P_1}[\![\alpha,\beta]\!]_{\nu(P_0)} = [\#_{P_0}\alpha, \#_{P_1}\beta] + [\#_{P_1}\alpha, \#_{P_0}\beta]$$

for all $\alpha, \beta \in \Omega^1(M)$. □

Let us recall the following

Proposition 2 (see [3]). *If (P,N) is a Poisson–Nijenhuis structure, then the $(2,0)$-tensor, NP, defined by $NP(\alpha,\beta) = P(\alpha, N^*\beta)$, is a Poisson bivector that is compatible with P.*

4 – The hierarchy of Poisson–Nijenhuis structures

In this section we shall obtain a noninductive proof of the existence of the hierarchy of Poisson–Nijenhuis structures constructed from an initial one.

Proposition 3. *Let N be a $(1,1)$-tensor field on M, and let $\Phi_t = exp(tN)$. The pair (P,N) is a Poisson–Nijenhuis structure if and only if (P,Φ_t) is a Poisson–Nijenhuis structure.*

Proof: We remark that we know that N is Nijenhuis if and only if Φ_t is also Nijenhuis. So we need only prove that the compatibility conditions are satisfied.

Let us suppose first that (P,Φ_t) is a Poisson–Nijenhuis structure. Then, by taking the first derivative at $t=0$ of the compatibility conditions between P and Φ_t, we obtain those for P and N.

Reciprocally, let us suppose that (P,N) is a Poisson–Nijenhuis structure. We shall consider the tensor field $\Phi_{-t}P\Phi_t^*$ defined by

$$\Phi_{-t}P\Phi_t^*(\alpha,\beta) := P(\Phi_{-t}^*\alpha, \Phi_t^*\beta) .$$

The first derivative of $\Phi_{-t}P\Phi_t^*$ is $\Phi_{-t}(PN^* - NP)\Phi_t^* = 0$. Therefore, $\Phi_{-t}P\Phi_t^*$ is constant. But since its value at $t=0$ is P, $\Phi_{-t}P\Phi_t^* = P$, or, equivalently, $\Phi_t P = P\Phi_t^*$.

Let us now study the second compatibility condition between P and Φ_t.

The first derivative of $C(P,\Phi_t)(\alpha,\beta)$ is $C(P,\Phi_t\circ N)(\alpha,\beta)$, and a simple computation using Eq. (3.1) shows that it is equal to

$$\Phi_t(C(P,N)(\alpha,\beta)) + C(P,\Phi_t)(\alpha,N^*\beta) + (\mathcal{L}_{PN\beta}\Phi_t^*)\alpha - (\mathcal{L}_{P\beta}\Phi_t^*)N^*\alpha . \tag{4.1}$$

The first term vanishes because we suppose that (P,N) is a Poisson–Nijenhuis structure. We shall see that the two last terms also vanish. Let us denote them by

$$H_t(\alpha,\beta) \;:=\; (\mathcal{L}_{PN\beta}\Phi_t^*)\alpha - (\mathcal{L}_{P\beta}\Phi_t^*)N^*\alpha \;.$$

The first derivative of H_t with respect to t is

$$\begin{aligned}(\mathcal{L}_{PN\beta}\Phi_t^* N^*)\alpha - (\mathcal{L}_{P\beta}\Phi_t^* N^*)N^*\alpha \;=&\\ =\; H_t\Big(N^*\alpha,\beta) + \Phi_t^*((\mathcal{L}_{PN\beta}N^*)\alpha - (\mathcal{L}_{P\beta}N^*)N^*\alpha\Big) \;.\end{aligned}$$

Now, let us recall that the following identity (See formula 7.13 [6])

$$\mathcal{L}_{NX}(N^*) = \mathcal{L}_X(N^*)N^*$$

is a condition equivalent to the vanishing of the Nijenhuis torsion of N. It is now clear that the two last terms vanish.

Then, we find that H_t satisfies the equation $H_t'(\alpha,\beta) = H_t(N^*\alpha,\beta)$ with initial condition $H_0(\alpha,\beta) = (\mathcal{L}_{PN\beta}Id)\alpha - (\mathcal{L}_{P\beta}Id)\alpha = 0$. Therefore, $H_t = 0$ for all $t\in\mathbb{R}$.

Let us return to $C(P,\Phi_t)$. By Eq. (4.1), $C(P,\Phi_t)$ satisfies $C(P,\Phi_t)'(\alpha,\beta) = C(P,\Phi_t)(\alpha,N^*\beta)$, and the initial condition, $C(P,\Phi_0) = C(P,Id) = 0$. Therefore, $C(P,\Phi_t) = 0$ for all $t\in\mathbb{R}$. ∎

Corollary 3. *If (P,N) is a Poisson–Nijenhuis structure, then, for any $t,s\in\mathbb{R}$,*

(1) *$(\Phi_s P,\Phi_t)$ is a Poisson–Nijenhuis structure, and*

(2) *$\Phi_s P$ and $\Phi_t P$ are compatible Poisson bivectors.*

Proof: The first statement is just a consequence of the following relation, which can be obtained from equation (3.1),

$$C(P,\Phi_t)'(\alpha,\beta) \;=\; C(P,N)(\Phi_t^*\alpha,\beta) + C(P,N)(\alpha,\Phi_t^*\beta) - C(\Phi_t P,N)(\alpha,\beta) \;.$$

Then, if (P,N) is a Poisson–Nijenhuis structure, both $C(P,N)$ and $C(P,\Phi_t)$ vanish, and $C(\Phi_t P,N) = 0$. This means that $(\Phi_t P,N)$ is a Poisson–Nijenhuis structure (note that by Propositions 2 and 3, $\Phi_t P$ is a Poisson bivector). By applying Proposition 3 to $(\Phi_t P,N)$ we find that $(\Phi_t P,\Phi_s)$ is a Poisson–Nijenhuis structure for all $t,s\in\mathbb{R}$.

The second statement is a consequence of the first. ∎

Remark 3. In [10] it is observed that if (P, N) is a Poisson–Nijenhuis structure, then not only the elements of the hierarchy are again Poisson–Nijenhuis structures, but so is any structure of the kind $((\sum_{i=0}^{\infty} a_i N^i) \circ P,\ \sum_{j=0}^{\infty} b_j N^j)$, where the series involved are convergent power series with constant coefficients. So, the first statement of the corollary is a particular case of this observation. The novelty here is that we have obtained it before proving the existence of the hierarchy, whereas in [10], it is a consequence of the existence of such a hierarchy. In fact, Corollary 3 is a condensed way of writing the hierarchy, as the next corollary will show. □

Remark 4. What we have found is a kind of surface in the set of all Poisson–Nijenhuis structures. If we write $x(t,s) = (\Phi_t P, \Phi_s)$ then

$$x(0,0) = (P, Id), \quad x_t(0,0) = (NP, Id), \quad x_s(0,0) = (P, N)\ ,$$
$$x_{tt}(0,0) = (N^2 P, Id), \quad x_{ts}(0,0) = (NP, N), \quad x_{ss}(0,0) = (P, N^2)\ . \ \square$$

Corollary 4 (The hierarchy of Poisson–Nijenhuis structures)**.** *If (P, N) is a Poisson–Nijenhuis structure, then, for any $k, \ell \in \mathbb{N}$,*

(**1**) *$(N^k P, N^\ell)$ is a Poisson–Nijenhuis structure, and*

(**2**) *$N^k P$ and $N^\ell P$ are compatible Poisson bivectors.*

Proof: As before, we need only take partial derivatives with respect to t and s of the compatibility conditions between $\Phi_s P$ and Φ_t and evaluate them at $t = 0$ and $s = 0$. ∎

Corollary 5. *If (P, N) is a Poisson–Nijenhuis structure, then*

$$[\![\alpha, \beta]\!]_{\nu(\Phi_t P)} = [\![\alpha, \beta]\!]_{\Phi_t^* \cdot \nu(P)} = \Phi_{-t}^* [\![\Phi_t^* \alpha, \Phi_t^* \beta]\!]_{\nu(P)}\ .$$

Proof: It is a consequence of the fact that (P, Φ_t) is a Poisson–Nijenhuis structure and of the fact that, for any Poisson–Nijenhuis structure, (P, N), N^* is Nijenhuis with respect to the bracket $[\![\ ,\ ]\!]_{\nu(P)}$ (see [3] lemma 4.2), and then of Formula 2.2. ∎

This last result has an interpretation in terms of generating operators. Let us recall that a generating operator of the Koszul–Schouten bracket, $[\![\ ,\ ]\!]_{\nu(P)}$, is $\mathcal{L}_P = [i_P, d]$, where d denotes the exterior derivative (see [5]). Now, as a consequence of Lemma 1, we can state the following

Corollary 6. *If (P,N) is a Poisson–Nijenhuis structure then,*

$$\mathcal{L}_{\Phi_t P} \;=\; \Phi^*_{-t} \circ \mathcal{L}_P \circ \Phi^*_t \;.$$

Proof: Two generating operators of the same Gerstenhaber bracket differ by a derivation of degree -1. In this case, it is easy to check that $\mathcal{L}_{\Phi_t P} - \Phi^*_{-t} \circ \mathcal{L}_P \circ \Phi^*_t$ is the null derivation. ∎

Remark 5. We have worked here with the definition of Poisson–Nijenhuis manifolds given in [3], but the same results can be obtained for similar, but not fully equivalent, definitions, for example, the one given in [9]. The key point is to observe that the statement in Proposition 1 is also valid in the following form: Let $F \subset E$ be a vector subspace, then $[N,N]_{FN}$ vanishes on F if and only if $[\Phi_t, \Phi_t]_{FN}$ vanishes on F, Φ_t being a one-parameter group of graded automorphisms and N is its infinitesimal generator. □

Remark 6. Recently, the notion of Jacobi-Nijenhuis structure has also been studied, see for example [7], [8] or [2]. It is not difficult to see that a proof of the existence of hierarchies of Jacobi-Nijenhuis manifolds can also be obtained using similar arguments to those advanced in this note. □

5 – An example

The initial Poisson bivector, taken from [1] for $n=5$, is

$$P = \begin{pmatrix} 0 & 0 & -a_1 & a_1 & 0 \\ 0 & 0 & 0 & -a_2 & a_2 \\ a_1 & 0 & 0 & 0 & 0 \\ -a_1 & a_2 & 0 & 0 & 0 \\ 0 & -a_2 & 0 & 0 & 0 \end{pmatrix},$$

where $\{a_1, a_2, b_1, b_2, b_3\}$ are the coordinates for $\mathbb{R}^5$.

Let us consider the tensor fields given by

$$N = \begin{pmatrix} f & 0 & 0 & 0 & 0 \\ 0 & L(a_2,b_3) & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 \\ 0 & 0 & 0 & f & f - L(a_2,b_3) \\ 0 & 0 & 0 & 0 & L(a_2,b_3) \end{pmatrix},$$

where f is a constant and L a function depending on the variables a_2, b_3. An easy computation shows that (P, N) is a Poisson–Nijenhuis structure.

The integral flow of N is

$$\Phi_t = \begin{pmatrix} e^{ft} & 0 & 0 & 0 & 0 \\ 0 & e^{L(a_2,b_3)t} & 0 & 0 & 0 \\ 0 & 0 & e^{ft} & 0 & 0 \\ 0 & 0 & 0 & e^{ft} & e^{ft} - e^{L(a_2,b_3)t} \\ 0 & 0 & 0 & 0 & e^{L(a_2,b_3)t} \end{pmatrix}.$$

And the 1-parameter family, $\Phi_s P$ of compatible Poisson bivectors is

$$\begin{pmatrix} 0 & 0 & -a_1 e^{fs} & a_1 e^{fs} & 0 \\ 0 & 0 & 0 & -a_2 e^{L(a_2,b_3)s} & a_2 e^{L(a_2,b_3)s} \\ a_1 e^{fs} & 0 & 0 & 0 & 0 \\ -a_1 e^{fs} & a_2 e^{L(a_2,b_3)s} & 0 & 0 & 0 \\ 0 & -a_2 e^{L(a_2,b_3)s} & 0 & 0 & 0 \end{pmatrix}.$$

By our results, each pair $(\Phi_s P, \Phi_t)$ is a Poisson–Nijenhuis structure for all $t, s \in \mathbb{R}$.

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J. Monterde,
Dept. de Geometria i Topologia, Universitat de València,
Avda. Vicent Andrés Estellés, 1, E-46100-Burjassot (València) – SPAIN
E-mail: `monterde@uv.es`