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BOUTROUX'S METHOD VS. RE-SCALING
Lower estimates for the orders of growth of the second and fourth Painlevé transcendents

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Abstract: We give a new proof of Shimomura's sharp lower estimates for the orders of growth of the Painlevé transcendents II and IV: $\varrho_{II} \ge 3/2$ and $\varrho_{IV} \ge 2$.

1 – Introduction

We are concerned with the transcendental solutions of *Painlevé's second* and *fourth equation,*

$$w'' = \alpha + zw + 2w^3 \tag{1.1}$$

and

$$2ww'' = w'^2 + 3w^4 + 8zw^3 + 4(z^2-\alpha)w^2 + 2\beta \ , \tag{1.2}$$

the *second* and *fourth transcendents*. In [Sh1] and [St1] it was shown that any second and fourth Painlevé transcendent w has order of growth $\varrho(w) \le 3$ and $\varrho(w) \le 4$, respectively. More precisely, if (p_n) denotes the sequence of non-zero poles of w, it was shown in [St1] that, in the respective cases,

$$\sum_{|p_n|\le r} |p_n|^{-1} = O(r^2) \quad \text{and} \quad \sum_{|p_n|\le r} |p_n|^{-2} = O(r^2)$$

hold. In the other direction, Shimomura [Sh3] recently derived the sharp lower estimates $\varrho(w) \ge 3/2$, resp. $\varrho(w) \ge 2$. Equality is attained for particular solutions, called *Airy-* and *Hermite–Weber–*solutions, respectively. For more details

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concerning these functions we refer to [GLS]. For $2\alpha \in \mathbb{Z}$ proofs of $\varrho(w) \geq 3/2$ in case (II) can be found in [Sh2] and [St2].

Combining the *re-scaling method* with some modified Shimomura approach [Sh3] we will be able to give a different proof of Shimomura's lower estimates, which may be stated as follows:

Theorem. *Let w be a transcendental solution of one of the differential equations (1.1) or else (1.2), with sequence (p_n) of non-zero poles. Then, for some $\kappa = \kappa(w) > 0$ and $r > r_0$*

$$\sum_{|p_n| \leq r} |p_n|^{-3/2} \geq \kappa \log r \quad \text{and} \quad \sum_{|p_n| \leq r} |p_n|^{-1/2} \geq \kappa\, r$$

or else

$$\sum_{|p_n| \leq r} |p_n|^{-2} \geq \kappa \log r \quad \text{and} \quad \sum_{|p_n| \leq r} |p_n|^{-1} \geq \kappa r$$

holds in the respective case.

2 – Two local methods

We start by describing two methods of investigating the Painlevé transcendents locally, and restrict ourselves to equation (1.1).

a) Re-scaling

Let w be any transcendental solution of (II) and let (p_n) be any sub-sequence of the sequence of poles of w. Then

$$y_n(\mathfrak{z}) \;=\; p_n^{-1/2} w\,(p_n + p_n^{-1/2}\mathfrak{z})$$

has the series expansion

$$y_n(\mathfrak{z}) \;=\; \frac{\epsilon_n}{\mathfrak{z}} - \frac{\epsilon_n}{6}\mathfrak{z} - p_n^{-3/2}\,\frac{\alpha+\epsilon_n}{4}\mathfrak{z}^2 + h_n\, p_n^{-2}\mathfrak{z}^3 + \cdots\,, \qquad \epsilon_n = \pm 1\ ,$$

about $\mathfrak{z} = 0$ and satisfies

$$y_n''(\mathfrak{z}) \;=\; p_n^{-3/2}\alpha + (p_n^{-3/2}\mathfrak{z} + 1)\, y_n(\mathfrak{z}) \;+\; 2\, y_n^3(\mathfrak{z})\ ,$$

where now $'$ denotes differentiation with respect to $\mathfrak{z}$. One of the major results of the re-scaling method developed in [St1, St2] was that the sequence $(h_n p_n^{-2})$ has a *uniform bound* only depending on the solution w. Thus choosing a sub-sequence of (p_n), again denoted (p_n), such that $\epsilon_n = \epsilon$ is constant and $h_n p_n^{-2} \to h$, we obtain $y_n(\mathfrak{z}) \to \mathfrak{y}_{rs}(\mathfrak{z})$ ($_{rs}$ stands for *re-scaled*), locally uniformly in $\mathbb{C}$, where $\mathfrak{y}_{rs}$ is the unique solution of

$$\mathfrak{y}_{rs}'' = \mathfrak{y}_{rs} + 2\,\mathfrak{y}_{rs}^3 \quad \text{with} \quad \mathfrak{y}_{rs}(\mathfrak{z}) = \frac{\epsilon}{\mathfrak{z}} - \frac{\epsilon}{6}\mathfrak{z} + h\mathfrak{z}^3 + \cdots \quad \text{about } \mathfrak{z} = 0 \; .$$

We note that

$$\mathfrak{y}_{rs}'^2 \;=\; \tfrac{7}{36} - 10\,\epsilon h + \mathfrak{y}_{rs}^2 + \mathfrak{y}_{rs}^4 \;=\; c + \mathfrak{y}_{rs}^2 + \mathfrak{y}_{rs}^4 \; ,$$

with $|c|$ uniformly bounded, independent of the sequence (p_n).

Finally, application of Hurwitz' Theorem yields the following

Remark a. Every pole $\mathfrak{z}_0$ of $\mathfrak{y}_{rs}$ is the limit of poles $\mathfrak{z}_n$ of y_n; thus $p_n' = p_n + p_n^{-1/2}\mathfrak{z}_n$ is a pole of w, and any such sequence p_n' gives rise to a pole $\mathfrak{z}_0 = \lim_{n\to\infty}(p_n' - p_n)p_n^{1/2}$ of $\mathfrak{y}_{rs}$. □

b) Boutroux's method

Again let w be any transcendental solution of (II). The change of variables

$$\xi = \tfrac{2}{3}\, z^{3/2} = \phi(z) \, , \qquad \Theta(\xi) = z^{-1/2}\, w(z) \; ,$$

see Boutroux's paper [B], leads to the differential equation (where now $'$ denotes $d/d\xi$)

$$\Theta''(\xi) \;=\; 2\,\Theta^3(\xi) + \Theta(\xi) - \frac{\Theta'(\xi)}{\xi} + \frac{2\alpha}{3\xi} + \frac{\Theta(\xi)}{9\xi^2} \; .$$

To be more precise, let $\mathbb{H}$ be any half-plane with $0 \in \partial\mathbb{H}$. Then any branch of $\psi(\xi) = (\frac{3}{2}\,\xi)^{2/3}$ maps $\mathbb{H}$ conformally onto some sector S of angular width $2\pi/3$, and ϕ will denote the inverse map $\psi^{-1} \colon S \to \mathbb{H}$.

If $p_n \neq 0$ denotes any pole of w in the sector S, we obtain for $v_n(\mathfrak{z}) = \Theta(\phi(p_n) + \mathfrak{z})$ the differential equation

$$v_n''(\mathfrak{z}) \;=\; 2\,v_n^3(\mathfrak{z}) + v_n(\mathfrak{z}) - \frac{v_n'(\mathfrak{z})}{\phi(p_n) + \mathfrak{z}} + \frac{2\alpha}{3(\phi(p_n) + \mathfrak{z})} + \frac{v_n(\mathfrak{z})}{9(\phi(p_n) + \mathfrak{z})^2} \; .$$

If we choose $p_n \to \infty$ (the same sub-sequence as was chosen above) we obtain in the limit the differential equation

$$\mathfrak{v}_B'' \;=\; 2\mathfrak{v}_B^3 + \mathfrak{v}_B \; ,$$

where $_B$ stands for *Boutroux.* It is obvious that

$$\Theta\Big(\phi(p_n)+\mathfrak{z}\Big) \;=\; \Big(\psi(\phi(p_n)+\mathfrak{z})\Big)^{-1/2} w\Big(\psi(\phi(p_n)+\mathfrak{z})\Big) \sim p_n^{-1/2}\, w\Big(p_n+p_n^{-1/2}\mathfrak{z}\Big)$$

holds as $n\to\infty$, and hence the functions $\mathfrak{y}_{rs}$ and $\mathfrak{v}_B$ agree. This phenomenon was already observed in [St1] for Painlevé's first equation. The re-scaling method yields the additional information that Θ and Θ' are *uniformly bounded* outside the union of disks $|\xi-\phi(p_n)|<\delta$ about the poles $\phi(p_n)$ of Θ; $\delta>0$ is arbitrary. We thus have

Remark b. Every pole $\mathfrak{z}_0$ of $\mathfrak{y}_B$ is the limit of poles $\mathfrak{z}_n$ of v_n; thus $p'_n = \psi(\phi(p_n)+\mathfrak{z}_n)$ is a pole of w, and any such sequence p'_n gives rise to a pole $\mathfrak{z}_0=\lim\limits_{n\to\infty}(\phi(p'_n)-\phi(p_n))$ of $\mathfrak{y}_B$. ▫

3 – Proof of the Theorem

To start with the proof we need the following Lemma, which in similar form also was proved in [Sh2, Lemma 2.2.].

Lemma. *Let $\mathfrak{L}_c$ denote the (possibly degenerate) period lattice for the differential equation $\mathfrak{y}'^2=\mathfrak{y}^4+\mathfrak{y}^2+c$, and let Σ be any open sector with vertex at the origin and containing $\{1,i\}$ (or $\{-1,i\}$ or $\{-1,-i\}$ or $\{1,-i\}$). Then given $K>0$ there exists $R>0$, such that $\Sigma\cap\{\omega:|\omega|\le R\}\cap\mathfrak{L}_c\ne\emptyset$ for every c satisfying $|c|\le K$.*

Remark. For $c\ne 0,1/4$, every non-constant solution of $\mathfrak{y}'^2=\mathfrak{y}^4+\mathfrak{y}^2+c$ is an elliptic function, closely related to Jacobi's *sinus amplitudinis.* If $\{\omega,\tilde\omega\}$ is a suitably chosen basis of the period lattice $\mathfrak{L}$ and if $\mathfrak{y}$ has a pole at $\mathfrak{z}=0$, then it has simple poles exactly at $m\omega+(n+\frac12)\tilde\omega$, $m,n\in\mathbb{Z}$, see the famous book [HC, p. 215] by Hurwitz and Courant. ▫

Proof of Lemma: We have to consider separately the points of degeneration, namely $c=0$, $c=1/4$ and $c=\infty$. For $c=0$ and $c=1/4$ the non-constant solutions $\mathfrak{y}$ are simply periodic with primitive periods $\omega_0=\pm\pi/\sqrt2$ and $\omega_{1/4}=\pm i\pi$, respectively. Hence, for $\delta>0$ sufficiently small, we have in the respective cases $|c|<\delta$ and $|c-1/4|<\delta$ that, by continuity, one of the periods $\pm\omega_c$ belong to $\Sigma\cap\{\omega:|\omega|\le 4\}$, say.

In case $c\to\infty$ we set $\mathfrak{u}_a(z)=a\mathfrak{y}(az)$ with $a^4c=1$, to obtain ${\mathfrak{u}'_a}^2=\mathfrak{u}_a^4+a^2\mathfrak{u}_a^2+1$, and hence, in the limit $c\to\infty$, the differential equation ${\mathfrak{u}'_0}^2=\mathfrak{u}_0^4+1$.

Thus, for $|c|$ large, the period lattice $\mathfrak{L}_c$ is approximately a square lattice with mesh size $\asymp |c|^{-1/4}$. We again note, however, that in our case $|c|$ is uniformly bounded.

In the compact parameter set $\{c\colon\ \delta \le |c| \le K,\ |c-1/4| \ge \delta\}$ each lattice $\mathfrak{L}_c$ has a basis $\{\omega_c, \tilde{\omega}_c\}$ such that $\kappa R \le |\omega_c| \le |\tilde{\omega}_c| \le |\omega_c \pm \tilde{\omega}_c| \le R$ for some constants $R \ge 4, \kappa > 0$, independent of c. The problem now is equivalent to the following: Let $\mathfrak{L}$ be the lattice spanned by 1 and τ with

$$\operatorname{Im}\tau > 0\,, \qquad -1/2 < \operatorname{Re}\tau \le 1/2 \quad \text{and} \quad 1 \le |\tau| \le M\ ,$$

$M > 1$ some fixed constant, and let Σ be *any* open sector with vertex at the origin and with angular width $> \pi/2$. Then we have to show that

$$\Sigma \cap \{1, 1+\tau, \tau, -1+\tau, -1, -1-\tau, -\tau, 1-\tau\} \ne \emptyset\ .$$

This, however, follows immediately from the fact that the angle between any two consecutive points in the sequence $(1, 1+\tau, \ldots, 1-\tau, 1)$ is $< \pi/2$. ∎

Proof of the Theorem in case (II): To fix ideas we consider (the branches of) $\psi(\xi) = (\frac{3}{2}\xi)^{2/3}$ in the half-plane $\mathbb{H}\colon\ -\pi/4 < \arg\xi < 3\pi/4$ with $\psi(\mathbb{H}) = S = \{z\,:\, -\pi/6 < \arg z < \pi/2\}$ (the other possibilities being $\psi(\mathbb{H}) = e^{2\pi i/3}S$ and $\psi(\mathbb{H}) = e^{4\pi i/3}S$). We also set, for $z_0 \in \psi(\mathbb{H})$, $D(z_0) = \psi(\phi(z_0)+\mathbb{H})$. Then, if $r > 0$ is sufficiently large, it follows from the Lemma and Remarks a. and b. about the distribution of poles, that to any pole p of w in $D(re^{\pi i/6})$ there exists a pole $\phi(p')$ of Θ in $\phi(p)+\mathbb{H}$ with $|\phi(p')-\phi(p)| \le 2R$, say. Hence $p' \in D(p) \subset D(re^{\pi i/6})$ is a pole of w satisfying $\frac{3}{2}|p'^{3/2} - p^{3/2}| \ge \frac{1}{2}|p|^{1/2}|p'-p|$, for r and thus $|p|$ sufficiently large, and this gives $|p'-p| \le 4\,R|p|^{-1/2}$.

Since $D(p') \subset D(p) \subset D(re^{\pi i/6})$, this process may be repeated to obtain a sequence $\tilde{p}_1 = p,\ \tilde{p}_2 = \tilde{p}_1',\ \tilde{p}_3 = \tilde{p}_2', \ldots$ of *different* poles[1] of w such that $|\tilde{p}_{n+1}| \le |\tilde{p}_n| + O(|\tilde{p}_n|^{-1/2}) = |\tilde{p}_n|(1+O(|\tilde{p}_n|^{-3/2})$ as $n \to \infty$. This gives $|\tilde{p}_{n+1}| - |\tilde{p}_1| = O(\sum_{\nu=1}^{n} |\tilde{p}_\nu|^{-1/2})$ and $\log|\tilde{p}_{n+1}| - \log|\tilde{p}_1| = O(\sum_{\nu=1}^{n} |\tilde{p}_\nu|^{-3/2})$ for every $n \in \mathbb{N}$. The assertion of our theorem in case (II) now follows, since the same method applies to the open half-plane $i\mathbb{H}$ with associated sectors $e^{\pi i/3}S$, $-S$ and $e^{5\pi i/3}S$; then the domain $\bigcup_{\nu=0}^{5} D(re^{(2\nu+1)\pi i/6})$ is some punctured neighbourhood of ∞. ∎

[1] The crucial point was to prove that the construction leads to an *infinite* sequence of poles. It was Shimomura's paper [Sh3] which inspired me to compare Boutroux's method with re-scaling and so to overcome this difficulty.

The **proof** in case (IV) is almost the same, details will be omitted. We just note that, after some simple calculation, the re-scaling process $y_n(\mathfrak{z}) = p_n^{-1} w(p_n + p_n^{-1}\mathfrak{z})$ leads to the differential equation

$$\mathfrak{y}_{rs}'^2 = \mathfrak{y}^4 + 4\mathfrak{y}^3 + 4\mathfrak{y}^2 + c\mathfrak{y} ,$$

with $|c|$ uniformly bounded. The degenerate cases correspond to the parameters $c = 0$, $c = 32/27$ and $c = \infty$; by the substitution $\mathfrak{u}(\mathfrak{z}) = a\mathfrak{y}(a\mathfrak{z})$, $a^3 c = 1$, the latter case again reduces in the limit $c \to \infty$ to $\mathfrak{u}'^2 = \mathfrak{u}^4 + 1$. One also has to work with (the branches of) $\psi(\xi) = (2\xi)^{1/2}$ in the half-planes $\mathbb{H}:\ -\pi/4 < \arg\xi < 3\pi/4$, $i\mathbb{H}$, $-\mathbb{H}$ and $-i\mathbb{H}$. ∎

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