Fractional Gaussian fields: A survey

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Abstract: We discuss a family of random fields indexed by a parameter $s \in \mathbb{R}$ which we call the *fractional Gaussian fields*, given by

$$\mathrm{FGF}_s(\mathbb{R}^d) = (-\Delta)^{-s/2}W,$$

where W is a white noise on $\mathbb{R}^d$ and $(-\Delta)^{-s/2}$ is the fractional Laplacian. These fields can also be parameterized by their Hurst parameter $H = s - d/2$. In one dimension, examples of FGF_s processes include Brownian motion ($s = 1$) and fractional Brownian motion ($1/2 < s < 3/2$). Examples in arbitrary dimension include white noise ($s = 0$), the Gaussian free field ($s = 1$), the bi-Laplacian Gaussian field ($s = 2$), the log-correlated Gaussian field ($s = d/2$), Lévy's Brownian motion ($s = d/2+1/2$), and multidimensional fractional Brownian motion ($d/2 < s < d/2+1$). These fields have applications to statistical physics, early-universe cosmology, finance, quantum field theory, image processing, and other disciplines.

We present an overview of fractional Gaussian fields including covariance formulas, Gibbs properties, spherical coordinate decompositions, restrictions to linear subspaces, local set theorems, and other basic results. We also define a discrete fractional Gaussian field and explain how the FGF_s with $s \in (0, 1)$ can be understood as a long range Gaussian free field in which the potential theory of Brownian motion is replaced by that of an isotropic $2s$-stable Lévy process.

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Contents

1. Introduction

The d-dimensional **fractional Gaussian field** h on $\mathbb{R}^d$ with index $s \in \mathbb{R}$ (abbreviated as $\mathrm{FGF}_s(\mathbb{R}^d)$) is given by

$$h := (-\Delta)^{-s/2} W, \tag{1.1}$$

where W is a real white noise on $\mathbb{R}^d$ and $(-\Delta)^{-s/2}$ is the fractional Laplacian on $\mathbb{R}^d$. In Sections 2 and 3, we will review classical and recent literature on the fractional Laplacian (see, e.g., [LD72, Sil07, CSS08, CG11]) and show how to assign rigorous meaning to (1.1).

Our goal is to provide a mathematically rigorous, unified, and accessible account of the $\mathrm{FGF}_s(\mathbb{R}^d)$ processes, treating the full range of values $s \in \mathbb{R}$ and $d \in \mathbb{N}$. This paper is fundamentally a survey, but we also present several basic facts that we have not found articulated elsewhere in the literature. Many of these are generalizations of classical results that had previously only been formulated for specific d and s values.

(a) White Noise, $s = 0$ (b) GFF, $s = 1$

(c) Bi-Laplacian, $s = 2$ (d) FGF$_s$ with $s = 3$
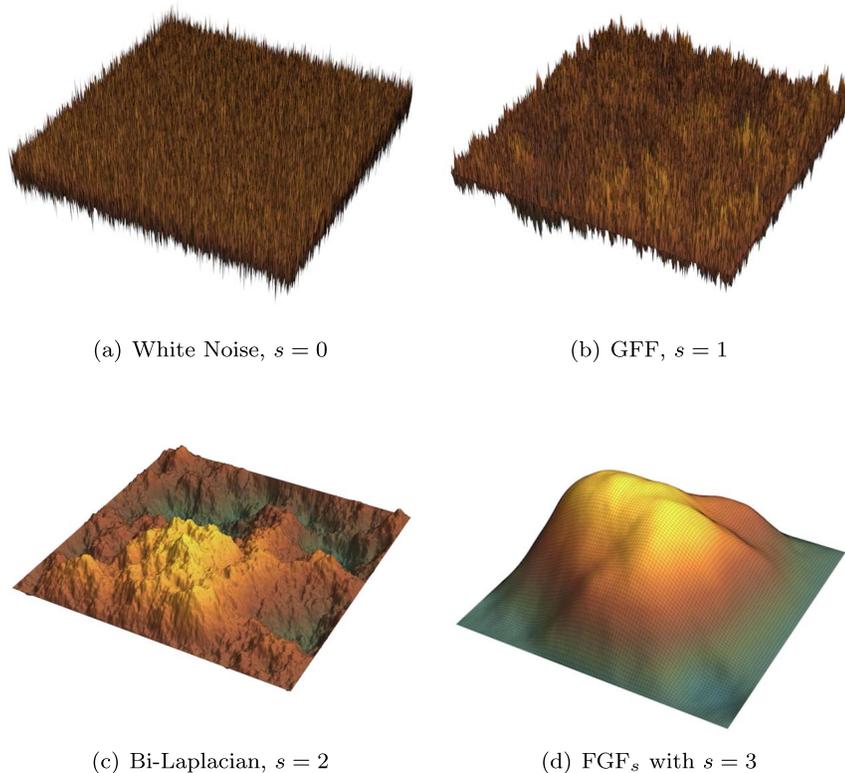
FIG 1.1. *Surface plots of discrete fractional Gaussian fields as defined on a bounded domain $D = [0,1]^2 \subset \mathbb{R}^2$ with zero boundary conditions, where $s = 0$, 1, 2, and 3 respectively. These discrete random functions are defined on a 500×500 grid and linearly interpolated. The corresponding continuum limit, $\mathrm{FGF}_s([0,1]^2)$, is not a function when $s = 0$ or $s = 1$, is α-Hölder continuous for all $\alpha < 1$ when $s = 2$, and has α-Hölder continuous first-order derivatives for all $\alpha < 1$ when $s = 3$.*

We hope that this survey will increase the circulation of basic information about fractional Gaussian fields in the mathematical community. For example, the vocabulary and content of the following statements should arguably be well known to probabilists, but the authors were unaware of much of it until recently:

- In dimension 3, the Gaussian field with logarithmic correlations has been used as an approximate model for the gravitational potential of the early universe; its Laplacian is a $\mathrm{FGF}_{-1/2}(\mathbb{R}^3)$ and has been used to model the perturbation from uniformity of the mass/energy density of the early universe[1].
- In dimension 4, the so-called bi-Laplacian field has logarithmic correlations, and its Laplacian is white noise.

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[1]An overview of this story appears in the reference text [Dod03] and a few additional notes and references appear in [DRSV].

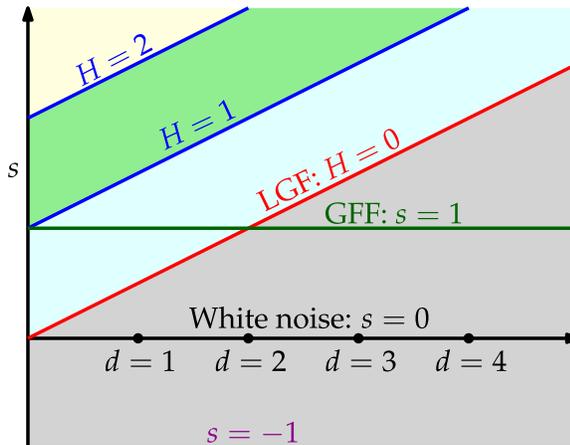FIG 1.2. *When $H < 0$ (grey shaded region) the FGF is defined as a random tempered distri-bution, not a function. When $H \in (0,1)$, the FGF is defined as a random continuous function modulo a global additive constant. Generally, for integers $k > 0$ and $H \in (k, k+1)$, the FGF is a translation invariant random k-times-differentiable function defined modulo polynomials of degree k. For integer $H = k \geq 0$, the FGF is a random $(k-1)$-times-differentiable function (or distribution if $k = 0$) defined modulo polynomials of degree k.*

- In any dimension, Lévy Brownian motion can be defined as a random continuous function whose restriction to any line has the law of a Brownian motion (modulo additive constant). In dimension 5, the Laplacian of Lévy Brownian motion is the Gaussian free field.

We also hope that this text will be a useful reference for experts in the study of Gaussian fields; to this end, we provide a robust account of the regularity of FGF fields, the long and short range correlation formulae, conditional expectations given field values outside of fixed domains, the Fourier transforms and spherical coordinate decompositions of the FGF, and various bounded-domain definitions of the FGF.

The family of fractional Gaussian fields includes several well-known Gaussian fields such as Brownian motion ($d = 1$ and $s = 1$), white noise ($s = 0$), the Gaussian free field ($s = 1$), and the log-correlated Gaussian field ($s = \frac{d}{2}$).

Given $s \in \mathbb{R}$ and $d \geq 1$, the **Hurst parameter** H is defined by

$$H := s - \frac{d}{2}. \tag{1.2}$$

The Hurst parameter describes a scaling relation satisfied by $h \sim \mathrm{FGF}_s(\mathbb{R}^d)$: for $a > 0$, the field $x \mapsto h(ax)$ has the same law as $x \mapsto a^H h(x)$.[2] Fields satisfying such a relation are said to be **self-similar**, and they arise naturally in the study of statistical physics models [New80]. The FGFs belong to a more

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[2]When s and d are such that h is a random tempered distribution, but not a random function, we interpret $x \mapsto h(ax)$ as a distribution via $(x \mapsto h(ax), \phi) = a^{-d}(h, x \mapsto \phi(x/a))$.

general class of translation-invariant self-similar Gaussian random fields which were investigated and classified in [Dob79]. When $d = 1$ and $H \in (0, 1)$, the $\text{FGF}_s(\mathbb{R}^d)$ process is commonly known as **fractional Brownian motion with Hurst parameter** H, and is the subject of an extensive literature (see the survey [CI13]). Brownian motion itself corresponds to $H = 1/2$ and $s = 1$.

The law of a Brownian motion or fractional Brownian motion B_t, indexed by $t \in \mathbb{R}$ and defined so that $B_0 = 0$, is not translation invariant. However, the law of Brownian motion is translation invariant if we consider Brownian motion as a random process defined only modulo a global additive constant. In other words, Brownian motion has stationary increments. Similarly, the indefinite integral of a Brownian motion can be interpreted, in a translation invariant way, as a random function defined modulo the space of linear functions. We generally interpret all of the FGF_s processes as translation invariant random distributions, but in some cases they are defined modulo a space of polynomials. More precisely, when $H < 0$, $\text{FGF}_s(\mathbb{R}^d)$ is a translation invariant random tempered distribution (that is, a generalized function) on $\mathbb{R}^d$. When $H > 0$, $\text{FGF}_s(\mathbb{R}^d)$ is a translation invariant random element of the space $C^{\lceil H \rceil - 1}(\mathbb{R}^d)$ modulo the space of polynomials on $\mathbb{R}^d$ of degree no greater than $\lfloor H \rfloor$. This means that h is defined as a linear functional on the subspace of test functions ϕ satisfying $\int_{\mathbb{R}^d} \phi(x) L(x) dx = 0$ for all polynomials L of degree $\lfloor H \rfloor$. Alternatively, at the cost of breaking translation invariance, we may define $\text{FGF}_s(\mathbb{R}^d)$ as a random element of $C^{\lceil H \rceil - 1}(\mathbb{R}^d)$ by fixing the derivatives of h at 0 up to order $\lceil H \rceil - 1$. The FGF covariance structure is described by the Hurst parameter H. When H is a positive non-integer, we have

$$\text{Cov}[(h, \phi_1), (h, \phi_2)] = C(s, d) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^{2H} \phi_1(x) \phi_2(y) dx dy,$$

for some constant $C(s, d)$. A variant of this statement applies for negative and integer values of H (see Theorem 3.3).

Note that H is an affine function of s and can be used instead of s to parameterize the family of FGFs. We use the parameter s in part to highlight the connection to the fractional Laplacian and white noise. With our convention, white noise is $\text{FGF}_0(\mathbb{R}^d)$ and the Gaussian free field is $\text{FGF}_1(\mathbb{R}^d)$. However, in many of our formulas and theorems H will be the more natural parameter to use; thus, we fix the relationship (1.2) and reference both H and s throughout the paper. We note that the fields $\{\text{FGF}_s(\mathbb{R}^d) : s \in \mathbb{R}\}$ may be coupled with the same white noise so that (1.1) holds for all $s \in \mathbb{R}$ (Proposition 6.3).

1.1. Examples

The simplest example of a fractional Gaussian field is $\text{FGF}_0(\mathbb{R}^d)$, which is white noise. We denote by $\mathcal{S}(\mathbb{R}^d)$ the space of Schwartz functions on $\mathbb{R}^d$, and we let $\mathcal{S}'(\mathbb{R}^d)$ be its dual, the space of tempered distributions (see Section 2 for details). If $h \in \mathcal{S}'(\mathbb{R}^d)$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$, we use the notation (h, ϕ) for h evaluated at ϕ. White noise (surveyed in [Kuo96]) is a random element of $\mathcal{S}'(\mathbb{R}^d)$ with the

property that for $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d)$, the random variables (h, ϕ_1) and (h, ϕ_2) are centered Gaussians with covariance

$$\text{Cov}[(h, \phi_1), (h, \phi_2)] = \int_{\mathbb{R}^d} \phi_1(x)\phi_2(x)\, dx.$$

Taking $d = 1$ and $s = 1$, we see that $(-\Delta)^{-s/2}$ is the antiderivative operator. It follows that $\text{FGF}_1(\mathbb{R})$ is the antiderivative of one dimensional white noise, which is a Brownian motion interpreted as a real-valued function modulo constant. If we fix the constant by setting the value at 0, we get ordinary Brownian motion.

If $s = 1$ and $d \in \mathbb{N}$, then $\text{FGF}_1(\mathbb{R}^d)$ is a d-dimensional generalization of Brownian motion called the Gaussian free field (GFF). As surveyed in [She07], the GFF is a random tempered distribution on $\mathbb{R}^d$ (defined modulo additive constant if $d = 2$) with covariance given by

$$\text{Cov}[(h, \phi_1), (h, \phi_2)] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi(x - y)\phi_1(x)\phi_2(y)\, dx\, dy,$$

where Φ is the fundamental solution of the Laplace equation in $\mathbb{R}^d$. The two-dimensional GFF (which is the same as $\text{FGF}_1(\mathbb{R}^2)$) has been studied in a wide range of contexts in recent years. It can be obtained as a scaling limit of random discrete models, such as domino tilings [Ken01], as well as continuum models, such as those arising in random matrix theory [RV06]. It is central to conformal field theory and Liouville quantum gravity [She10, DS11] and has many connections to the Schramm Loewner evolution [Dub09, SS10, MS12a, MS12b, MS, MS13]. The 2D GFF is also known in the geostatistics literature as the *de Wijs process* or the *logarithmic variogram model*, where it was introduced in the early 1950s to describe ore deposits [dW51, dW53, Mon15, CD09]. More recently, variations in crop yields have been modeled using the GFF [McC02, MC06].

For all $d \in \mathbb{N}$, the d-dimensional GFF exhibits a certain Markov property: For each fixed domain $D \subset \mathbb{R}^d$, if we are given the restriction a GFF h to $\mathbb{R}^d \setminus D$, then the conditional law of h restricted to D is given by a conditionally deterministic function (the harmonic extension of the field from ∂D to D)[3] plus an independent zero-boundary GFF defined on D.

In Section 5 we will establish an analogous property that applies when h is an $\text{FGF}_s(\mathbb{R}^d)$ with $s \geq 0$. Namely, if we are given the restriction of h to $\mathbb{R}^d \setminus D$, then the conditional law of h restricted to D is given by a conditionally deterministic function (the so-called s-**harmonic extension** of the field from $\mathbb{R}^d \setminus D$ to D) plus a random function (the so-called **zero-boundary-condition** FGF_s **on** D). If $s \in \mathbb{N}$, then the conditionally deterministic function depends on the restriction to ∂D of h and its derivatives up to a certain order. This follows from the fact that $(-\Delta)^s$ is a local operator when $s \in \mathbb{N}$.

As previously mentioned, another generalization of Brownian motion is the fractional Brownian motion (FBM). Fractional Brownian motion appears to

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[3]Since the GFF is not defined pointwise, some care is needed to define the harmonic extension of the values of the GFF on $\mathbb{R}^d \setminus D$. Nevertheless, this can be made rigorous [SS10].

have been first introduced by Kolmogorov in 1940 [Kol40], and the term "fractional Brownian motion" was introduced by Mandelbrot and Van Ness in 1968 [MVN68]. As motivation, Mandelbrot and Van Ness discuss various empirical studies of real world processes (the price of wheat, water flowing through the Nile, etc.) that had been made by Hurst, who found different scaling exponents in different settings[4].

The definition of fractional Brownian motion can be extended to describe a random function modulo additive constant on $\mathbb{R}^d$ when $d > 1$. Given $H \in (0, 1)$ we define the FBM (also called the fractional Brownian field) on $\mathbb{R}^d$ as a mean-zero Gaussian process $(B_t^H)_{t \in \mathbb{R}}$ with covariance

$$\mathrm{Cov}(B_t^H B_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}),$$

where H is the Hurst parameter of the field. We will prove in Section 6 that the multidimensional fractional Brownian motion defined this way is equivalent to $\mathrm{FGF}_s(\mathbb{R}^d)$, where $H = s - \frac{d}{2} \in (0, 1)$.

In the case $H = 1/2$, this multidimensional process was introduced by Lévy in 1940 and is known as Lévy Brownian motion [Lév40]. General processes including multidimensional fractional Brownian motion are discussed in Yaglom in 1957 and by Gangolli in 1967 [Yag57, Gan67]. (Gangolli gives general analytic arguments for positive definiteness of covariance kernels that apply in this case.) Fractional Brownian motion is studied in more detail in works of Mandelbrot, as referenced in [Man75]. More detailed and modern discussions of fractional Brownian motion (including topics such as excursion set theory, Hausdorff dimension, Hölder regularity, etc.) can be found in [AT07, Adl10].

The log-correlated Gaussian field (LGF) is a random element h of the space of tempered distributions modulo constants and has covariance given by

$$\mathrm{Cov}[(h, \phi_1), (h, \phi_2)] = -\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \log|x - y| \phi_1(x)\phi_2(y) \, dx \, dy,$$

In two dimensions, the LGF coincides with the GFF (up to a constant factor). We will see in Section 3 that the d-dimensional LGF is a multiple of $\mathrm{FGF}_{d/2}(\mathbb{R}^d)$. In recent years the log-correlated Gaussian field has enjoyed renewed interest because of its relationship to Gaussian multiplicative chaos. For a survey article of Gaussian multiplicative chaos see [RV13]. Furthermore, the LGF in $\mathbb{R}^3$ plays an important role in early universe cosmology, where it approximately describes the gravitational potential function of the universe at a fixed time shortly after the big bang; see [DRSV] for more discussion and references.

Another noteworthy subclass of the fractional Gaussian fields is $\mathrm{FGF}_2(\mathbb{R}^d)$, which is known as the bi-Laplacian Gaussian field. The discrete counterpart of the bi-Laplacian Gaussian field is called the membrane model in physics literature; for a mathematical point of view see [Sak03], [Kur07], [Kur09], and [Sak12]. In dimension at least five, there is a natural discrete field associated with the uniform spanning forest on $\mathbb{Z}^d$ whose scaling limit is $\mathrm{FGF}_2(\mathbb{R}^d)$ [SW13].

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[4]FBM is not the only model exhibiting the scaling behavior observed by Hurst. See [BGW83] for a model which uses drift rather than long-range dependence.

1.2. Fractional Gaussian fields in one dimension

The $\mathrm{FGF}_s(\mathbb{R}^d)$ processes are easiest to classify and explain when $d = 1$. We first consider $H = s - \frac{d}{2} \in (0, 1)$ (so that $s \in (1/2, 3/2)$), in which case the $\mathrm{FGF}_s(\mathbb{R})$ is a Gaussian random function $h : \mathbb{R} \to \mathbb{R}$ which we interpret as being defined modulo an additive constant. This means that while the quantity $h(t)$ is not a well-defined random variable for $t \in \mathbb{R}$, the quantity $h(t_1) - h(t_2)$ is a well-defined random variable for $t_1, t_2 \in \mathbb{R}$. When $H \in (0, 1)$, the $\mathrm{FGF}_s(\mathbb{R})$ is the stationary-increment form of the **fractional Brownian motion with Hurst parameter** H. The law of the fractional Brownian motion is determined by the variance formula
$$\mathrm{Var}\big(h(t_1) - h(t_2)\big) = |t_1 - t_2|^H.$$
When $H = 0$, so that $s = 1/2$, the $\mathrm{FGF}_s(\mathbb{R})$ is the log-correlated Gaussian field (LGF), which is defined as a random tempered distribution modulo additive constant.

When $d = 1$ the weak derivative of an $\mathrm{FGF}_s(\mathbb{R})$ is an $\mathrm{FGF}_{s-1}(\mathbb{R})$. Thus all $\mathrm{FGF}_s(\mathbb{R})$ processes may be obtained by either integrating or differentiating fractional Brownian motion (with $s \in (1/2, 3/2)$) or the LGF ($s = 1/2$) an integral number of times. From this, it is clear that if an $\mathrm{FGF}_s(\mathbb{R})$, for $s \in (1/2, 3/2]$, is defined modulo additive constant in a translation invariant way, then the distributional derivatives $\mathrm{FGF}_{s-1}(\mathbb{R})$, $\mathrm{FGF}_{s-2}(\mathbb{R})$, etc. are defined without an additive constant. Thus the $\mathrm{FGF}_s(\mathbb{R})$ is defined as a random tempered distribution without an additive constant when $s \leq 1/2$. Similarly, if the $\mathrm{FGF}_s(\mathbb{R})$, for $s \in (1/2, 3/2]$ is defined modulo additive constant (in a translation invariant way), then the indefinite integrals $\mathrm{FGF}_{s+1}(\mathbb{R})$, $\mathrm{FGF}_{s+2}(\mathbb{R})$, etc. are respectively defined modulo linear polynomials, quadratic polynomials, etc.

The following proposition, rephrased and proved as Theorem 7.1 in Section 7, is one reason that the one-dimensional case is significant.

Proposition 1.1. *If $H \geq 0$, then the restriction of the d-dimensional FGF with Hurst parameter H (i.e., with $s = H + \frac{d}{2}$) to any fixed k-dimensional subspace (with $1 \leq k < d$) is a k-dimensional FGF with Hurst parameter H (up to multiplicative constant).*

1.3. Interpretation as a long range GFF

The Gaussian free field $\mathrm{FGF}_1(\mathbb{R}^d)$ can be approximated by the discrete Gaussian free field, which only has nearest neighbor interactions. This discrete Markov property gives rise to the domain Markov property of the Gaussian free field in the limit [She07]. In Section 12, we construct a discrete version of FGF_s for $s \in (0, 1)$ by introducing a discrete fractional gradient to play the role of the discrete gradient in the definition of the discrete GFF. The fractional gradient involves long range interactions, which may be viewed as the reason that the Markov property fails for FGF_s when s is not an integer.

The comparison between the short range $\mathrm{FGF}_s(\mathbb{R}^d)$ (when $s \in \mathbb{Z}$) and the long range $\mathrm{FGF}_s(\mathbb{R}^d)$ (when $s \notin \mathbb{Z}$) may also be seen from the point of view

of the corresponding potential theories. As an illustration, consider GFF and FGF_s for $0 < s < 1$. The covariance kernel for the Gaussian free field is given by the solution of the ordinary Laplace equation $-\Delta f = \phi$. As we will see, the counterpart for FGF_s with $0 < s < 1$ is the fractional Laplacian equation $(-\Delta)^s f = \phi$. The Laplacian is a local differential operator, while $(-\Delta)^s$ for $s \in (0, 1)$ is a non-local pseudo-differential operator and $(-\Delta)^s f(x)$ depends on the values of $f(x)$ for all $x \in \mathbb{R}^d$. Another way to see the distinction between the $s = 1$ and $s \in (0, 1)$ cases is to recall that the Green's function for the Dirichlet Laplacian is given by the density of the occupation measure of a Brownian motion (see [MP10], for example), which is continuous. The corresponding process when $s \in (0, 1)$ is an isotropic $2s$-stable Lévy motion, which is a jump process.

2. Preliminaries

In this section we remind the reader of some definitions and facts regarding tempered distributions and homogeneous Sobolev spaces. Some of the following notation and ideas are from [Tri83], to which we refer the reader for more discussion on homogeneous spaces. We will introduce and construct several linear spaces. To aid the reader in keeping track of the various definitions, we include a glossary of these definitions in the appendix on page 51.

2.1. Tempered distributions and Sobolev spaces

Fix a positive integer d, and denote by $\mathcal{S}(\mathbb{R}^d)$ the real Schwartz space, defined to be the set of real-valued functions on $\mathbb{R}^d$ whose derivatives of all orders exist and decay faster than any polynomial at infinity. A multi-index $\beta = (\beta_1, \ldots, \beta_d)$ is an ordered d-tuple of nonnegative integers, and the order of β is defined to be $|\beta| := \sum_{j=1}^d \beta_j$. We equip $\mathcal{S}(\mathbb{R}^d)$ with the topology generated by the family of seminorms

$$\left\{ \|f\|_{n,\beta} := \sup_{x \in \mathbb{R}^d} |x|^n |\partial^\beta f(x)| \, : \, n \geq 0, \ \beta \text{ is a multi-index} \right\}.$$

The space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions is defined to be the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$.

We take the convention that the Fourier transform $\mathcal{F}$ acting on a Schwartz function ϕ on $\mathbb{R}^d$ is the function

$$\mathcal{F}[\phi](\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \phi(x) e^{-i\xi \cdot x} \, dx$$

which we will often abbreviate as $\widehat{\phi}(\xi)$. The complex Schwartz space (the space of functions whose real and imaginary parts are in $\mathcal{S}(\mathbb{R}^d)$) is closed under the operation of taking the Fourier transform [Tao10, Section 1.13], so the inverse

Fourier transform $\mathcal{F}^{-1}$ is well-defined on the complex Schwartz space and satisfies the formula

$$\mathcal{F}^{-1}[\phi](x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \phi(\xi) e^{ix\cdot\xi} \, d\xi.$$

We define the Fourier transform $\widehat{f}$ of a tempered distribution f by setting $(\widehat{f}, \phi) := (f, \widehat{\phi})$, so that $\mathcal{F}$ and $\mathcal{F}^{-1}$ may be interpreted as operators from $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. Regarding $\phi \in \mathcal{S}(\mathbb{R}^d)$ as a tempered distribution via $\phi(\psi) := \int_{\mathbb{R}^d} \phi(x)\,\psi(x)\,dx$, we have the continuous, dense inclusion $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$. For the fundamentals of the theory of distributions, we refer the reader to [Lax02, Appendix B] or [Tao10]. For a more detailed introduction to distribution theory we refer to [FJ98] and [Hör03].

For $r \in \mathbb{R}$, define $\mathcal{S}_r(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ to be the set of Schwartz functions ϕ such that $(\partial^\alpha \widehat{\phi})(0) = 0$ (or, equivalently, $\int_{\mathbb{R}^d} x^\alpha \phi(x)\,dx = 0$) for all multi-indices α such that $|\alpha| \leq r$. We equip $\mathcal{S}_r(\mathbb{R}^d)$ with the subspace topology inherited from $\mathcal{S}(\mathbb{R}^d)$ and denote by $\mathcal{S}'_r(\mathbb{R}^d)$ the topological dual of $\mathcal{S}_r(\mathbb{R}^d)$. Observe that $\mathcal{S}'_r(\mathbb{R}^d)$ is canonically isomorphic to $\mathcal{S}'(\mathbb{R}^d)/\mathcal{T}_r(\mathbb{R}^d)$, where $\mathcal{T}_r(\mathbb{R}^d)$ denotes the space of polynomials of degree at most r on $\mathbb{R}^d$. Observe also that $\mathcal{S}_r(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)$ whenever r is negative, and that $\mathcal{S}_0(\mathbb{R}^d) = \{\phi \in \mathcal{S}(\mathbb{R}^d) \,:\, \int_{\mathbb{R}^d} \phi(x)\,dx = 0\}$.

Given $r \in \mathbb{R}$, we also consider the space

$$\widetilde{\mathcal{S}}_r(\mathbb{R}^d) = \{\phi \in \mathcal{S}(\mathbb{R}^d) \,:\, (\partial^\alpha \phi)(0) = 0 \text{ for all } |\alpha| \leq r\},$$

which is equal to the image of $\mathcal{S}_r(\mathbb{R}^d)$ under the inverse Fourier transform operator. We define the Fourier transform of an element of $\mathcal{S}'_r(\mathbb{R}^d)$ as an element of $\widetilde{\mathcal{S}}'_r(\mathbb{R}^d)$ via $(\widehat{f}, \phi) := (f, \widehat{\phi})$ whenever $f \in \mathcal{S}'_r(\mathbb{R}^d)$ and $\phi \in \widetilde{\mathcal{S}}_r(\mathbb{R}^d)$.

Define the space

$$\mathring{H}^s(\mathbb{R}^d) := \left\{ f \in \mathcal{S}(\mathbb{R}^d) \,:\, \xi \mapsto |\xi|^s \widehat{f}(\xi) \in L^2(\mathbb{R}^d) \right\}$$

and equip $\mathring{H}^s(\mathbb{R}^d)$ with the inner product

$$(f,g)_{\mathring{H}^s(\mathbb{R}^d)} := \left(\xi \mapsto |\xi|^s \widehat{f}(\xi), \xi \mapsto |\xi|^s \overline{\widehat{g}(\xi)} \right)_{L^2(\mathbb{R}^d)}.$$

We define the Sobolev space $\dot{H}^s(\mathbb{R}^d)$ to be the Hilbert space completion of $\mathring{H}^s(\mathbb{R}^d)$, which we continuously embed in $\mathcal{S}'_H(\mathbb{R}^d)$ as follows. If $\{f_n\}_{n\geq 1}$ is a Cauchy sequence in $\dot{H}^s(\mathbb{R}^d)$ and $\phi \in \mathcal{S}_H(\mathbb{R}^d)$, then by Plancherel and Cauchy-Schwarz we have

$$|(f_m - f_n, \phi)_{L^2(\mathbb{R}^d)}| \leq \tag{2.1}$$
$$\left(\int |\widehat{f}_m(\xi) - \widehat{f}_n(\xi)|^2 |\xi|^{2s}\,d\xi \right)^{1/2} \left(\int |\widehat{\phi}(\xi)||\xi|^{-2s}\,d\xi \right)^{1/2}.$$

The first factor on the right-hand side tends to 0 as $\min(m,n) \to \infty$ and the second factor is finite since $\phi \in \mathcal{S}_H(\mathbb{R}^d)$. It follows that $(f_m - f_n, \phi)_{L^2(\mathbb{R}^d)}$ is

Cauchy in $\mathbb{R}$, which implies that we can define a linear map $f : \mathcal{S}_H(\mathbb{R}^d) \to \mathbb{R}$ by $(f, \phi) := \lim_{n \to \infty} (f_n, \phi)$ for all $\phi \in \mathcal{S}_H(\mathbb{R}^d)$. Observing that $\phi_k \to 0$ in $\mathcal{S}_H(\mathbb{R}^d)$ implies

$$\limsup_{k \to \infty} |(f, \phi_k)|^2 \leq$$

$$\limsup_{k \to \infty} \limsup_{n \to \infty} \int |\widehat{f_n}(\xi)|^2 |\xi|^{2s} \, d\xi \times \int |\widehat{\phi_k}(\xi)|^2 |\xi|^{-2s} \, d\xi = 0,$$

we conclude that f is a continuous functional on $\mathcal{S}_H(\mathbb{R}^d)$. Therefore, we may realize $\dot{H}^s(\mathbb{R}^d)$ as a subset of $\mathcal{S}'_H(\mathbb{R}^d)$ by identifying each Cauchy sequence $\{f_n\}_{n \geq 1}$ with its $\dot{H}^s(\mathbb{R}^d)$-limit $f \in \mathcal{S}'_H(\mathbb{R}^d)$.

We can characterize $\dot{H}^s(\mathbb{R}^d)$ in another way which will be useful for the following section. Note that if $(\phi_n)_{n \in \mathbb{N}}$ is an $\dot{H}^s(\mathbb{R}^d)$-Cauchy sequence of Schwartz functions converging to f in $\mathcal{S}'_H(\mathbb{R}^d)$, then $(\widehat{\phi}_n)_{n \in \mathbb{N}}$ is Cauchy in $L^2(\mathbb{R}^d, |\xi|^{2s} \, d\xi)$, where $|\xi|^{2s} \, d\xi$ denotes the measure whose density with respect to Lebesgue measure is $\xi \mapsto |\xi|^{2s}$. Therefore, there exists $g \in L^2(\mathbb{R}^d, |\xi|^{2s} \, d\xi)$ to which $\widehat{\phi}_n$ converges with respect to the $L^2(\mathbb{R}^d, |\xi|^{2s} \, d\xi)$ norm. Furthermore, it is straightforward to verify using the Cauchy-Schwartz inequality that $g = \widehat{f} \in \widetilde{\mathcal{S}}'_H(\mathbb{R}^d)$. Therefore,

$$\dot{H}^s(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'_H(\mathbb{R}^d) \, : \, \widehat{f} \in L^2(|\xi|^{2s} \, d\xi) \right\},$$

where $\widehat{f} \in L^2(|\xi|^{2s} \, d\xi)$ means that there exists $g \in L^2(|\xi|^{2s} \, d\xi)$ such that $(\widehat{f}, \phi) = \int_{\mathbb{R}^d} g(x)\phi(x) \, dx$ for all $\phi \in \widetilde{\mathcal{S}}_H(\mathbb{R}^d)$.

2.2. The fractional Laplacian

The fractional Laplacian generalizes the notion of a power $(-\Delta)^s$ of the Laplacian from nonnegative integer values of s, for which it is defined as a local operator by iterating the Laplacian, to all real values of s. A standard reference for the fractional Laplacian is [LD72]. Here we use ideas from Section 2 of [Sil07]. Let $k \in \{-1, 0, 1, 2, \ldots\}$, and let $\phi \in \mathcal{S}_k(\mathbb{R}^d)$. If $s > -\frac{1}{2}(d + k + 1)$, then we set

$$(-\Delta)^s \phi := \mathcal{F}^{-1}\left[\xi \mapsto |\xi|^{2s} \widehat{\phi}(\xi) \right], \tag{2.2}$$

which is well-defined because $\xi \mapsto |\xi|^{2s} \widehat{\phi}(\xi)$ is in $L^1(\mathbb{R}^d)$. Note that (2.2) agrees with the local definition of $-\Delta$ when $s = 1$. Because of the singularity at the origin in its Fourier transform, $(-\Delta)^s \phi$ is not necessarily Schwartz. However, it is real-valued, smooth, and has polynomial decay at infinity:

Proposition 2.1. *Let $k \in \{-1, 0, 1, 2, \ldots\}$, $\phi \in \mathcal{S}_k(\mathbb{R}^d)$, and $s > -\frac{1}{2}(d+k+1)$. If α is a multi-index, then there exists a constant C such that $\phi \in \mathcal{S}_k(\mathbb{R}^d)$ implies*

$$\sup_{x \in \mathbb{R}^d} (1 + |x|^{d+2s+k+1}) |\partial^\alpha (-\Delta)^s \phi(x)| \leq C \sup_{|\beta| \leq \max(k+1, |\alpha|)} \|\partial^\beta \phi\|_{L^\infty(\mathbb{R}^d)}. \tag{2.3}$$

Furthermore, $(-\Delta)^s \phi$ is real-valued and smooth.

Proof. The proof of smoothness is routine: we write the inverse Fourier transform using its definition and differentiate under the integral sign. To show that $(-\Delta)^s\phi$ is real-valued, we note that a function $\psi \in L^1(\mathbb{R}^d)$ is the Fourier transform of a real-valued function in $L^1(\mathbb{R}^d)$ if and only if $\psi(-\xi)$ and $\psi(\xi)$ are complex conjugates for all $\xi \in \mathbb{R}^d$. The function $\xi \mapsto |\xi|^{-2s}\widehat{\phi}(\xi)$ has this property whenever $\widehat{\phi}$ does, so we may conclude that $(-\Delta)^s\phi$ is real-valued.

To prove (2.3), let $\{f_1, f_2\}$ be a partition of unity subordinate to the open cover $\{\mathbb{R}^d \setminus B(0, \frac{1}{|x|}), B(0, \frac{2}{|x|})\}$ of $\mathbb{R}^d$, and define $\phi_i(\xi) = f_i(\xi)\widehat{\phi}(\xi)/|\xi|^{k+1}$ for $i \in \{1, 2\}$. We calculate

$$\partial^\alpha(-\Delta)^s\phi(x) = C \int_{\mathbb{R}^d} e^{ix\cdot\xi}\xi^\alpha|\xi|^{2s+k+1}\widehat{\phi}(\xi)/|\xi|^{k+1} \, d\xi$$

$$= C \int_{\mathbb{R}^d \setminus B(0,1/|x|)} e^{ix\cdot\xi}\xi^\alpha|\xi|^{2s+k+1}\phi_1(\xi) \, d\xi +$$

$$C \int_{B(0,2/|x|)} e^{ix\cdot\xi}\xi^\alpha|\xi|^{2s+k+1}\phi_2(\xi) \, d\xi,$$

where C is some constant. To obtain the desired bound for the first integral, we write the integral in spherical form and apply integration-by-parts with respect to the radial coordinate. For the second integral, we bound $\phi_2(x)$ by a constant times $\sup_{|\beta|=k+1} |\partial^\beta\phi(0)||\xi|^{-k-1}$ near the origin, using Taylor's theorem. □

For $s > -d/2$, we define[5] the space $\mathcal{U}_s(\mathbb{R}^d)$ to be the space of all functions $\phi \in C^\infty(\mathbb{R}^d)$ such that

$$x \mapsto (1 + |x|^{d+2s})(\partial^\alpha f)(x)$$

is bounded for all multi-indices α. These spaces interpolate between $C^\infty(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$, as the derivatives of their elements decay polynomially at a rate indexed by s. In particular, $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{U}_s(\mathbb{R}^d) \subset \mathcal{U}_{s'}(\mathbb{R}^d)$ whenever $s > s'$. We equip $\mathcal{U}_s(\mathbb{R}^d)$ with the topology induced by the family of seminorms $f \mapsto \sup_{x \in \mathbb{R}^d} |(1 + |x|^{d+2s})(\partial^\alpha f)(x)|$. By Proposition 2.1, $(-\Delta)^s$ is a continuous map from $\mathcal{S}_k(\mathbb{R}^d)$ to $\mathcal{U}_{s+(k+1)/2}(\mathbb{R}^d)$. Furthermore, $(-\Delta)^s\phi = 0$ for $\phi \in \mathcal{S}_k(\mathbb{R}^d)$ implies that $\widehat{\phi}$ vanishes except possibly at the origin. This implies that ϕ is a polynomial, which in turn implies that $\phi = 0$. Therefore, $(-\Delta)^s$ is injective. For all f in the topological dual of the image $(-\Delta)^s\mathcal{S}_k(\mathbb{R}^d) \subset \mathcal{U}_{s+(k+1)/2}$, we define $(-\Delta)^sf \in \mathcal{S}'_k(\mathbb{R}^d)$ by

$$((-\Delta)^sf, \phi) = (f, (-\Delta)^s\phi),$$

It is straightforward to verify that this definition agrees with (2.2) when $f \in \mathcal{S}(\mathbb{R}^d)$. Observe that $(-\Delta)^{s_1}(-\Delta)^{s_2} = (-\Delta)^{s_1+s_2}$ for all $s_1, s_2 \in \mathbb{R}$. We will consider two important examples of elements of $((-\Delta)^s\mathcal{S}_k(\mathbb{R}^d))'$:

(i) Elements of homogeneous Sobolev spaces. Let $s \in \mathbb{R}$ and $H = s - d/2$. It is straightforward to verify that $f \in \dot{H}^s(\mathbb{R}^d)$ determines an element of $((-\Delta)^s\mathcal{S}_H(\mathbb{R}^d))'$ via $(f, \phi) := (\widehat{f}, \widehat{\phi})$. Furthermore, the definition of $(-\Delta)^sf$

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[5]These spaces are denoted $\overline{\mathcal{S}}_s(\mathbb{R}^d)$ in [Sil07].

arising from this correspondence satisfies $\widehat{(-\Delta)^s}f(\xi) = |\xi|^{2s}\widehat{f}(\xi)$. It follows that $(-\Delta)^s$ is an isometric isomorphism from $\dot{H}^{s_0}(\mathbb{R}^d)$ to $\dot{H}^{s_0-2s}(\mathbb{R}^d)$.

(ii) Measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ satisfying

$$\int_{\mathbb{R}^d} |f(x)|(1+|x|^{d+2s+k+1})^{-1}\,dx < \infty. \tag{2.4}$$

Interpreting f as a linear functional on $(-\Delta)^s\mathcal{S}_k(\mathbb{R}^d)$ by integration against a test function, the continuity of f with respect the $\mathcal{U}_{s+(k+1)/2}(\mathbb{R}^d)$ topology follows from Proposition 2.1.

The following proposition gives an alternative representation of the fractional Laplacian in the case $0 < s < 1$.

Proposition 2.2. *For all $f \in \mathcal{S}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, and $s \in (0,1)$, we have*

$$(-\Delta)^s f(x) = -\frac{1}{2}C(d,s)\int_{\mathbb{R}^d} \frac{f(x+y) - 2f(x) + f(x-y)}{|y|^{d+2s}}\,dy,$$

where $1/C(d,s) = \int_{\mathbb{R}^d}(1 - \cos x_1)|x|^{-d-2s}\,dx$.

Proof. Combine Lemma 3.2 and Proposition 3.3 in [DNPV]. $\qquad\square$

When $s \in \{0,1,2,\ldots\}$, the fractional Laplacian $(-\Delta)^s$ coincides with the poly-Laplacian, a fundamental example of a higher order elliptic operator, obtained by iterating the Laplacian operator. For $s \in (0,1)$, $(-\Delta)^s$ is a classical example of a non-local pseudo-differential operator. These two classes generate all the operators of the form $(-\Delta)^s$ for $s \geq 0$ in the sense that $(-\Delta)^s$ can be written as a composition of $(-\Delta)^{s-\lfloor s \rfloor}$ and $(-\Delta)^{\lfloor s \rfloor}$.

For the properties of the poly-Laplacian, we refer the reader to [GGS10] and references therein. For more properties of $(-\Delta)^s$ where $s \in (0,1)$, see [Sil07] and reference therein.

2.3. White noise

On a finite dimensional Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)_{\mathcal{H}}$, one characterization of standard Gaussian h on $\mathcal{H}$ is that h is a standard Gaussian in $\mathcal{H}$ if and only if for all $v \in \mathcal{H}$, $(h,v)_{\mathcal{H}}$ is a centered Gaussian variable with variance $(v,v)_{\mathcal{H}}$. If $\mathcal{H}$ is infinite dimensional, then it is not possible to define a random element of $\mathcal{H}$ that satisfies this condition [Jan97, She07]. Nevertheless, we can still say that a random functional (which we will denote by $(h,\cdot)_{\mathcal{H}}$) is a standard Gaussian on $\mathcal{H}$ if for all $v \in \mathcal{H}$, $(h,v)_{\mathcal{H}}$ is a centered Gaussian variable with variance $(v,v)_{\mathcal{H}}$. Note that such a functional cannot be almost surely continuous with respect to $\|\cdot\|_{\mathcal{H}}$.

White noise on $\mathbb{R}^d$ can be regarded as a standard Gaussian on $L^2(\mathbb{R}^d)$. We will define W to be a random generalized function such that (W,f) is a centered Gaussian with variance $\|f\|^2_{L^2(\mathbb{R}^d)}$ for all $f \in \mathcal{S}(\mathbb{R}^d)$. However, it is not obvious that there exists a measure on $\mathcal{S}'(\mathbb{R}^d)$ satisfying these conditions. Since we will

rigorously construct the FGF in Section 3.1 in the same manner, we will review
a construction of white noise following [Sim79].

We say that a complex-valued function Φ on $\mathcal{S}(\mathbb{R}^d)$ is the characteristic func-
tion of a probability measure ν on $\mathcal{S}'(\mathbb{R}^d)$ if

$$\Phi(\phi) = \int_{\mathcal{S}'(\mathbb{R}^d)} e^{i(x,\phi)} \, d\nu(x), \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^d). \tag{2.5}$$

Theorem 2.3 (Bochner-Minlos theorem for $\mathcal{S}'(\mathbb{R}^d)$). *A complex-valued function
Φ on $\mathcal{S}(\mathbb{R}^d)$ is the characteristic function of a probability measure ν on $\mathcal{S}'(\mathbb{R}^d)$
if and only if $\Phi(0) = 1$, Φ is continuous, and Φ is positive definite, that is,*

$$\sum_{j,k=1}^{n} z_j \overline{z_k} \Phi(\phi_j - \phi_k) \geq 0,$$

*for all $\phi_1, \ldots, \phi_n \in \mathcal{S}(\mathbb{R}^d)$, and $z_1, \ldots, z_n \in \mathbb{C}$. Furthermore, Φ determines ν
uniquely.*

Proof. We briefly sketch the proof given in [Sim79, Theorem 2.3] for the case
$d = 1$; the case $d > 1$ may be proved similarly. We introduce coordinates to
the space $\mathcal{S}(\mathbb{R})$ by writing each function $\phi \in \mathcal{S}(\mathbb{R})$ as $\phi = \sum_{n=1}^{\infty} (\phi, \phi_n)_{L^2(\mathbb{R})} \phi_n$,
where $\{\phi_n\}_{n=0}^{\infty}$ is the Hermite basis of $L^2(\mathbb{R})$ defined by

$$\phi_n(x) = \frac{(-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}[e^{-x^2}]}{\pi^{1/4}\sqrt{2^n n!}}.$$

Identifying $\phi \in \mathcal{S}(\mathbb{R})$ with $\{(\phi, \phi_n)_{L^2(\mathbb{R})}\}_{n=0}^{\infty}$ and using the fact that ϕ_n is an
eigenfunction of $-\frac{d^2}{dx^2} + x^2$, we find that $\mathcal{S}(\mathbb{R})$ is isomorphic to the sequence
space

$$s = \bigcap_{m \in \mathbb{Z}} \left\{ x \in \mathbb{R}^{\mathbb{N}_0} \, : \, \sum_n (1 + n^2)^m |x_n| =: \|x\|_m < \infty \right\},$$

and the topology of $\mathcal{S}(\mathbb{R})$ is equivalent to the one induced by the family of
seminorms $\| \cdot \|_m$. Furthermore, $\mathcal{S}'(R^d)$ is isomorphic to

$$s' = \bigcup_{m \in \mathbb{Z}} \left\{ x \in \mathbb{R}^{\mathbb{N}_0} \, : \, \|x\|_m < \infty \right\}$$

if we interpret a sequence x as a linear functional L_x via $L_x(y) = \sum_{n=0}^{\infty} x_n y_n$.

Bochner's theorem states that characteristic functions of $\mathbb{R}^n$-valued random
variables are in one-to-one correspondence with normalized, continuous, positive
definite functions on $\mathbb{R}^n$. Using Bochner's theorem, we conclude for all $n \in \mathbb{N}_0$,
there is a measure μ_n on $\text{span}(\phi_1, \ldots, \phi_n)$ such that

$$\Phi(\phi) = \int e^{i(x,\phi)} \, d\mu_n(x) \quad \text{for all } \phi \in \text{span}(\phi_1, \ldots, \phi_n).$$

By the uniqueness part of Bochner's theorem, these measure are consistent. By the Kolmogorov extension theorem, there exists a measure μ on $\mathbb{R}^{\mathbb{N}_0}$ such that (2.5) holds for all ϕ in the linear span of $\{\phi_n\}_{n=0}^{\infty}$ (that is, when at most finitely many of ϕ's coordinates are nonzero). It may be shown using the continuity of Φ that $\mu(s') = 1$ (see [Sim79] for details), which allows us to restrict μ to obtain a probability measure on s' and conclude that (2.5) holds for all $\phi \in \mathcal{S}(\mathbb{R})$. $\square$

We will use Theorem 2.3 in conjunction with the following proposition, which gives sufficient conditions for a functional to be positive definite.

Proposition 2.4. *Let* $(S, (\cdot, \cdot))$ *be an inner product space. Then the functional* $\Phi : S \to \mathbb{R}$ *defined by* $\Phi(v) := \exp(-\frac{1}{2}(v, v))$ *is positive definite.*

Proof. Let $v_1, \ldots v_n$ be elements of S, and choose an orthonormal basis $e_1, \ldots, e_m$ of the span of $\{v_1, \ldots v_n\}$. Let $Z = (Z_1, \ldots, Z_m)$ be a vector of independent standard normal real random variables, and note that for all $u \in \mathbb{R}^m$, we have

$$\Phi\left(\sum_{j=1}^{m} u_j e_j\right) = \exp\left(-\frac{1}{2}\sum_{i=1}^{m} u_i^2\right) = \mathbb{E}[e^{iu \cdot Z}],$$

which implies that

$$\sum_{j,k=1}^{n} z_j \overline{z_k} \Phi(v_j - v_k) = \sum_{j,k=1}^{n} z_j \overline{z_k} \mathbb{E}[e^{i(v_j - v_k)\cdot Z}] = \mathbb{E}\left|\sum_{j=1}^{n} z_j e^{iv_j \cdot Z}\right|^2 \geq 0,$$

as desired. $\square$

We will apply Theorem 2.3 to construct a measure μ on $\mathcal{S}'(\mathbb{R}^d)$ which we will refer to as white noise W. Recall that $\mathcal{S}(\mathbb{R}^d)$ is a nuclear space and let us define the functional

$$\Phi_0(\phi) = \exp\left(-\frac{1}{2}\|\phi\|_{L^2(\mathbb{R}^d)}^2\right), \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^d).$$

By Proposition 2.4, this functional is positive definite. Since it is also continuous and satisfies $\Phi_0(0) = 1$, Theorem 2.3 implies that there is a unique probability measure on $\mathcal{S}'(\mathbb{R})$ having Φ_0 as its characteristic function, which we define as white noise W. In particular we have the relation

$$\int_{\mathcal{S}'(\mathbb{R}^d)} e^{i(x,\phi)} d\mu(x) = \exp\left(-\frac{1}{2}\|\phi\|_{L^2(\mathbb{R}^d)}^2\right), \quad \phi \in \mathcal{S}(\mathbb{R}^d),$$

which implies for every $f \in \mathcal{S}(\mathbb{R}^d)$ the random variable (W, f) is a centered Gaussian with variance $\|f\|_{L^2(\mathbb{R}^d)}^2$.

An alternative to the preceding view of white noise as a random tempered distribution is to regard white noise as a collection of random variables $\{(W, f) : f \in \mathcal{S}(R^d)\}$. The advantage of this perspective is that we may extend this collection so that (W, f) is a well-defined random variable for all $f \in L^2(\mathbb{R}^d)$. However, in this construction $f \mapsto (W, f)$ is no longer almost surely continuous. Recall the following definition from [Jan97] or [She07].

Definition 2.5. A **Gaussian Hilbert space** is a collection of Gaussian random variables on a common probability space $(\Omega, \mathcal{F}, \mu)$ which is equipped with the $L^2(\Omega, \mathcal{F}, \mu)$ inner product and is closed with respect to the norm of the $L^2(\Omega, \mathcal{F}, \mu)$ inner product.

To define a Gaussian Hilbert space $\{(W, f) : f \in L^2(\mathbb{R}^d)\}$ where W is a white noise, we consider the map from $\mathcal{S}(\mathbb{R}^d)$ to $L^2(\Omega)$ which sends $\phi \in \mathcal{S}(\mathbb{R}^d)$ to the random variable (W, ϕ) (here Ω denotes the underlying probability space). Since $\mathbb{E}[(W, \phi)^2] = \|\phi\|^2_{L^2(\mathbb{R}^d)}$, this map is an isometry. Since $L^2(\Omega)$ is complete, we may extend this isometry to an operator from $L^2(\mathbb{R}^d)$ to $L^2(\Omega)$ by defining $(W, f) := \lim_{n \to \infty}(W, \phi_n)$ where $\phi_n \in \mathcal{S}(\mathbb{R}^d)$ and $\phi_n \to f$ in $L^2(R^d)$ as $n \to \infty$. Since $\mathbb{E}[e^{i\xi(h, \phi_n)}] \to \mathbb{E}[e^{i\xi(h, \phi)}]$ by the bounded convergence theorem, we have $(W, f) \sim \mathcal{N}(0, \|f\|^2_{L^2(\mathbb{R}^d)})$ for all $f \in L^2(\mathbb{R}^d)$. We call $\{(W, f) : f \in L^2(\mathbb{R}^d)\}$ a white noise Gaussian Hilbert space. Given $f, g \in L^2(\mathbb{R}^d)$ we may apply this fact to $(W, f + g)$ to see that

$$\mathrm{Cov}[(W, f), (W, g)] = (f, g)_{L^2(\mathbb{R}^d)},$$

so if f and g are orthogonal with respect to the $L^2(\mathbb{R}^d)$ inner product, then (W, f) and (W, g) are independent. We may rewrite the above expression as

$$\mathrm{Cov}[(W, f), (W, g)] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \delta(x - y) f(x) g(y) \, dx \, dy,$$

and say that W has covariance kernel $\delta(x - y)$ (here $\delta(x) \, dx$ is notation for the Dirac measure which assigns unit mass to the origin). In Section 3.2, we will compute the covariance kernel of the $\mathrm{FGF}_s(\mathbb{R}^d)$ for general s and d.

3. The FGF on $\mathbb{R}^d$

We provide the construction of $\mathrm{FGF}_s(\mathbb{R}^d)$ following the same procedure as used in Section 2.3 for white noise. We also compute the covariance kernel for the $\mathrm{FGF}_s(\mathbb{R}^d)$.

3.1. Definition of $\mathrm{FGF}_s(\mathbb{R}^d)$

We begin with some heuristic motivation for the rigorous construction that follows. We want to define h to be a standard Gaussian on $\dot{H}^s(\mathbb{R}^d)$. As a first guess, we might try to define a random element h of $\dot{H}^s(\mathbb{R}^d)$ so that for all $f \in \dot{H}^s(\mathbb{R}^d)$, we have

$$(h, f)_{\dot{H}^s(\mathbb{R}^d)} \sim \mathcal{N}\left(0, \|f\|^2_{\dot{H}^s(\mathbb{R}^d)}\right). \tag{3.1}$$

However, since $\dot{H}^s(\mathbb{R}^d)$ is infinite dimensional, no such random element exists [Jan97, She07]. However, we note that when $h, f \in \mathcal{S}_H(\mathbb{R}^d)$, we have

$$(h, f)_{\dot{H}^s(\mathbb{R}^d)} = (h, (-\Delta)^s f)_{L^2(\mathbb{R}^d)}. \tag{3.2}$$

Therefore, substituting (3.2) into (3.1) and defining $\phi := (-\Delta)^s f$, we find that it is reasonable to change the desired relation from (3.1) to

$$\mathbb{E}\left[(h,\phi)^2_{L^2(\mathbb{R}^d)}\right] = \mathbb{E}\left[(h,(-\Delta)^s f)^2_{\dot{H}^s(\mathbb{R}^d)}\right] = \|(-\Delta)^{-s}f\|^2_{\dot{H}^s(\mathbb{R}^d)}. \tag{3.3}$$

The advantage of this formulation is that we may reinterpret it by replacing the inner product $(h,\phi)^2_{L^2(\mathbb{R}^d)}$ with the evaluation of a continuous linear functional $(h,\cdot)$ at $\phi \in \mathcal{S}_H(\mathbb{R}^d)$. The norm on the right-hand side can be rewritten as

$$\|(-\Delta)^{-s}\phi\|^2_{\dot{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\xi|^{2s}|\xi|^{-4s}|\widehat{\phi}(\xi)|^2 \, d\xi = \|\phi\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}.$$

So, if h is a random element of $\mathcal{S}'_H(\mathbb{R}^d)$ with the property that

$$(h,\phi) \sim \mathcal{N}\left(0, \|\phi\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}\right) \quad \text{for all } \phi \in \mathcal{S}_H(\mathbb{R}^d), \tag{3.4}$$

then we say that h is a **fractional Gaussian field with parameter** s on $\mathbb{R}^d$ and write $h \sim \mathrm{FGF}_s(\mathbb{R}^d)$; note that by abuse of notation we refer to either h or its law as $\mathrm{FGF}_s(\mathbb{R}^d)$. We note that when $h \sim \mathrm{FGF}_s(\mathbb{R}^d)$ and $a > 0$, the scaling relation

$$x \mapsto h(ax) \stackrel{d}{=} a^{s-d/2}h$$

follows from (3.4) (here we are interpreting $x \mapsto h(ax)$ as a distribution via $(x \mapsto h(ax), \phi) = a^{-d}(h, x \mapsto \phi(x/a))$). For more discussion of FGF scaling and its relationship to the scaling properties of statistical physics models, see [New80, Dob79].

We now provide a construction establishing the existence of fractional Gaussian fields. We would like to apply the Bochner-Minlos theorem with the functional $\phi \mapsto \exp(-\frac{1}{2}\|\phi\|^2_{H^{-s}(\mathbb{R}^d)})$, but this functional is only finite when $\phi \in \mathcal{S}_H(\mathbb{R}^d)$, not for all $\phi \in \mathcal{S}(\mathbb{R}^d)$. Therefore, we define a functional (3.5) which is finite for all Schwartz functions and which reduces to $\phi \mapsto \exp(-\frac{1}{2}\|\phi\|^2_{H^{-s}(\mathbb{R}^d)})$ whenever $\phi \in \mathcal{S}_H(\mathbb{R}^d)$.

Let $\{\phi_\alpha : \alpha \text{ is a multi-index}\}$ be a collection Schwartz functions such that $\int_{\mathbb{R}^d} x^\alpha \phi_\beta(x) \, dx = \mathbf{1}_{\{\alpha=\beta\}}$. Such a collection may be obtained via a Gram-Schmidt procedure. Define the functional $C_s : \mathcal{S}_H(\mathbb{R}^d) \to \mathbb{R}$ by

$$C_s(\phi) = \exp\left(-\frac{1}{2}\left\|\phi - \sum_{|\alpha| \leq \lfloor H \rfloor} \phi_\alpha \int_{\mathbb{R}^d} x^\alpha \phi(x) \, dx\right\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}\right). \tag{3.5}$$

By Proposition 2.4, C_s is positive definite. Since C_s is also continuous and satisfies $C_s(0) = 1$, we may apply the Bochner-Minlos theorem to conclude that there is a random tempered distribution h such that $\mathbb{E}[e^{i(h,\phi)}] = C_s(\phi)$ for all $\phi \in \mathcal{S}(R^d)$. Considering h as a random element of $\mathcal{S}'_H(\mathbb{R}^d)$ by restricting its domain to $\mathcal{S}_H(\mathbb{R}^d)$, we obtain a random element of $\mathcal{S}'_H(\mathbb{R}^d)$ which satisfies (3.4)

(note that this restriction is necessary so that the definition does not depend on the arbitrary choice of functions ϕ_α).

As we did for white noise (see page 15), we may define a Gaussian Hilbert space $\{(h, \phi) \; : \; \phi \in T_s(\mathbb{R}^d)\}$ for a class $T_s(\mathbb{R}^d)$ of test functions larger than $\mathcal{S}_H(\mathbb{R}^d)$. In particular, we define $T_s(\mathbb{R}^d)$ to be the closure of $\mathcal{S}_H(\mathbb{R}^d)$ in $\dot{H}^{-s}(\mathbb{R}^d)$. Consider the isometry from $T_s(\mathbb{R}^d)$ to $L^2(\Omega)$ which sends $\phi \in \mathcal{S}_H(\mathbb{R}^d)$ to the random variable (h, ϕ); we extend this isometry to an operator from $T_s(\mathbb{R}^d)$ to $L^2(\Omega)$. Writing $\phi \in T_s(\mathbb{R}^d)$ as a limit of functions in $\mathcal{S}_H(\mathbb{R}^d)$ and considering the limit of the corresponding characteristic functions, we conclude that

$$(h, \phi) \sim \mathcal{N}\left(0, \|\phi\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}\right) \text{ for all } \phi \in T_s(\mathbb{R}^d).$$

We call $\{(h, \phi) \; : \; \phi \in T_s(\mathbb{R}^d)\}$ an $\mathrm{FGF}_s(\mathbb{R}^d)$ Gaussian Hilbert space.

We now make sense of the expression $h = (-\Delta)^{-s/2}W$ (see (1.1)). Let W be a white noise on $\mathbb{R}^d$. Observe that $(-\Delta)^{-s/2}\phi \in L^2(\mathbb{R}^d)$ for all $\phi \in T_s(\mathbb{R}^d)$. Therefore, we may define for all $\phi \in T_s(\mathbb{R}^d)$ the random variable $(h, \phi) = (W, (-\Delta)^{-s/2}\phi)$. In this way, we have constructed a coupling between an $\mathrm{FGF}_s(\mathbb{R}^d)$ Gaussian Hilbert space $\{(h, \phi) \; : \; \phi \in T_s(\mathbb{R}^d)\}$ and a white noise Gaussian Hilbert space $\{(W, \phi) \; : \; \phi \in L^2(\mathbb{R}^d)\}$ so that $(h, \phi) = (W, (-\Delta)^{-s/2}\phi)$. In this sense we can say that $h = (-\Delta)^{-s/2}W$. For a coupling in which this equation holds almost surely, see Proposition 6.3.

Remark 3.1. Computing $\|\phi\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}$ amounts to computing the covariance kernel of the $\mathrm{FGF}_s(\mathbb{R}^d)$, which will be done in Section 3.2.

Remark 3.2. Since $C_c^\infty(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$, the $\mathrm{FGF}_s(\mathbb{R}^d)$ is uniquely determined by the random variables $\{(h, \phi_n)\}_{n \geq 1}$ where ϕ_n is a dense (in $\mathcal{S}(\mathbb{R}^d)$) sequence of $C_c^\infty(\mathbb{R}^d)$ functions.

3.2. The FGF covariance kernel

Given $h \sim \mathrm{FGF}_s(\mathbb{R}^d)$ with Hurst parameter $H = s - d/2$, let $G^s(x, y)$ be a function (or generalized function) such that for $\phi_1, \phi_2 \in C_c^\infty(\mathbb{R}^d) \cap T_s(\mathbb{R}^d)$ we have

$$\mathrm{Cov}[(h, \phi_1), (h, \phi_2)] = (\phi_1, \phi_2)_{\dot{H}^{-s}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G^s(x, y)\phi_1(x)\phi_2(y)\, dx\, dy. \tag{3.6}$$

We call $G^s(x, y)$ a **covariance kernel** of the $\mathrm{FGF}_s(\mathbb{R}^d)$. We point out that there can be more than one function G^s satisfying (3.6). For example, if $H \geq 0$ and $G^s(x, y)$ satisfies (3.6), then so does $G^s(x, y) + g(x, y)$ for any polynomial g in x or in y of degree no greater than $\lfloor H \rfloor$.

In this section we compute covariance kernels for the fractional Gaussian fields on $\mathbb{R}^d$. For most positive values of s, we find that $G^s(x, y) = C(s, d)|x - y|^{2H}$ for some constant $C(s, d)$. When $s < 0$ the formula is similar but involves some derivatives of the delta function, and when H is a nonnegative integer there is a logarithmic correction. The constant $C(s, d)$, and therefore also the correlation

of $\mathrm{FGF}_s(\mathbb{R}^d)$, is positive when $s \in (0, d/2)$, is $(-1)^{\lfloor s \rfloor}$ when s is a negative non-integer, and is $(-1)^{1+\lfloor H \rfloor}$ when H is a positive non-integer. The statement and proof of the following theorem are adapted from [LD72, Chapter 1, §1].

Theorem 3.3. *Each of the following holds.*

(i) *If $H \in (-\frac{d}{2}, \infty)$ (that is, $s > 0$) and H is not a nonnegative integer, then*

$$G^s(x, y) = C(s, d)|x - y|^{2H}$$

satisfies (3.6), *where*

$$C(s, d) = \frac{2^{-2s}\pi^{-d/2}\Gamma\left(\frac{d}{2} - s\right)}{\Gamma(s)}.$$

(ii) *If $s < 0$ (that is, $H < -d/2$) and $s \in (-k-1, -k)$ where k is a nonnegative integer, then $\mathrm{Cov}[(h, \phi_1), (h, \phi_2)]$ is given by*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} C(s, d)|x - y|^{2H} \left[\phi_1(x)\phi_2(y) - \sum_{j=0}^{k} \phi_1(x)H_j\Delta^j\phi_2(x)|x - y|^{2j} \right],$$

where

$$H_j = \frac{\Omega_d}{2^j j! d(d+2)\cdots(d+2j-2)},$$

and $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the surface area of the unit sphere in $\mathbb{R}^d$.

(iii) *If $s = -k$ where k is a nonnegative integer, then $\mathrm{Cov}[(h, \phi_1), (h, \phi_2)]$ is given by*

$$\int_{\mathbb{R}^d} \phi_1(x)(-\Delta)^k\phi_2(x)\,dx.$$

(iv) *If H is a nonnegative integer k, then*

$$G^s(x, y) = 2\,c_{-1}^{(\frac{d}{2}+k)}|x - y|^{2H}\log|x - y|,$$

satisfies (3.6), *where $c_{-1}^{(\frac{d}{2}+k)}$ is the residue at $\frac{d}{2} + k$ of $s \mapsto C(s, d)$:*

$$c_{-1}^{(\frac{d}{2}+k)} = \frac{(-1)^{k+1}2^{-2k-d}\pi^{-d/2}}{k!\,\Gamma(\frac{d}{2}+k)}.$$

Remark 3.4. In case (ii) above, we can also write

$$G^s(x, y) = C(s, d)|x - y|^{2H} \left[1 - \sum_{j=0}^{k} |x - y|^{2j} H_j\Delta^j\delta(x - y) \right],$$

Similarly, in case (iii),

$$G^s(x, y) = (-\Delta)^k\delta(x - y).$$

Proof of Theorem 3.3. (i) We first assume $H \in (-\frac{d}{2}, 0)$ and let $h \sim \mathrm{FGF}_s(\mathbb{R}^d)$. Let $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d)$. Then we may compute the covariance:

$$
\begin{aligned}
\mathrm{Cov}[(h, \phi_1), (h, \phi_2)] &= \int_{\mathbb{R}^d} |\xi|^{-2s} \hat{\phi}_1(\xi) \overline{\hat{\phi}_2(\xi)} \, d\xi, \\
&= (|\xi|^{-2s} \hat{\phi}_1, \hat{\phi}_2)_{L^2(\mathbb{R}^d)}, \\
&= \left(\mathcal{F}^{-1}(|\xi|^{-2s}) * \phi_1, \phi_2 \right)_{L^2(\mathbb{R}^d)}, \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} C(s, d) |x - y|^{2H} \phi_1(x) \phi_2(y) \, dx \, dy,
\end{aligned}
$$

where in the third line we used the Plancherel theorem, and in the last line we used the following Fourier transform formula given in [LD72, Chapter 1, §1]:

$$
\mathcal{F}\left[C(s, d) |x|^{2H} \right] = |\xi|^{-2s}. \tag{3.7}
$$

It is important to note that (3.7) is only valid for $0 < s < d/2$ when the class of test functions is taken to be $\mathcal{S}(\mathbb{R}^d)$. Indeed, $|\xi|^{-2s}$ is not a tempered distribution when $s \geq d/2$ (due to the singularity at the origin), and $C(s, d)|x|^{2H}$ is not a tempered distribution when $s \leq 0$. Therefore, we extend the Fourier transform formula (3.7) outside of the region $H \in (-\frac{d}{2}, 0)$. Now for $H \geq 0$ and non-integral, since $\phi_2(y) \in \mathcal{S}_H(\mathbb{R}^d)$ it follows that for all $N \geq 0$, $\phi_2(y) = O(|y|^{-N})$ as $|y| \to \infty$, thus

$$
\psi(x, s) := C(s, d) \int_{\mathbb{R}^d} |x - y|^{2H} \phi_2(y) \, dy
$$

is a smooth function of x and an analytic function of s for all H in the range under consideration [LD72, p. 48]. Furthermore, as $|x| \to \infty$ we have $\psi(x, s) = O(|x|^{2H})$ so that $\phi_1(x)\psi(x, s)$ is integrable in x and analytic in s for all H in the range under consideration. By an analytic continuation argument as in [LD72, Chapter 1, §1], (i) follows.

Formulas (ii) and (iii) follow directly from equation (1.1.10) in [LD72]:

$$
\psi(x, s) = C(s, d) \int_{\mathbb{R}^d} \left[\phi_2(y) - \sum_{j=0}^{k} H_j \Delta^j \phi_2(x) |x - y|^{2j} \right] |x - y|^{2H} \, dy,
$$

where $\psi(x, s)$ is an analytic continuation from $0 < s < d/2$ to $s \in (-k-1, -k]$. The result for (iii) follows from the equality

$$
\psi(x, -k) = (-1)^k \Delta^k \phi_2(x).
$$

Finally, to obtain (iv) we will take a limit as $t \to s$ of both sides of

$$
\|\phi\|_{\dot{H}^{-t}(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} C(t, d) |x - y|^{2t - d} \phi(x) \phi(y) \, dx \, dy; \tag{3.8}
$$

see [LD72, p. 50] for more details. Since ϕ_1 and ϕ_2 are in $\mathcal{S}_k(\mathbb{R}^d)$, we have $\int_{\mathbb{R}^d} x^j \phi_1(x) \, dx = \int_{\mathbb{R}^d} y^j \phi_2(y) \, dy = 0$ for all $0 \leq j \leq k$. This implies

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x-y|^{2t-d} \phi_1(x)\phi_2(y)\, dx\, dy =$$

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (|x-y|^{2t-d} - |x-y|^{2s-d})\phi_1(x)\phi_2(y)\, dx\, dy.$$

We use Taylor's theorem to write

$$|x-y|^{2t-d} - |x-y|^{2s-d}$$
$$= 2(t-s)|x-y|^{2k} \ln|x-y| + O\left(\left((t-s)|x-y|^{2k}\ln|x-y|\right)^2\right),$$

and substitute into (3.8). Taking $t \to s$ and using $\lim_{t \to s}(t-s)C(t,d) = \mathrm{Res}_{t=s}\,C(t,d)$, we obtain (iv). The formula for $c_{-1}^{(\frac{d}{2}+k)}$ follows from the fact that the residue of Γ at a negative integer $-n$ is $(-1)^n/n!$. $\qquad\square$

4. The FGF on a domain

4.1. The space $\dot{H}_0^s(D)$

Let $s \geq 0$, and let $D \subset \mathbb{R}^d$ be a domain. Recall that $C_c^\infty(D)$ denotes the set of smooth functions supported on a compact subset of D. We have $C_c^\infty(D) \subset \dot{H}^s(\mathbb{R}^d)$ from the definition of $\dot{H}^s(\mathbb{R}^d)$ and the closure of the complex Schwartz functions under the Fourier transform (see Section 2.1). We may therefore define the set $\dot{H}_0^s(D)$ to be the closure of $C_c^\infty(D)$ in $\dot{H}^s(\mathbb{R}^d)$ and equip it with the $\dot{H}^s(\mathbb{R}^d)$ inner product.

Definition 4.1. We call a domain $D \subset \mathbb{R}^d$ **allowable** for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ there exists $C = C(D, d, \phi) < \infty$ such that for all $g \in C_c^\infty(D)$, we have

$$|(\phi, g)_{L^2(\mathbb{R}^d)}| \leq C\|g\|_{\dot{H}^s(\mathbb{R}^d)}.$$

We will construct a fractional Gaussian field $\mathrm{FGF}_s(D)$ for all allowable domains $D \subset \mathbb{R}^d$ (see Remark 4.3). The following lemma gives sufficient conditions for a domain to be allowable.

Lemma 4.2. *Let $s \geq 0$. If $H = s - d/2$ is not a nonnegative integer, then every proper subdomain of $\mathbb{R}^d$ is allowable. If $H = s - d/2$ is a nonnegative integer, then a domain D is allowable if $\mathbb{R}^d \setminus D$ contains an open set.*

Proof. Let $D \subset \mathbb{R}^d$ be a domain, and let $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $g \in C_c^\infty(D)$. We have

$$|(\phi, g)_{L^2(\mathbb{R}^d)}| = \left|\int_{\mathbb{R}^d} |\xi|^{-s}\widehat{\phi}(\xi)|\xi|^s\widehat{g}(\xi)\, d\xi\right|$$
$$\leq \|\phi\|_{\dot{H}^{-s}(\mathbb{R}^d)} \|g\|_{\dot{H}^s(\mathbb{R}^d)},$$

by the Plancherel formula and Cauchy-Schwarz. If $0 \leq s < d/2$, we conclude that $\|\phi\|_{\dot{H}^{-s}(\mathbb{R}^d)}$ is finite and therefore that D is allowable.

If $H = s - d/2 \in \{0, 1, \ldots\}$ and $\mathbb{R}^d \setminus D$ contains an open set, then let B be a ball contained in $\mathbb{R}^d \setminus D$, and let $\eta \in C_c^\infty(\mathbb{R}^d)$ be supported on B and satisfy $\int_{\mathbb{R}^d} \eta(x)x^\alpha \, dx = \int_{\mathbb{R}^d} \phi(x)x^\alpha \, dx$ for every multi-index α satisfying $|\alpha| \leq H$ (such a function may be constructed via a Gram-Schmidt procedure). Since $\eta g = 0$, we have

$$(\phi, g)_{L^2(\mathbb{R}^d)} = (\phi - \eta, g)_{L^2(\mathbb{R}^d)} \leq \|\phi - \eta\|_{\dot{H}^{-s}(\mathbb{R}^d)} \|g\|_{\dot{H}^s(\mathbb{R}^d)}$$

By our choice of η, the Fourier transform of $\phi - \eta$ vanishes to order H at the origin, so $\|\phi - \eta\|_{\dot{H}^{-s}(\mathbb{R}^d)}$ is finite. Therefore D is allowable.

Suppose that $s - d/2 > 0$ is not an integer and that $D \subsetneq \mathbb{R}^d$. Without loss of generality, we may assume D does not contain the origin. Let P^ϕ be the unique polynomial of degree $\lfloor H \rfloor$ such that all the derivatives up to order $\lfloor H \rfloor$ of $\mathcal{F}(\phi - P^\phi(D)\delta)$ are zero, where $P(D)$ denotes the differential operator corresponding to a polynomial P and δ denotes a unit Dirac mass at 0. Then

$$|(\phi, g)_{L^2(\mathbb{R}^d)}| = |(\phi - P^\phi(D)\delta, g)_{L^2(\mathbb{R}^d)}| \leq \left\|\phi - P^\phi(D)\delta\right\|_{\dot{H}^{-s}(\mathbb{R}^d)} \|g\|_{\dot{H}^s(\mathbb{R}^d)}.$$

The expression $\|\phi - P^\phi(D)\delta\|_{\dot{H}^{-s}(\mathbb{R}^d)}$ is finite since $\mathcal{F}(\phi - P^\phi(D)\delta)$ is bounded by a constant times $|\xi|^{\lfloor H \rfloor + 1}$ near the origin and by a constant times $|\xi|^{\lfloor H \rfloor}$ as $\xi \to \infty$. $\qquad\square$

Let $\phi \in \mathcal{S}(\mathbb{R}^d)$, and let D be an allowable domain. By the definition of allowability, $(\phi, \cdot)_{L^2(\mathbb{R}^d)}$ is a continuous linear functional on $\dot{H}_0^s(D)$. Therefore, by the Riesz representation theorem for Hilbert spaces, there exists a unique $f \in \dot{H}_0^s(D)$ such that $(\phi, g)_{L^2(\mathbb{R}^d)} = (f, g)_{\dot{H}^s(\mathbb{R}^d)}$ for all $g \in \dot{H}_0^s(D)$. Writing out the definition of $(f, g)_{\dot{H}^s(\mathbb{R}^d)}$ and using the Plancherel formula, we see that this implies that f is the unique solution of the distributional equation

$$(-\Delta)^s f = \phi, \qquad f \in \dot{H}_0^s(D). \tag{4.1}$$

For $s > 0$, we define the semi-norm $\|\phi\|_{\dot{H}^{-s}(D)} := \|f\|_{\dot{H}^s(\mathbb{R}^d)}$, where f is determined by ϕ via (4.1).

Denote by $\mathcal{S}(D)$ the space of functions on D which can be realized as the restriction of a Schwartz function to D. Then $d(\phi, \psi) := \|\phi - \psi\|_{\dot{H}^{-s}(D)}$ defines a metric on $\mathcal{S}(D)$. Taking the completion under this metric as we did at the end of Section 2.1, we get a Hilbert space $T_s(D) \subset \mathcal{S}'(\mathbb{R}^d)$ which will serve as a space of test functions for $\mathrm{FGF}_s(D)$.

4.2. The zero-boundary FGF in a domain

Let $D \subsetneq \mathbb{R}^d$ be an allowable domain, let $s \geq 0$, and define the functional

$$C_{D,s}(\phi) := \exp\left(-\frac{1}{2}\|\phi\|_{\dot{H}^{-s}(D)}^2\right)$$

for $\phi \in \mathcal{S}(\mathbb{R}^d)$. Since $C_{D,s}$ is continuous by the definition of allowability, we may use Proposition 2.4 and the Bochner-Minlos Theorem to $C_{D,s}$ to conclude that

there is a unique random element h_D of $\mathcal{S}'(\mathbb{R}^d)$ such that (h_D, ϕ) is a mean-zero Gaussian with variance $\|\phi\|^2_{\dot{H}^{-s}(D)}$. Since $\mathbb{E}[(h_D, \phi)^2] = 0$ whenever ϕ is supported in $\mathbb{R}^d \setminus D$, the support of h_D is almost surely contained in $\overline{D}$. We call[6] h_D the *zero-boundary FGF* on D, abbreviated as $\mathrm{FGF}_s(D)$.

Remark 4.3. We construct $h_D \sim \mathrm{FGF}_s(D)$ only when D is allowable because we want to ensure that h_D is a tempered distribution (rather than a tempered distribution modulo a space of polynomials).

We can also define a Gaussian Hilbert space version of $\mathrm{FGF}_s(D)$, following the corresponding discussion $\mathrm{FGF}_s(\mathbb{R}^d)$ in Section 3.1. In this way we obtain a collection of random variables $\{(h_D, f) : f \in T_s(D)\}$ so that (h_D, f) is a centered Gaussian with variance $\|f\|^2_{\dot{H}^{-s}(D)}$.

If s is an even positive integer, then $\|f\|_{\dot{H}^s_0(D)} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^d)}$ for all $f \in C^\infty_0(D)$. If s is an odd positive integer, then

$$\|f\|_{\dot{H}^s_0(D)} = \|(-\Delta)^{\frac{s-1}{2}} f\|_{\dot{H}^1_0(D)}$$

for all $f \in C^\infty_0(D)$. Therefore, if $s = 0$ then h_D is white noise on D, and if $s = 1$ then h_D is the GFF on D. Thus $\mathrm{FGF}_s(D)$ generalizes the domain versions of white noise and the Gaussian free field.

4.3. Covariance kernel for the FGF on the unit ball

Let $s \geq 0$, and let D be an allowable domain. As usual, we say that a function $G^s_D : D \times D \to \mathbb{R}$ is the $\mathrm{FGF}_s(D)$ covariance kernel if it satisfies

$$\mathrm{Cov}[(h_D, \phi_1), (h_D, \phi_2)] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G^s_D(x, y)\phi_1(x)\phi_2(y)\, dx\, dy. \qquad (4.2)$$

for $h_D \sim \mathrm{FGF}_s(D)$ and for all $\phi_1, \phi_2 \in C^\infty_c(D)$. We treat each of the cases

(i) s is an integer,
(ii) $s \in (0, 1)$, and
(iii) s is a non-integer greater than 1.

Suppose that s is a positive integer, and let $\phi \in \mathcal{S}(B)$. By (2.65) in Chapter 2 of [GGS10], the unique solution of (4.1) is $f(x) = \int G^s_B(x, y)\phi(y)dy$, where

$$G^s_B(x, y) = k_{s,d}|x - y|^{2H} \int_1^{\frac{\left| |x|y - \frac{x}{|x|} \right|}{|x-y|}} (v^2 - 1)^{s-1} v^{1-d} dv, \quad x, y \in B \qquad (4.3)$$

and

$$k_{s,d} = \frac{\Gamma(1 + d/2)}{d\pi^{d/2} 4^{d-1}((s-1)!)^2}.$$

---

[6]We use the word *boundary* instead of *complement* for consistency with the GFF terminology. Note, however, that due to the nonlocal nature of the fractional Laplacian, the relevant boundary data include the values on $\mathbb{R}^d \setminus \overline{D}$.

It follows that for all $\phi \in C_c^\infty(D)$, we have

$$\mathbb{E}[(h_D, \phi)^2] = \|\phi\|_{\dot{H}^{-s}(D)}^2 = \|f\|_{\dot{H}_0^s(D)}^2 = \iint G_B(x,y)\phi(x)\phi(y)\,dx\,dy, \quad (4.4)$$

which shows that G_B^s is the $\mathrm{FGF}_s(D)$ covariance kernel.

Suppose that $0 < s < 1$. Let X_t denote a $2s$-stable symmetric Lévy process, and let τ_B be the first time X exits B. Recall the definition of the constant $C_{d,s}$ in Theorem 3.3, and define $u(x,y) = (2/\pi)^{2s}C_{d,s}|x-y|^{2H}$. By the potential theory of $2s$-symmetric stable processes, (see, for example, [CS98]), the function

$$G_B^s(x,y) = u(x,y) - \mathbb{E}^x[u(X_{\tau_B}, y)]$$

is the $\mathrm{FGF}_s(D)$ covariance kernel. The following explicit formula for G_s^B is given as Corollary 4 in [BGR61]:

$$G_B^s(x,y) = \tilde{k}_{s,d}|x-y|^{2H}\int_0^{\frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}} (v+1)^{-d/2}v^{s-1}dv, \quad x,y \in B, \quad (4.5)$$

where

$$\tilde{k}_{s,d} = \frac{\Gamma(d/2)}{4^s\pi^{d/2}\Gamma(s)^2}.$$

Suppose that $s > 1$ is not an integer and $\phi \in \mathcal{S}(\mathbb{R}^d)$. We claim that $G_B^s(x,y) = \int_B G^{\lfloor s\rfloor}(x,u)G^{s-\lfloor s\rfloor}(u,y)du$ is the covariance kernel for $\mathrm{FGF}_s(\mathbb{D})$. Indeed, we may write $(-\Delta)^s = (-\Delta)^{\lfloor s\rfloor}(-\Delta)^{s-\lfloor s\rfloor}$ and calculate

$$(-\Delta)^s \iint G_B^{s-\lfloor s\rfloor}(x,u)G_B^{\lfloor s\rfloor}(u,y)\phi(y)\,dy$$
$$= (-\Delta)^{\lfloor s\rfloor}\int G_B^{\lfloor s\rfloor}(u,y)\phi(y)\,dy$$
$$= \phi(x),$$

which implies that G_B^s is the $\mathrm{FGF}_s(D)$ covariance kernel by (4.4).

Remark 4.4. Similar results may be obtained for a more general class of domains D. The ingredients are the corresponding potential theory of the poly-Laplacian and fractional Laplacian for $s \in (0,1)$.

5. Projections of the FGF

Given a domain $D \subset \mathbb{R}^d$ and a distribution f defined on $\mathbb{R}^d \setminus D$, if a distribution $g : \mathbb{R}^d \to \mathbb{R}$ satisfies the condition

$$f|_{\mathbb{R}^d \setminus D} = g|_{\mathbb{R}^d \setminus D}$$
$$((-\Delta)^s g)|_D = 0,$$

then we call g the **s-harmonic extension** of f. In this section we decompose $h \sim \mathrm{FGF}_s(\mathbb{R}^d)$ as a sum of two random fields, one of which is supported on D

and the other of which may be interpreted as the s-harmonic extension of the values of h on $\mathbb{R}^d \setminus D$.

Let $s > 0$, let $D \subsetneq \mathbb{R}^d$ be an allowable domain, and define

$$\mathrm{Har}_s(D) = \{f \in \dot{H}^s(\mathbb{R}^d) \,:\, ((-\Delta)^s f)|_D = 0\}.$$

Proposition 5.1. $\dot{H}^s(\mathbb{R}^d) = \mathrm{Har}_s(D) \oplus \dot{H}_0^s(D)$.

Proof. If $f \in \mathrm{Har}_s(D)$ and $g \in \dot{H}_0^s(D)$, then $(f,g)_{\dot{H}^s(\mathbb{R}^d)} = ((-\Delta)^s f, g) = 0$. Therefore, $\mathrm{Har}_s(D)$ and $\dot{H}_0^s(D)$ are orthogonal subspaces of $\dot{H}^s(\mathbb{R}^d)$.

Let $f \in \dot{H}^s(\mathbb{R}^d)$. Since D is allowable, $((-\Delta)^s f, \cdot)_{L^2(\mathbb{R}^d)}$ is a continuous functional on $\dot{H}_0^s(D)$. Therefore, there exists $f_D \in \dot{H}_0^s(D)$ such that for all $g \in \dot{H}_0^s(D)$, we have $(f,g)_{\dot{H}^s(\mathbb{R}^d)} = (f_D, g)_{\dot{H}^s(\mathbb{R}^d)}$. In particular, this implies that $((-\Delta)^s(f - f_D), g) = 0$ for all $g \in C_c^\infty(D)$, which means that

$$(-\Delta)^s(f - f_D)|_D = 0.$$

Thus we can write f as a sum of elements of $\mathrm{Har}(D)$ and $\dot{H}_0^s(D)$ as $f = (f - f_D) + f_D$. $\qquad\square$

Observe that Proposition 5.1 implies that $\mathrm{Har}_s(D)$ is a closed subspace of $\dot{H}^s(\mathbb{R}^d)$. We define the projection operators $P_D f = f_D$ and $P_D^{\mathrm{Har}} f = f - f_D$. We will make sense of $P_D h$ and $P_D^{\mathrm{Har}} h$ almost surely, although these are defined a priori only for $h \in \dot{H}^s(\mathbb{R}^d)$ and not for arbitrary elements of $\mathcal{S}_H'(\mathbb{R}^d)$.

We begin by observing that the solution f of (4.1) is given by $f = P_D(-\Delta)^{-s}\phi$. Indeed, $P_D(-\Delta)^{-s}\phi \in \dot{H}_0^s(\mathbb{R}^d)$, and

$$(P_D(-\Delta)^{-s}\phi, g)_{\dot{H}^s(\mathbb{R}^d)} = ((-\Delta)^{-s}\phi, g)_{\dot{H}^s(\mathbb{R}^d)} = (\phi, g)_{L^2(\mathbb{R}^d)},$$

since $(P_D^{\mathrm{Har}}(-\Delta)^{-s}\phi, g) = 0$ for all $g \in C_c^\infty(D)$. Therefore, we may apply the Bochner-Minlos theorem to the functional

$$\Phi(\phi) = \exp\left(-\frac{1}{2}\|P(-\Delta)^{-s}\phi\|_{\dot{H}^s(\mathbb{R}^d)}^2\right)$$

for $P = P_D$ and for $P = P_D^{\mathrm{Har}}$ to obtain random tempered distributions h_D and h_D^{Har}, respectively. We call h_D^{Har} the s-harmonic extension of h restricted to $\mathbb{R}^d \setminus D$. In Section 8, we will show that h_D^{Har} is smooth in D almost surely.

Remark 5.2. Like the fractional Gaussian field in $\mathbb{R}^d$, h_D^{Har} is a random element of $\mathcal{S}_H'(\mathbb{R}^d)$. But h_D is a random element of $\mathcal{S}'(\mathbb{R}^d)$, as mentioned in Remark 4.3.

Now sample h_D^{Har} and h_D independently and define $h = h_D^{\mathrm{Har}} + h_D$. By the uniqueness part of the Bochner-Minlos theorem, h is an $\mathrm{FGF}_s(\mathbb{R}^d)$. For all $f \in \dot{H}_0^s(D)$, we have $(h, f)_{\dot{H}^s(\mathbb{R}^d)} = (h_D, f)_{\dot{H}^s(\mathbb{R}^d)}$ almost surely. Therefore, h_D is almost surely determined by h. Thus $h_D^{\mathrm{Har}} = h - h_D$ is also almost surely determined by h. So we can define measurable maps P_D and P_D^{Har} on $\mathcal{S}_H'(\mathbb{R}^d)$ such that $h_D = P^D h \sim \mathrm{FGF}_s(\mathbb{R}^d)$ and $h_D^{\mathrm{Har}} = P_D^{\mathrm{Har}} h$ is the harmonic extension of h restricted to $\mathbb{R}^d \setminus D$.

Remark 5.3. We will sometimes describe the relationship between h_D and h_D^{Har} by saying that h_D^{Har} is the conditional expectation of $h \sim \text{FGF}_s(\mathbb{R}^d)$ given the values of h on $\mathbb{R}^d \setminus D$.

Because $(-\Delta)$ commutes with P_D and P_D^{Har}, by the Bochner-Minlos theorem, we have

$$(-\Delta)h_D^s \overset{d}{=} h_D^{s-2} \quad \text{and} \quad (-\Delta)h_{\text{Har}}^{D,s} \overset{d}{=} h_{\text{Har}}^{D,s-2} \tag{5.1}$$

where $\overset{d}{=}$ denotes equality in distribution.

Suppose $U \subset D$ is another allowable domain. Since projection operators in $\dot{H}^s(\mathbb{R}^d)$ commute,

$$P_U P_D h = P_D P_U h = P_U h,$$

$$h_D = P_D h = P_D(P_U^{\text{Har}}h + P_U h) = P_U^{\text{Har}}h_D + P_U h$$

almost surely. Moreover, $P_{\text{Har}}^U h_D$ and $P_U h$ are independent. As discussed above, h_U and $P_U^{\text{Har}}h_D$ are determined by h_D almost surely. Thus we have the following proposition.

Proposition 5.4. *Given allowable domains U and D such that $U \subset D$, there is a coupling $(h_D, h_{U,D}^{\text{Har}}, h_U)$ such that*

(i) $h_D = h_{U,D}^{\text{Har}} + h_U$,
(ii) h_D *is a zero boundary* FGF *on D,*
(iii) h_U *is zero boundary* FGF *on U, and*
(iv) $h_{U,D}^{\text{Har}}$ *and h_U are independent and both determined by h_D almost surely.*

We call $h_{U,D}^{\text{Har}}$ the harmonic extension of h_D given its values on D/U.

By the definition of h_D^{Har}, given $\phi \in C_c^\infty(D) \cap \mathcal{S}_H(\mathbb{R}^d)$ and $f = (-\Delta)^{-s}\phi \in \dot{H}^s(\mathbb{R}^d)$, we have $(h_D^{\text{Har}}, \phi) = (h, (-\Delta)^s f_D^{\text{Har}})$. Since $\text{supp}((-\Delta)^s f_D^{\text{Har}}) \subset \mathbb{R}^d \setminus D$, we can say that the value of h_{Har}^D on D modulo a polynomial of degree at most $\lfloor H \rfloor$ is determined by values of h on $\mathbb{R}^d \setminus D$. More precisely, the random variable $h_D^{\text{Har}}|_D$ is determined by $\{(h, \phi) : \phi \in T_s(\mathbb{R}^d), \text{supp}(\phi) \subset \mathbb{R}^d \setminus D\}$.

When s is a positive integer, the operator $(-\Delta)^s$ is local, in that case we have a stronger result: $h_D^{\text{Har}}|_D$ is measurable with respect to the σ-algebra generated by the intersection of the value of h on every neighborhood of the boundary (that is, the action of h on test functions supported on a neighborhood of the boundary). This is a generalization of the corresponding Markov property for the Gaussian free field [She07].

6. Fractional Brownian motion and the FGF

The d-dimensional fractional Brownian motion B with Hurst parameter $H > 0$ is defined to be the centered Gaussian process on $\mathbb{R}^d$ with

$$\mathbb{E}[B(x)B(y)] = |x-y|^{2H} - |x|^{2H} - |y|^{2H} \quad \text{for all } x, y \in \mathbb{R}^d. \tag{6.1}$$

The existence of such a process is guaranteed by the general theory of Gaussian processes (for example, see Theorem 12.1.3 in [Dud02]), because the right-hand

side of (6.1) is positive definite [OW89]. The special case $H = \frac{1}{2}$ is called Lévy Brownian motion [Lév40], [Lév45].

Proposition 6.1. *If $s \in (d/2, d/2+1)$ (that is, $H \in (0,1)$) and $h \sim \mathrm{FGF}_s(\mathbb{R}^d)$, then the process defined by $\widetilde{h}(x) = (h, \delta_x - \delta_0)$ has the same distribution as the fractional Brownian motion with Hurst parameter H (up to multiplicative constant).*

Proof. Let $x \in \mathbb{R}^d$. Since the Fourier transform of δ_x is $\xi \mapsto e^{2\pi i x \cdot \xi}$, one may verify from the definition of the $\dot{H}^{-s}(\mathbb{R}^d)$ norm that $\delta_x - \delta_0$ is an element of $\dot{H}^{-s}(\mathbb{R}^d)$ and therefore an element of $T_s(\mathbb{R}^d)$. So if $h \sim \mathrm{FGF}_s(\mathbb{R}^d)$, then we may define $\widetilde{h}(x) = (h, \delta_x - \delta_0)$. Then by Theorem 3.3(i) we have

$$\mathbb{E}[\widetilde{h}(x)\widetilde{h}(y)] = G^s(x, y) - G^s(0, y) - G^s(x, 0), \tag{6.2}$$

where $G^s(x, y) = C(s, d)|x - y|^{2H}$. Combining (6.1) and (6.2), we see that $C(s, d)B$ and $\widetilde{h}$ have the same covariance structure. Since both are centered Gaussian processes, this implies that they have the same law. □

Since (6.1) and (6.2) show that $\widetilde{h}(0) = B(0) = 0$ almost surely, Proposition 6.1 establishes that the $\mathrm{FGF}_s(\mathbb{R}^d)$ can be identified as (a constant multiple of) the fractional Brownian motion by fixing its value to be zero at the origin.

Denote by $C^{k,\alpha}(\mathbb{R}^d)$ the space of functions on $\mathbb{R}^d$ all of whose derivatives of order up to k exist and are α-Hölder continuous. Note that the differentiability and Hölder continuity of a function-modulo-polynomials is well-defined, because adding a polynomial to a function does not affect its regularity properties.

Proposition 6.2. *Let h be an FGF on $\mathbb{R}^d$ with Hurst parameter $H > 0$, and define $k = \lceil H \rceil - 1$. Then $h \in C^{k,\alpha}(\mathbb{R}^d)$ almost surely for all $0 < \alpha < H - \lceil H \rceil$.*

Proof. We consider several cases:

(i) Suppose that $0 < H < 1$. By Theorem 8.3.2 in [Adl10], fractional Brownian motion is α-Hölder continuous for all $\alpha < H$. The result then follows from Proposition 6.1.

(ii) Suppose that $1 < H < 2$, and let $s = d/2 + H$. As in the case $H \in (0,1)$, it is straightforward to verify that $\partial^\alpha \delta_x - \partial^\alpha \delta_0 \in T_s(\mathbb{R}^d)$ when $|\alpha| \leq 1$ and $x \in \mathbb{R}^d$. Therefore, if $h \sim \mathrm{FGF}_s(\mathbb{R}^d)$, we may fix all derivatives of h of order up to 1 to vanish at the origin. In this way we obtain a scale-invariant function h_0 whose restriction to $\mathcal{S}_1(\mathbb{R}^d)$ coincides with h. Since $|h_0(x)|$ has the same law as $|x|^H h_0(1)$ by scale invariance, we have $\mathbb{E}|h_0(x)| = c|x|^H$ for all $x \in \mathbb{R}^d$, where $c = \mathbb{E}[|h_0(1)|]$. Thus

$$\mathbb{E}\left[\int_{|x|>1} \frac{|h_0(x)|}{|x|^{d+2}}\,dx\right] = \int_{|x|>1} \frac{\mathbb{E}|h_0(x)|}{|x|^{d+2}}\,dx = \int_{|x|>1} \frac{c}{|x|^{d+2-H}} < \infty,$$

which implies that h_0 satisfies condition (2.4) with $s = 1/2$ almost surely (see Section 2.2). Therefore, $\widetilde{h} := (-\Delta)^{1/2}h_0$ is well-defined as a random element of

$\mathcal{S}_0'(\mathbb{R}^d)$. Furthermore, since

$$(\widetilde{h}, \phi) = (h, (-\Delta)^{1/2}\phi) \sim \mathcal{N}\left(0, \|(-\Delta)^{1/2}\phi\|_{\dot{H}^{-s}(\mathbb{R}^d)}\right)$$
$$= \mathcal{N}\left(0, \|\phi\|_{\dot{H}^{-(s-1)}(\mathbb{R}^d)}\right)$$

for all $\phi \in \mathcal{S}_0(\mathbb{R}^d)$, we see that $\widetilde{h} \sim \mathrm{FGF}_{s-1}(\mathbb{R}^d)$. Thus $\widetilde{h}$ is α-Hölder continuous for all $\alpha < s - 1$ by the preceding case. By the proof[7] of [Sil07, Proposition 2.8], h is almost surely in $C^{1,\alpha}(\mathbb{R}^d)$.

(iii) If $H = 1$, we may apply the same argument with $(-\Delta)^{(1-\alpha)/2}$ in place of $(-\Delta)^{1/2}$, which means that $\widetilde{h} \sim \mathrm{FGF}_{(1+\alpha)/2}(\mathbb{R}^d)$.

(iv) If $H = 2$, then we may apply the same reasoning we applied in case (ii), leveraging the $H = 1$ case.

(v) For $H > 2$, we note that $f \in C^{k+2,\alpha}(\mathbb{R}^d)$ whenever $\Delta f \in C^{k,\alpha}(\mathbb{R}^d)$ [Fol99, Theorem 2.28]. Therefore, the result follows from the case $H \in (0,2]$ by induction.   □

As an application of the ideas presented in this section, we construct a coupling of all the fractional Gaussian fields on $\mathbb{R}^d$.

Proposition 6.3. *There exists a coupling of the random fields* $\{h_s : s \in \mathbb{R}\}$ *such that* $h_s \sim \mathrm{FGF}_s(\mathbb{R}^d)$ *and* $h_s = (-\Delta)^{\frac{s'-s}{2}} h_{s'}$ *for all* $s, s' \in \mathbb{R}$. *Furthermore, in this coupling* h_s *determines* $h_{s'}$ *for all* $s, s' \in \mathbb{R}$.

Proof. We will start with an FGF with Hurst parameter 2 and apply the fractional Laplacian to obtain FGFs with Hurst parameters in $(0, 2)$. The remaining FGFs are then obtained by applying integer powers of the Laplacian to FGFs with Hurst parameter in $(0, 2]$.

Let $h_{2+d/2} \sim \mathrm{FGF}_{2+d/2}(\mathbb{R}^d)$. As discussed in the proof of Proposition 6.2 case (ii), we can fix the values and first-order derivatives of h to vanish at the origin to obtain a scale-invariant random function h_0 whose restriction to $\mathcal{S}_1(\mathbb{R}^d)$ agrees with h. Furthermore, we have

$$\mathbb{E}\left[\int_{|x|>1} \frac{|h_0(x)|}{|x|^{d+2s+k+1}}\, dx\right] = \int_{|x|>1} \frac{\mathbb{E}|h_0(x)|}{|x|^{d+2s+k+1}}\, dx = \int_{|x|>1} \frac{c|x|^2}{|x|^{d+2s+k+1}} < \infty,$$

whenever $s \in (0, 1/2]$ and $k = 1$ or when $s \in (1/2, 1)$ and $k = 0$. Therefore, we may define $h_{s'+d/2} = (-\Delta)^{1-s'/2} h_{2+d/2}$ for all $s' \in (0, 2)$. If $s' + d/2 \in \mathbb{R} \setminus (0, 2]$, define $h_{s'+d/2} = (-\Delta)^{\frac{s-s'}{2}} h_{s+d/2}$, where s is the unique real number in $(0, 2]$ for which $s - s'$ is an even integer.

It follows from the construction that $h_s \sim \mathrm{FGF}_s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$ and that $h_s = (-\Delta)^{\frac{s'-s}{2}} h_{s'}$ for all $s, s' \in \mathbb{R}$, which in turn implies that h_s determines $h_{s'}$ for all $s, s' \in \mathbb{R}$.   □

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[7]Proposition 2.8 in [Sil07] includes a boundedness hypothesis which does not hold here. However, that hypothesis is only used for a norm bound also given in the proposition statement. The regularity assertion follows from the other hypotheses.

7. Restricting FGFs

In this section we study how fractional Gaussian fields behave when restricted to a lower dimensional subspace.

We regard $\mathbb{R}^{d-1}$ as a subspace of $\mathbb{R}^d$ by associating $(x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$ with $(x_1, \ldots, x_{d-1}, 0) \in \mathbb{R}^d$. For all $\phi \in \mathcal{S}_H(\mathbb{R}^{d-1})$, we define $\phi^\uparrow \in \mathcal{S}'(\mathbb{R}^d)$ by

$$(\phi^\uparrow, f) := \int_{\mathbb{R}^{d-1}} f(x)\phi(x)\, dx$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Theorem 7.1. *Fix $s > \frac{1}{2}$, suppose $h^d \sim \mathrm{FGF}_s(\mathbb{R}^d)$. Then $\phi^\uparrow \in T_s(\mathbb{R}^d)$ for all $\phi \in \mathcal{S}_H(\mathbb{R}^{d-1})$, which means that $(h, \phi^\uparrow)$ is a well-defined random variable almost surely (see Section 3.1). Moreover, h^d almost surely determines a random distribution $h^{d-1} \sim \mathrm{FGF}_{s-1/2}(\mathbb{R}^{d-1})$ such that for all $\phi \in C_c^\infty(\mathbb{R}^{d-1}) \cap \mathcal{S}_H(\mathbb{R}^d)$ fixed, the relation*

$$(h^d, \phi^\uparrow) = C(h^{d-1}, \phi), \tag{7.1}$$

holds almost surely, where C is a constant depending only on d and s.

We refer to h^{d-1} as the **restriction** of h^d to $\mathbb{R}^{d-1}$.

Proof. Let $\{\eta_k\}_{k \in \mathbb{N}}$ be an approximation to the identity, which means that

(i) η_k is smooth for all $k \in \mathbb{N}$,
(ii) $\eta_k \geq 0$,
(iii) $\mathrm{supp}(\eta_k) \subset B(0, 1/k)$, and
(iv) $\int_{\mathbb{R}^d} \eta_k(x)\, dx = 1$.

Then $\phi_k^\uparrow := \eta_k * \phi^\uparrow \in \mathcal{S}_H(\mathbb{R}^d)$, because applying the definition of a convolution and making a substitution $w = x - y$ yields

$$\int_{\mathbb{R}^d} x^\alpha (\eta_k * \phi^\uparrow)(x)\, dx = \int_{\mathbb{R}^d} \eta_k(w) \overbrace{\int_{\mathbb{R}^{d-1}} (w + y)^\alpha \phi(y)\, dy}^{0}\, dw = 0,$$

since $\int x^\alpha \phi(x)\, dx = 0$ whenever $|\alpha| \leq H$. Moreover we can use Theorem 3.3 to check that $\{\phi_k^\uparrow\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $\dot{H}^{-s}(\mathbb{R}^d)$. Since $\phi_k^\uparrow \to \phi^\uparrow$ in $\mathcal{S}'(\mathbb{R}^d)$, we have $\phi_k \to \phi^\uparrow$ in $\dot{H}^{-s}(\mathbb{R}^d)$ and therefore $\phi^\uparrow \in T_s(\mathbb{R}^d)$.

Since $\phi_k^\uparrow \to \phi^\uparrow$ in $\dot{H}^{-s}(\mathbb{R}^d)$, we have $\mathrm{Var}[(h^d, \phi)] = \lim_{k \to \infty} \|\phi_k\|_{\dot{H}^{-s}(\mathbb{R}^d)}^2$. By definition of $\{\eta_k\}$, $\mathrm{Var}[(h^d, \phi)]$ satisfies the formula in Theorem 3.3 where we replace $\mathbb{R}^d$ by $\mathbb{R}^{d-1}$ and set $\phi_1 = \phi_2 = \phi$. This is the covariance structure of FGF on $\mathbb{R}^{d-1}$ with the same Hurst parameter as h^d, up multiplicative constant. In other words, there is a constant C so that if we define $(h^{d-1}, \phi) := C^{-1}(h^d, \phi^\uparrow)$ for all ϕ in a countable dense subset $\Phi \subset C_c^\infty(\mathbb{R}^d)$, then h^{d-1} has the law of an $\mathrm{FGF}_{s-1/2}(\mathbb{R}^{d-1})$ restricted to Φ. Therefore, h^{d-1} extends uniquely to a tempered distribution on $\mathbb{R}^{d-1}$, and it satisfies (7.1) for all $\phi \in C_c^\infty(\mathbb{R}^d) \cap \mathcal{S}_H(\mathbb{R}^d)$ by continuity. $\square$

Since h^{d-1} is a function of h^d almost surely, we can define a measurable function $\mathfrak{R}$ on $\mathcal{S}'$ such that $h^{d-1} = \mathfrak{R}h^d$. We call $\mathfrak{R}$ the restriction operator. We can see that $\mathfrak{R}$ maps an FGF to a lower dimensional FGF with the same Hurst parameter. By applying $\mathfrak{R}$ repeatedly, we can restrict an $\mathrm{FGF}(\mathbb{R}^d)$ to an $\mathrm{FGF}(R^{d'})$ with the same Hurst parameter, as long as $d' > -2H$.

When the Hurst parameter is positive, $\mathrm{FGF}_s(\mathbb{R}^d)$ is a pointwise-defined random function, so $\mathfrak{R}$ agrees with the usual restriction of functions. In particular, we note that the restriction of a multidimensional fractional Brownian motion with Hurst parameter H to a line through the origin is a linear fractional Brownian motion with Hurst parameter H.

8. Regularity of FGF(D)

Let $s \geq 0$, and let $h \sim \mathrm{FGF}_s(\mathbb{R}^d)$ be coupled with $h_D \sim \mathrm{FGF}_s(D)$ and h_D^{Har} as in Section 5, so that $h = h_D + h_D^{\mathrm{Har}}$. In this section, we will show that h_D^{Har} is smooth in D almost surely. First, we record some results on the fractional Laplacian following Section 2 of [Sil07].

Lemma 8.1. *If s is a positive integer and g is a distribution on D such that $(-\Delta)^s g$ is smooth on D, then g is smooth on D.*

Proof. Let $s = 1$; the case $s > 1$ follows by induction. If $\Delta g = 0$, the desired result is Weyl's lemma (see Appendix B of [Lax02]). If Δg is not zero, suppose that $U \subset D$ is an arbitrary ball, and let g_1 be a function which is smooth on U such that $(-\Delta)g_1|_U = (-\Delta)g|_U$ [Fol99, Corollary 2.20]. Applying the result for the case $\Delta g = 0$, we see that $g - g_1$ is smooth on U, and hence so is g. Since U was arbitrary, g is smooth on D. $\square$

Lemma 8.2. *Let $0 < s < 1$, and let $B \subset \mathbb{R}^d$ be an open ball. If $(-\Delta)^s f$ is smooth in B, then f is smooth in B.*

Proof. By [LD72, (1.6.11), p. 121] (see also Section 5.1 in [Sil07]) the solution u to $(-\Delta)^s u|_B = 0$ and $f|_{\mathbb{R}^d \setminus B} = f|_{\mathbb{R}^d \setminus B}$ is given by the convolution $u(y) = \int_{\mathbb{R}^d \setminus B} f(x) P(x,y)\, dy$ where $P(x,y)$, the Poisson kernel of $(-\Delta)^s$, is proportional to
$$\frac{(1 - |x|^2)^s}{(|y|^2 - 1)^s |x - y|^d}.$$
Since P is smooth, we see that

$$g \text{ is smooth in } B \text{ whenever } (-\Delta)^s g = 0 \text{ in } B. \tag{8.1}$$

By convolving with the Green's function (4.5) for the fractional Laplacian on B, we see that there exists a continuous solution g of the equation $(-\Delta)^s g = (-\Delta)^s f$ on B and $g = 0$ on $\mathbb{R}^d \setminus B$ which is smooth in B. Since $f - g$ is also smooth in B by (8.1), we conclude that f is smooth in B. $\square$

We can now prove the main result in this section.

Theorem 8.3. *If $D \subsetneq \mathbb{R}^d$ is an allowable domain, then h_D^{Har} is smooth on D almost surely. If $U \subset D$ is a domain, then $h_{U,D}^{\mathrm{Har}}$ is smooth on U almost surely.*

Proof. We consider several cases:

(i) We first suppose d is even, $0 < H < 1$, and D is a ball. The argument in case (ii) of Proposition 6.2 shows that h satisfies condition (2.4) with $s = H$ almost surely. Since $h_D^{\mathrm{Har}} = h - h_D$ and h_D is supported in D, we see that h_D^{Har} also satisfies (2.4) with $k = -1$. Therefore, $(-\Delta)^H h_D^{\mathrm{Har}}$ is tempered distribution. By the definition of h_D^{Har} as a random field with $(h, \phi) \sim \mathcal{N}(0, \|P_D^{\mathrm{Har}}(-\Delta)^{-s}\phi\|_{\dot{H}^s(\mathbb{R}^d)})$, we have for all $\phi \in C_c^\infty(D)$,

$$((-\Delta)^{\frac{d}{2}}(-\Delta)^H h_D^{\mathrm{Har}}, \phi) = ((-\Delta)^s h_D^{\mathrm{Har}}, \phi) = 0$$

almost surely. Considering a countable dense subset of $C_c^\infty(D)$, we conclude that $(-\Delta)^{\frac{d}{2}}(-\Delta)^H h_D^{\mathrm{Har}}|_D = 0$ almost surely. Thus by Lemma 8.1, $(-\Delta)^H h_D^{\mathrm{Har}}$ is smooth in D almost surely. By Lemma 8.2, h_D^{Har} is smooth in D almost surely.

(ii) Suppose that d is even, $1 < H < 2$ and D is a ball whose closure does not contain the origin. By the scale invariance of h, there exists $c > 0$ so that have $\mathbb{E}|\nabla h(x)| = c|x|^{H-1}$ for all $x \in \mathbb{R}^d$. Thus

$$\mathbb{E}\left[\int_{|x|>1} \frac{|\nabla h(x)|}{|x|^{d+2H-2}}\, dx\right] = \int_{|x|>1} \frac{\mathbb{E}|\nabla h(x)|}{|x|^{d+2H-2}}\, dx < \infty,$$

which implies that $|\nabla h|$ satisfies condition (2.4) with $s = H - 1$ almost surely. Since $h \in C^1(\mathbb{R}^d)$ and $h_D \in C^1(\mathbb{R}^d)$, we have $h_D^{\mathrm{Har}} \in C^1(\mathbb{R}^d)$. Therefore, $|\nabla h_D^{\mathrm{Har}}|$ satisfies (2.4) almost surely. By the same argument as in Case (i) above, $\partial_{x_i} h_D^{\mathrm{Har}}$ is smooth in D for all $1 \le i \le d$. Therefore h_D^{Har} is smooth in D.

(iii) If d is even, $H \in \{0,1\}$ and D is a ball, then s is an integer. So h_D^{Har} is smooth by Lemma 8.1.

(iv) If $D \subsetneq \mathbb{R}^d$ is allowable, suppose that $U \subset D$ is an arbitrary ball. Since $\mathbb{E}[h_D^2(x)] \le \mathbb{E}[h^2(x)]$ and $\frac{\mathbb{E}[(h_D(x))^2]}{(\mathbb{E}|h_D(x)|)^2} = \frac{\mathbb{E}[(h(x))^2]}{(\mathbb{E}|h(x)|)^2}$, the arguments for the preceding cases imply that $h_{U,D}^{\mathrm{Har}}$ is smooth on U. By the formula

$$h_U^{\mathrm{Har}} = P_{\mathrm{Har}}^U(h_D + h_D^{\mathrm{Har}}) = h_{U,D}^{\mathrm{Har}} + h_D^{\mathrm{Har}},$$

we see that h_D^{Har} is also smooth on U. Since U is arbitrary ball contained in D, this implies that h_D^{Har} is smooth in D. If U is an arbitrary allowable domain in D, again by $h_U^{\mathrm{Har}} = h_{U,D}^{\mathrm{Har}} + h_D^{\mathrm{Har}}$, we see that $h_{U,D}^{\mathrm{Har}}$ is smooth on U.

(v) Suppose that d is even and $s > 0$. In the preceding cases we have established the result for $H \in [0,2)$. Since $(-\Delta)h_D^{\mathrm{Har}} \overset{d}{=} \widetilde{h}_D^{\mathrm{Har}}$ when $s > 2$, $h \sim \mathrm{FGF}_s(D)$, and $\widetilde{h} \sim \mathrm{FGF}_{s-2}(D)$, Lemma 8.1 establishes the result for all $s > 0$.

(vi) Suppose that d is odd, $H > 0$, and H is not an even integer. Suppose D is an allowable domain in $\mathbb{R}^d$ and regard $\mathbb{R}^d$ as a subspace of $\mathbb{R}^{d+1}$ by mapping $x \in \mathbb{R}^{d-1}$ to $(x_1, x_2, \ldots, x_{d-1}, 0)$. Since $h = h_D + h_D^{\mathrm{Har}}$ where h_D and h_D^{Har} are independent, h_D^{Har} is the conditional expectation of h given h on $\mathbb{R}^d \setminus D$. So if

we regard $\mathbb{R}^d \setminus D$ as a closed set in $\mathbb{R}^{d+1}$, the restriction of $h_{\mathbb{R}^{d+1} \setminus (\mathbb{R}^d \setminus D)}^{\text{Har}}$ has the same law as h_D^{Har} on $\mathbb{R}^d$. Since the restriction of a smooth function is smooth, we conclude that h_D^{Har} is a smooth function in D almost surely.

(vii) If d is odd and $s > 0$, we apply the argument in Case (v) to the result from Case (vi). □

Since $h = h_D + h_D^{\text{Har}}$ and h_D^{Har} is smooth in D, the regularity of $\text{FGF}_s(D)$ is the same as the regularity of $\text{FGF}_s(\mathbb{R}^d)$. In other words, h_D is has α-Hölder derivatives of order up to k, where $k = \lceil H \rceil - 1$ and $\alpha < H - \lceil H \rceil$ (Proposition 6.2).

9. The eigenfunction FGF

Let $D \subset \mathbb{R}^d$ be a bounded domain, and let $s \in (0,1)$. In this section we discuss an different notion of a fractional Gaussian field on D, which we call the **eigenfunction FGF** and denote $\text{EFGF}_s(D)$.

The eigenfunction FGF is based on the following definition of a fractional Laplacian operator on D. Following Section 2.3 in [She07], we let $\{f_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of eigenfunctions of the Dirichlet Laplacian on D, arranged in increasing order of their corresponding eigenvalues $\lambda_n > 0$. We define for all $\phi = \sum (f_n, \phi)_{L^2(\mathbb{R}^d)} f_n \in L^2(D)$ the formal sum

$$(-\Delta)_D^s \phi = \sum_{n \in \mathbb{N}} \lambda_n^s (\phi, f_n)_{L^2(D)} f_n,$$

which converges if $\phi \in C_c^\infty(D)$ [She07]. We call $(-\Delta)_D^s$ the eigenfunction fractional Laplacian operator on D. This fractional Laplacian operator determines a Hilbert space, analogous to $\dot{H}_0^s(D)$, with inner product given by

$$\sum_{n \in \mathbb{N}} \lambda_n^s (\phi_1, f_n)_{L^2(D)} (\phi_2, f_n)_{L^2(D)}.$$

Note that $\{\lambda_n^{-s/2} f_n\}_{n \in \mathbb{N}}$ defines an orthonormal basis with respect to this inner product. We define $\text{EFGF}_s(D)$ to be a standard Gaussian on this space; more precisely, let $\{Z_n\}_{n \in \mathbb{N}}$ be an i.i.d. sequence of standard normal random variables and set for all $\phi \in C_c^\infty(D)$,

$$(h, \phi) := \sum_{n \in \mathbb{N}} Z_n \lambda_n^{-s/2} (f_n, \phi).$$

By Weyl's law, $\lambda_n = \Theta(n^{2/d})$ as $n \to \infty$, so the sum on the right-hand side converges almost surely for each ϕ. Furthermore, the functional h defined this way is a continuous functional by the same argument given for the GFF case in [She07]. We define $\text{EFGF}_s(D)$ to be the law of h.

Both the fractional Laplacian and the eigenfunction fractional Laplacian can be understood in terms of a local operator in $d + 1$ dimensions. In [CS07],

the fractional Laplacian is realized as a boundary derivative for an extension problem in $\mathbb{R}^d \times [0, \infty)$. A corresponding analysis for the eigenfunction Laplacian is developed in [CT10] and [CDDS11] by considering a similar extension problem in $D \times [0, \infty)$. We carry out an analogous comparison between $\mathrm{FGF}_s(\mathbb{R}^d)$, $\mathrm{FGF}_s(D)$, and $\mathrm{EFGF}_s(D)$ by realizing each as a restriction of a higher-dimensional random field that can be understood as a Gaussian free field with spatially varying resistance (see Propositions 9.1, 9.2 and 9.3 below).

Let $s \in (0, 1)$, and define $\alpha = \frac{1-2s}{1-s} \in (-\infty, 1)$. For simplicity, we will assume $d \geq 2$. We introduce the coordinates $(x_1, \ldots, x_d, z)$ for $\mathbb{R}^{d+1}$, and we define the following variant of the gradient operator. For $\phi \in \mathcal{S}(\mathbb{R}^{d+1})$, we set

$$\nabla_\alpha \phi := \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \ldots, \frac{\partial \phi}{\partial x_d}, |z|^{\alpha/2} \frac{\partial \phi}{\partial z} \right).$$

We will use

$$\mathcal{S}_{\mathrm{sym}}(\mathbb{R}^{d+1}) := \{ \phi \in \mathcal{S}_0(\mathbb{R}^{d+1}) \, : \, \phi(x, z) = \phi(x, -z) \text{ for all } x, z \}$$

as a space of test functions. Integrating by parts (see [Bas98, Chapter 7] for more details), we find that for all $\phi \in \mathcal{S}_{\mathrm{sym}}(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^{d+1}} |\nabla_\alpha \phi|^2 = - \int_{\mathbb{R}^{d+1}} \phi(L_\alpha \phi),$$

where the operator L_α is defined by

$$L_\alpha = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2} + \frac{\partial}{\partial z} \left(|z|^\alpha \frac{\partial}{\partial z} \right).$$

By the Bochner-Minlos theorem, we can define a random tempered distribution h_α for which

$$\mathbb{E}[\exp(i(h_\alpha, \phi))] = \exp \left(-\frac{1}{2} \int_{\mathbb{R}^{d+1}} \widetilde{\phi}(-L_\alpha)^{-1} \widetilde{\phi} \right) \tag{9.1}$$

$$= \exp \left(-\frac{1}{2} \int_{\mathbb{R}^{d+1}} |\nabla_\alpha(-L_\alpha)^{-1} \widetilde{\phi}|^2 \right),$$

where

$$\widetilde{\phi}(x, z) = \tfrac{1}{2}(\phi(x, z) + \phi(x, -z)),$$

and $(-L_\alpha)^{-1}\phi$ satisfies $-L_\alpha(-L_\alpha)^{-1}\phi = \phi$ and vanishes at infinity—see the proof of Proposition 9.1 for the existence of such a function. We then restrict the domain of h_α to $\mathcal{S}_{\mathrm{sym}}(\mathbb{R}^{d+1})$, so that (9.1) holds with ϕ in place of $\widetilde{\phi}$.

Since the right-hand side of (9.1) reduces when $\alpha = 0$ to the GFF characteristic function evaluated at $\widetilde{\phi}$, we may think of h as a symmetrized and re-weighted[8] version of the Gaussian free field on $\mathbb{R}^{d+1}$. We define restriction of h_α to $\mathbb{R}^d \times \{0\}$ by

$$(h_\alpha|_{\mathbb{R}^d \times \{0\}}, \phi) := (h_\alpha, (x, z) \mapsto \phi(x)\delta_0(z)) \quad \text{for } \phi \in \mathcal{S}(\mathbb{R}^d),$$

---

[8]We are using the term *weight* here in sense described for the disrete GFF in Section 4 of [She07].

where δ_0 denotes the unit Dirac mass at $z = 0$. See the proof of Theorem 7.1 for an explanation of why the random variable on the right-hand side is well-defined. More precisely, we will show that the covariance kernel of $h_\alpha|_{\mathbb{R}^d \times \{0\}}$ is that of an $\mathrm{FGF}_s(\mathbb{R}^d)$. It follows from continuity of $\mathrm{FGF}_s(\mathbb{R}^d)$ (as a functional on $\mathcal{S}(\mathbb{R}^d)$) that this restriction can be defined on a countable dense subset of $\mathcal{S}(\mathbb{R}^d)$ and continuously extended to obtain a random tempered distribution.

Proposition 9.1. *The restriction of h_α to $\mathbb{R}^d \times \{0\}$ is an $\mathrm{FGF}_s(\mathbb{R}^d)$, up to multiplicative constant.*

Proof. Let $(X_t)_{t \geq 0}$ be a diffusion in $\mathbb{R}^{d+1}$ with a standard Brownian motion B in the first d coordinates and the process $Z_t := (\frac{\sqrt{2}}{\delta} Y_t)^\delta$ in the last coordinate, where $\delta = 2(1-s)$ and Y is δ-dimensional Bessel process reflected symmetrically at 0. An application of Itō's formula reveals that L_α is the generator of X. We define the Green's function

$$G_\alpha(x_1, x_2) := \lim_{\epsilon \to 0} (2\epsilon)^{-d-1} \mathbb{E}^{x_1} \left[\int_0^\infty \mathbf{1}_{\{X_t \in Q(x_2, \epsilon)\}} \, dt \right], \quad x_1, x_2 \in \mathbb{R}^{d+1},$$

where $Q(x, \epsilon) := \{y \in \mathbb{R}^{d+1} : |x_k - y_k| < \epsilon \text{ for all } 1 \leq k \leq d+1\}$. Since $d+1 \geq 3$ implies that X is transient, the limit on the right-hand side is well-defined—the proof is similar to the proof for the case $\alpha = 0$ [MP10, Section 3.3]. Since L_α is the generator of X, we have

$$((-L_\alpha)^{-1} \phi)(x_1) = \int_{\mathbb{R}^{d+1}} G_\alpha(x_1, x_2) \, \phi(x_2) \, dx_2$$

for all $\phi \in \mathcal{S}_{\mathrm{sym}}(\mathbb{R}^{d+1})$ (see Chapter II in [Bas98]). Therefore,

$$\mathbb{E}[(h_\alpha, \phi)^2] = \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} G_\alpha(x_1, x_2) \, \phi(x_1) \phi(x_2) \, dx_1 \, dx_2.$$

We denote by $(\ell_s)_{s \geq 0}$ the local time of Z at 0 and by $(\tau_t)_{t \geq 0}$ the inverse function of ℓ [RY99]. Then $(X_{\tau_t})_{t \geq 0}$ is a $2s$-stable Lévy process in $\mathbb{R}^d$, and the integral $\int_0^s \frac{1}{2\epsilon} \mathbf{1}_{\{Z_u \in (-\epsilon, \epsilon)\}} \, du$ converges almost surely to ℓ_s as $\epsilon \to 0$ [MO69]. Therefore, the Green's function of the a $2s$-stable Lévy process in $\mathbb{R}^d$ evaluated at $x_1, x_2 \in \mathbb{R}^d \times \{0\}$ equals

$$\lim_{\epsilon \to 0} (2\epsilon)^{-d} \mathbb{E}^{x_1} \left[\int_0^\infty \mathbf{1}_{\{B_{\tau_t} \in Q(x_2, \epsilon)\}} \, dt \right]$$

$$= \lim_{\epsilon \to 0} (2\epsilon)^{-d} \mathbb{E}^{x_1} \left[\int_0^\infty \mathbf{1}_{\{B_s \in Q(x_2, \epsilon)\}} \frac{d\ell_s}{ds} \, ds \right]$$

$$= \lim_{\epsilon \to 0} (2\epsilon)^{-d-1} \mathbb{E}^{x_1} \left[\int_0^\infty \mathbf{1}_{\{B_s \in Q(x_2, \epsilon)\}} \mathbf{1}_{\{Z_s \in Q(0, \epsilon)\}} \, ds \right] = G_\alpha(x_1, x_2).$$

In other words, the restriction to $\{z = 0\}$ of the Green's function of X is equal to the Green's function of a $2s$-stable Lévy process in $\mathbb{R}^d$. The latter is proportional to $|x_1 - x_2|^{2s-d}$ [CS98, (1.1)], and the covariance kernel of $\mathrm{FGF}_s(\mathbb{R}^d)$
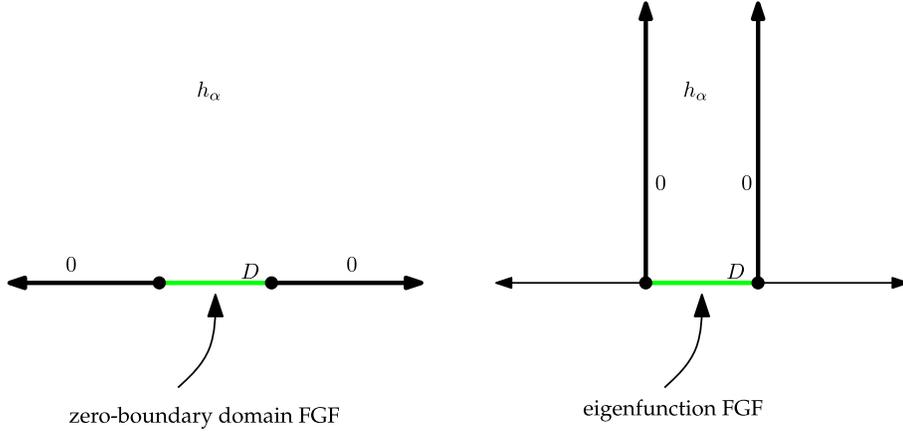
FIG 9.1. *Propositions 9.2 and 9.3 describe the relationship between* $\mathrm{FGF}_s(D)$ *and* $\mathrm{EFGF}_s(D)$. *We obtain an* $\mathrm{FGF}_s(D)$ *by subtracting from* h_α *its conditional expectation given its values on* $(\mathbb{R}^d \setminus D) \times \{0\}$ *and restricting to* $D \times \{0\}$, *and we obtain an* $\mathrm{EFGF}_s(D)$ *by subtracting from* h_α *its conditional expectation given its values on* $\partial D \times \mathbb{R}$ *and restricting to* $D \times \{0\}$.

is also proportional to $|x_1 - x_2|^{2s-d}$ by Theorem 3.3. Since the law of a centered Gaussian process is determined by its covariance kernel, this concludes the proof. □

In Propositions 9.2 and 9.3 below, we discuss projections of h_α onto certain subdomains of $\mathbb{R}^{d+1}$. These projections are analogous to those discussed for the FGF in Section 5. We state these propositions using terminology described in Remark 5.3, to which we refer the reader for a rigorous interpretation.

Proposition 9.2. *Let* $D \subset \mathbb{R}^d$ *and define* h_D *to be the restriction to* $D \times \{0\}$ *of* h_α *minus the conditional expectation of* h_α *given its values on* $(\mathbb{R}^d \setminus D) \times \{0\}$. *Then* $h_D \sim \mathrm{FGF}_s(D)$, *up to multiplicative constant.*

Proof. This result follows immediately from Proposition 9.1 and the fact that $h \sim \mathrm{FGF}_s(\mathbb{R}^d)$ minus its conditional expectation given its values on $\mathbb{R}^d \setminus D$ has the law of an $\mathrm{FGF}_s(D)$. □

Proposition 9.3. *Let* $D \subset \mathbb{R}^d$ *and define* $\widetilde{h}_D$ *to be the restriction to* $D \times \{0\}$ *of* h_α *minus the conditional expectation of* h_α *given its values on* $\partial D \times \mathbb{R}$. *Then* $\widetilde{h}_D \sim \mathrm{EFGF}_s(D)$, *up to multiplicative constant.*

Proof. By the definition of the eigenfunction FGF, it suffices to show that if f_1 and f_2 are $L^2(D)$-normalized eigenfunctions of the Dirichlet Laplacian on D with eigenvalues λ_1 and λ_2, then

$$\mathbb{E}[(\widetilde{h}_D, f_1)(\widetilde{h}_D, f_2)] = C\lambda_1^{-s}\mathbf{1}_{\{\lambda_1=\lambda_2\}}$$

for some constant C. (We will use C to denote a generic constant whose value may change throughout the proof.)

It is straightforward to verify that h_α minus the conditional expectation of h_α given its values on $\partial D \times \mathbb{R}$ is equal in law to the field h_α^{cyl} whose covariance kernel is given by the Green's function of the diffusion X defined in the proof of Proposition 9.1 stopped upon hitting the cylinder $\partial D \times \mathbb{R}$. Equivalently, the covariances of h_α^{cyl} are given in terms of the inverse L_α^{-1} of the operator L_α with zero boundary conditions on $\partial D \times \mathbb{R}$ via $\mathbb{E}[(h_\alpha^{\mathrm{cyl}}, \phi)^2] = \int_{D \times \mathbb{R}} \phi \widetilde{L}_\alpha^{-1} \phi$.

Let $w_\lambda(z)$ be the function on $\mathbb{R}$ which satisfies $w_\lambda(0) = 1$, $w_\lambda(\infty) = 0$,

$$-\lambda w_\lambda(z) + \frac{\partial}{\partial z}\left(z^\alpha \frac{\partial w_\lambda}{\partial z}\right) = 0 \quad \text{for all } z \in (0, \infty),$$

and $w_\lambda(z) = w_\lambda(-z)$ for all $z \in \mathbb{R}$. A symbolic ODE solver may be used to express w_λ in terms of the modified Bessel function of the second kind K_s as

$$w_\lambda(z) = C\lambda^{s/2} z^{s/(2-2s)} K_s\left(2(1-s)z^{\frac{1}{2-2s}}\sqrt{\lambda}\right).$$

We define the operator $L_{\alpha,\lambda} = -\lambda + \frac{\partial}{\partial z}(z^\alpha \frac{\partial}{\partial z})$. Integration by parts reveals that for all $\phi \in C_c^\infty(\mathbb{R})$, we have

$$(-L_{\alpha,\lambda} w_\lambda, \phi) := (w_\lambda, -L_{\alpha,\lambda}\phi) = \lim_{z \to 0} z^\alpha w_\lambda'(z)\phi(z) = C\lambda^s \phi(0) \qquad (9.2)$$

for some constant C, where in the last step we have used the expansion

$$K_s(t) = 2^{s-1}\Gamma(s)t^{-s} + 2^{-s-1}\Gamma(-s)t^s - \frac{2^{s-3}\Gamma(s)t^{2-s}}{s-1} + O(t^{2+s})$$

as $t \to 0^+$. We may restate (9.2) by writing $\delta_0 = C\lambda^{-s}(-L_{\alpha,\lambda})w_\lambda$, where δ_0 denotes the unit Dirac mass at the origin. Therefore, using the relation

$$(\widetilde{h}_D, \phi) := \lim_{k \to \infty} (h_\alpha^{\mathrm{cyl}}, (x, z) \mapsto \psi(x)\eta_k(z)),$$

where $\{\eta_k\}_{n \in \mathbb{N}}$ is an approximation to the identity, we have

$$\mathbb{E}[(\widetilde{h}_D, f_1)(\widetilde{h}_D, f_2)]$$

$$= C\lambda_1^{-s}\lambda_2^{-s} \int_{D \times \mathbb{R}} f_1(x)(L_{\alpha,\lambda_1} w_{\lambda_1})(z)(-L_\alpha)^{-1}[f_2(x)L_{\alpha,\lambda_2} w_{\lambda_2}(z)]\,dx\,dz$$

$$= C\lambda_1^{-s}\lambda_2^{-s} \int_{D \times \mathbb{R}} f_1(x)(L_{\alpha,\lambda_1} w_{\lambda_1})(z)f_2(x)w_{\lambda_2}(z)\,dx\,dz$$

$$= C\lambda_1^{-s}\lambda_2^{-s} \left(\int_D f_1(x)f_2(x)dx\right)\left(\int_{\mathbb{R}}(L_{\alpha,\lambda_1} w_{\lambda_1})(z)w_{\lambda_2}(z)dz\right)$$

$$= C\,\mathbf{1}_{\{\lambda_1 = \lambda_2\}}\lambda_1^{-2s}\lambda_1^s = C\,\mathbf{1}_{\{\lambda_1 = \lambda_2\}}\lambda_1^{-s},$$

as desired.   □

10. FGF local sets

10.1. FGF with boundary values

In Section 4, we defined the FGF on a domain with zero boundary conditions. It is also natural to consider other boundary conditions to give rigorous meaning to the idea that the conditional law of the FGF in D given the values of h outside D is an FGF on D with boundary value $h|_{\mathbb{R}^d \setminus D}$. For simplicity, we only consider the case where D is bounded and the boundary values are Schwartz.

Definition 10.1. Given a bounded domain D and a Schwartz function f which is s-harmonic in D, the random distribution $f + h_D$ is called the FGF on D with boundary values $f|_{\mathbb{R}^d \setminus D}$.

10.2. Local Sets of the FGF on a bounded domain

The concept of a *local set* of the Gaussian free field is developed in [SS10]. It turns out to be an important concept and tool in the study of couplings between the GFF and random closed sets such as SLE ([SS10], [MS12a], [MS12b], [MS], [MS13]). The theory of local sets of the Gaussian free field carries over to the FGF setting with minimal modification.

Let Γ_D be the space of all closed non-empty subsets of $\overline{D}$. We endow Γ with the Hausdorff metric induced by Euclidean distance: the distance between sets $S_1, S_2 \in \Gamma$ is

$$d_{\mathrm{Haus}}(S_1, S_2) := \max\Big\{ \sup_{x \in S_1} \mathrm{dist}(x, S_2), \sup_{y \in S_2} \mathrm{dist}(y, S_1) \Big\},$$

where $\mathrm{dist}(x, S) := \inf_{y \in S} |x - y|$. Note that Γ is naturally equipped with the Borel σ-algebra induced by this metric. Furthermore, Γ_D is a compact metric space [Mun99, pp. 280-281]. Note that the elements of Γ are themselves compact.

Given $A \subset \Gamma$, let A_δ denote the closed set containing all points in Γ whose distance from A is at most δ. Let $\mathcal{A}_\delta$ be the smallest σ-algebra in which A and the restriction of h (as a distribution) to the interior of A_δ are measurable. Let $\mathcal{A} = \bigcap_{\delta \in \mathbb{Q}, \delta > 0} \mathcal{A}_\delta$. Intuitively, this is the smallest σ-algebra in which A and the values of h in an infinitesimal neighbourhood of A are measurable.

Given a random closed set $A \subset D$ and deterministic open subset $B \subset D$, we define the event $S = \{A \cap B = \emptyset\}$ and the random set $\tilde{A} := A$ if S occurs and $\emptyset$ otherwise.

Lemma 10.2. *Let D be a bounded domain, suppose that (h, A) is a random variable which is a coupling of an instance h of the FGF with a random element A of Γ. Then the following are equivalent:*

(i) *For each deterministic open $B \subset D$, the event $A \cap B = \emptyset$ is conditionally independent, given the projection of h onto $\mathrm{Har}_s(B)$, of the projection of h onto $\dot{H}_0^s(B)$. In other words, the conditional probability that $A \cap B = \emptyset$ given h is a measurable function of the projection of h onto $\mathrm{Har}_s(B)$.*

   (ii) For each deterministic open $B \subset D$, we have that given *the projection of h onto* $\mathrm{Har}_s(B)$*, the pair $(S, \tilde{A})$ is independent of the projection of h onto* $\dot{H}_0^s(B)$.

  (iii) Conditioned on $\mathcal{A}$, (a regular version of) the conditional law of h is that of $h_1 + h_2$ where h_2 is a zero boundary FGF on $D \setminus A$ (extended to all of D by setting $h_1|_A = 0$) and h_1 is an $\mathcal{A}$-measurable random distribution (i.e., as a distribution-valued function on the space of distribution-set pairs (h, A), h_1 is $\mathcal{A}$-measurable) which is almost surely s-harmonic on $D \setminus A$.

  (iv) A sample with the law of (h, A) can be produced as follows. First choose the pair (h_1, A) according to some law where h_1 is almost surely s-harmonic on $D \setminus A$. Then sample an instance h_2 of zero boundary FGF on $D \setminus A$ and set $h = h_1 + h_2$.

    Lemma 10.2 may be proved by making minor modifications to the proof of Lemma 3.9 in [SS10] to generalize from the setting $s = 1, d = 2$ to arbitrary $s \in \mathbb{R}$ and $d \geq 1$.

    We say a random closed set A coupled with an instance h of the FGF, is **local** if one of the equivalent conditions in Lemma 10.2 holds. For any coupling of A and h, we use the notation C_A to describe the conditional expectation of the distribution h given $\mathcal{A}$. When A is local, C_A is the distribution h_1 described in (iii) above.

    Given two distinct random sets A_1 and A_2 (each coupled with a FGF h), we can construct a coupling (h, A_1, A_2) such that the marginal law of (h, A_i) (for $i \in \{1, 2\}$) is the given one, and conditioned on h, the sets A_1 and A_2 are independent of one another. This can be done by first sampling h and then sampling A_1 and A_2 independently from the regular conditional probabilities. The union of A_1 and A_2 is then a new random set coupled with h. We denote this new random set by $A_1 \breve{\cup} A_2$ and refer to it as the **c**onditionally independent union of A_1 and A_2. The following lemma is analogous to [SS10, Lemma 3.6],

Lemma 10.3. *If A_1 and A_2 are local sets coupled with the GFF h on D, then their conditionally independent union $A = A_1 \breve{\cup} A_2$ is also local. Moreover, given $\mathcal{A}$ and the pair (A_1, A_2), the conditional law of h is given by C_A plus an instance of the FGF on $D \setminus A$.*

10.3. An example of a local set

Certain level lines of the Gaussian free field are studied in [SS10] and shown to be local sets. We will show that certain level sets of fractional Gaussian fields with positive Hurst parameter are also local sets.

    Let $c_1, c_2 > 0$, let $s > d/2$ and let h be the FGF_s on the unit ball B in $\mathbb{R}^d$ with boundary values c_1 on the upper hemisphere, $-c_2$ on the lower hemisphere, and zero outside a compact set. Then there is a unique surface whose boundary equals between the boundary of the upper hemisphere and on which $h = 0$. This surface separates a region where h is positive and a region where h is negative. We call this interface the level set of h and denote it by L. To see that L is

a local set, fix $\delta > 0$ and let L_δ be the intersection of D with the union of all closed boxes of the grid $\delta \mathbb{Z}^d$ that intersect L. For each fixed closed set C, the event $\{L_\delta = C\}$ is determined by $h|_C$. Given a deterministic open set $U \cap C = \emptyset$, the projection of h to $\dot{H}_0^s(U)$ is independent of $h|_C$. Thus L_δ is local. Letting $\delta \to 0$, we see that L is local.

11. Spherical decomposition

Since the fractional Gaussian field on $\mathbb{R}^d$ is isotropic (that is, invariant under rotations), it is natural to consider its decomposition under spherical coordinates. There is a general theorem [Won70, Chapter 7] decomposing any isotropic Gaussian random field into a countable number of mutually uncorrelated single-parameter stochastic processes. However, since the FGF is a tempered distribution modulo a space of polynomials (rather than a tempered distribution) and since it has a special form, we will give the spherical decomposition directly.

11.1. FGF spherical average processes

Let S^{d-1} denote the unit sphere in $\mathbb{R}^d$, and define Ω_d to be the area of S^{d-1}. If f is a continuous function on $\mathbb{R}^d$, then we define the spherical average process $\overline{f}$: $(0, \infty) \to \mathbb{R}$ by $\overline{f}(r) = \frac{1}{\Omega_d} \int_{S^{d-1}} f(r\sigma) \, d\sigma$, where $d\sigma$ denotes $(d-1)$-dimensional Lebesgue measure on S^{d-1}. We calculate that for all $\phi \in C_c^\infty((0, \infty))$,

$$\int_0^\infty \overline{f}(r)\phi(r) \, dr = \frac{1}{\Omega_d} \int_{\mathbb{R}^d} f(x) \frac{\phi(|x|)}{|x|^{d-1}} \, dx. \tag{11.1}$$

Let $s \geq 0$, and let $h \sim \mathrm{FGF}_s(\mathbb{R}^d)$. Motivated by (11.1), we define the **spherical average process** $\overline{h}$ of h by

$$(\overline{h}, \phi) := \frac{1}{\Omega_d} \left(h, x \mapsto \frac{\phi(|x|)}{|x|^{d-1}} \right) \quad \text{for all } \phi \in C_c^\infty((0, \infty)) \cap \mathcal{S}_H(\mathbb{R}).$$

Note that if $\phi \in C_c^\infty((0, \infty)) \cap \mathcal{S}_H(\mathbb{R})$, then $x \mapsto \phi(|x|)/|x|^{d-1}$ is in $\mathcal{S}_H(\mathbb{R}^d)$, so this definition makes sense.

The sphere average process of an FGF is a random distribution, since h is a random tempered distribution and $\phi_n \to 0$ in $C_c^\infty((0, \infty))$ implies $x \mapsto \frac{\phi_n(|x|)}{|x|^{d-1}}$ converges to 0 in $\mathcal{S}(\mathbb{R}^d)$. To find the covariance kernel of $\overline{h}$, we calculate

$$\begin{aligned}
\mathbb{E}[(\overline{h}, \phi)^2] &= \frac{1}{\Omega_d^2} \mathbb{E}\left[\left(h, x \mapsto \frac{\phi(|x|)}{|x|^{d-1}} \right)^2 \right] \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G^s(x, y) \frac{\phi(|x|)\phi(|y|)}{\Omega_d^2 |x|^{d-1} |y|^{d-1}} \, dx \, dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{1}{\Omega_d^2} \int_{S^{d-1}} \int_{S^{d-1}} G^s(r_1\omega, r_2\sigma) \, d\omega d\sigma \right) \phi(r_1)\phi(r_2) \, dr_1 \, dr_2,
\end{aligned}$$

where G^s is the covariance kernel of h, given in Theorem 3.3. Therefore, the covariance kernel of $\overline{h}$ is

$$\overline{G}^s(r_1, r_2) := \frac{1}{\Omega_d^2} \int_{S^{d-1}} \int_{S^{d-1}} G^s(r_1\omega, r_2\sigma) \, d\omega d\sigma.$$

Applying spherical symmetries to simplify this integral, we obtain

$$\overline{G}^s(r_1, r_2) =$$
$$2\,C \int_0^\pi \left(\tfrac{1}{2}\log(r_1^2 + r_2^2 - 2r_1r_2\cos\theta)\right)^{\mathbf{1}_{\{H\in\mathbb{Z}_+\}}} \times$$
$$(r_1^2 + r_2^2 - 2r_1r_2\cos\theta)^H (\sin\theta)^{d-2} \, d\theta,$$

where C is a constant described in Theorem 3.3 and $\mathbb{Z}_+$ is the set of nonnegative integers. In the case $H \notin \mathbb{Z}^+$, we make a substitution to obtain an integral in Euler form whose solution may be expressed in terms of the Gauss hypergeometric function $_2F_1(a, b, c; z)$. In particular, we get

$$\overline{G}^s(r_1, r_2) = C\,2^{d-1}\pi^{-1/2}\frac{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(d-1)} \times$$
$$(r_1 + r_2)^{2H} \, _2F_1\left(\frac{d-1}{2}, -H, d-1; \frac{4r_1r_2}{(r_1+r_2)^2}\right).$$

The hypergeometric function $_2F_1(a, b, c; z)$ satisfies

$$|_2F_1(a, b, c; z) - {_2F_1}(a, b, c; 1)| \asymp |z-1|^{c-a-b}$$

as $z \to 1$ whenever $c - a - b \in (0, 1)$, because the indicial polynomial at $z = 1$ of the hypergeometric equation satisfied by $_2F_1$ has roots 0 and $c - a - b$ (see [Kri10] for details). When $c - a - b > 1$, $_2F_1(a, b, c; z)$ is differentiable at $z = 1$. Since $\frac{4r_1r_2}{(r_1+r_2)^2} = 1 + O(|r_1 - r_2|^2)$ as $r_1 \to r_2$, it follows that when $s > 1/2$, we have $\overline{G}^s(r_1, r_2) - \overline{G}^s(r_1', r_2) \asymp |r_1 - r_1'|^{\min(1,\,2s-1)}$ as r_1' approaches r_1.

When r_1 and r_2 are far apart, $\overline{G}^s(r_1, r_2)$ is approximately a constant times $(r_1 + r_2)^{2H}$ since $_2F_1(a, b, c; z)$ approaches a constant as $z \to 0$. So we see that long-range covariances of h are determined by H, while local covariances are dictated by the parameter $s - 1/2$. In the following proposition, we show that in fact $s - 1/2$ also governs the almost-sure regularity of sample paths of $\overline{h}$. We prove such a statement only for $s - \frac{1}{2} \in (0, 1)$, but we remark that in general the spherical average process is differentiable $\lceil s - \frac{1}{2}\rceil - 1$ times, and those derivatives are α-Hölder continuous for all α less than the fractional part of $s - \frac{1}{2}$.

Proposition 11.1. *When $s \in (1/2, 3/2)$, there exists a version of the spherical average process $\overline{h}$ which is α-Hölder continuous for all $\alpha < s - 1/2$.*

Proof. Since the spherical average covariance kernel $\overline{G}^s(r_1, r_2)$ is finite for all r_1 and r_2 when $s \in (1/2, 3/2)$, there exists a pointwise defined Gaussian process

$\widetilde{h}$ on $(0, \infty)$ which agrees in law with $\overline{h}$ [Dud02, Theorem 12.1.3]. Furthermore, the regularity of the covariance kernel implies that for $m = 1$, we have

$$\mathbb{E}[|\widetilde{h}(r_1) - \widetilde{h}(r_2)|^{2m}] \leq C_m |r_1 - r_2|^{m(2s-1)}, \tag{11.2}$$

where C_1 is some constant. Since $\widetilde{h}(r_1) - \widetilde{h}(r_2)$ is Gaussian, (11.2) holds for all $m \in \mathbb{N}$, for some constants C_m. Applying the Kolmogorov-Chentsov continuity theorem with suitably large m, we conclude that $\widetilde{h}$, and therefore also $\overline{h}$, has a version which is almost surely α-Hölder continuous for all $\alpha < s - 1/2$. □

11.2. Background on spherical harmonic functions

We write the Laplacian in spherical coordinates as

$$\Delta = r^{1-d} \frac{\partial}{\partial r} r^{d-1} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{d-1}}, \tag{11.3}$$

where $\Delta_{S^{d-1}}$ is the Laplacian on the unit sphere $S^{d-1} \subset \mathbb{R}^d$. A polynomial $\phi \in \mathbb{R}[x_1, x_2, \cdots, x_d]$ is said to be harmonic if $\Delta \phi = 0$. Suppose that ϕ is harmonic and homogeneous of degree k. Let $f = \phi|_{S^{d-1}}$, and note that we have $\phi(ru) = f(u)r^k$ for all $u \in S^{d-1}$ and $r \geq 0$. Writing $\Delta \phi = 0$, using (11.3), and setting $r = 1$ yields

$$\Delta_{S^{d-1}} f = -k(k + d - 2)f. \tag{11.4}$$

In other words, f is an eigenfunction of $\Delta_{S^{d-1}}$ with eigenvalue $-k(k + d - 2)$.

We mention a few basic results about spherical harmonics that appear, for example, in [SW71, Chapter IV, §2]. Assume $d \geq 2$, let A_k be the set of homogeneous degree k harmonic polynomials on $\mathbb{R}^d$ and let H_k be the space of functions on S^{d-1} obtained by restricting functions in A_k. An important property is that the spaces H_k are pairwise orthogonal (for the $L^2(S^{d-1})$ inner product) and their union is dense in $L^2(S^{d-1})$. This means that we can define, for each fixed k, an orthonormal basis $\{\phi_{k,j} : 1 \leq j \leq \dim(H_k)\}$ of H_k which is the restriction of the harmonic polynomials $\{P_{k,j} : 1 \leq j \leq \dim(H_k)\} \subset A_k$, so that the collection of all $\phi_{k,j}$ is an orthonormal basis of $L^2(S^{d-1})$.

We will need the following important theorem concerning the behaviour of harmonic polynomials under the Fourier transform [Ste70, pg. 72]. We say that a function $f : \mathbb{R}^d \to \mathbb{C}$ is radial if $f(x) = f(y)$ whenever $|x| = |y|$. We occasionally abuse notation and write $f(r)$ where f is radial and $r \geq 0$, with the understanding that we mean $f((r, 0, \ldots, 0))$.

Theorem 11.2. *Let $P_k(x)$ be a homogeneous harmonic polynomial of degree k in $\mathbb{R}^d$. Suppose that f is radial and that $P_k f \in L^2(\mathbb{R}^d)$. Then the Fourier transform of $P_k f$ is of the form $P_k g$, where g is a radial function. Moreover, the induced transform $\mathcal{F}_{d,k}(f) := g$ depends only on $d + 2k$. More precisely, we have $\mathcal{F}_{d,k} = i^k \mathcal{F}_{d+2k,0}$.*

Remark 11.3. If $P_{k,j} f \in \dot{H}^s(\mathbb{R}^d)$, then

$$\mathcal{F}\left[(-\Delta)^{s/2}(P_{k,j}f)\right](\xi) = |\xi|^s i^k \mathcal{F}_{d+2k,0}[f](\xi) P_{k,j}(\xi),$$

Applying the Fourier transform on both sides (which is the inverse Fourier transform evaluated at $-x$) and using the theorem again, we obtain

$$(-\Delta)^{s/2}(P_{k,j}f) = [(-\Delta)^{s/2}_{\mathbb{R}^{d+2k}}f]P_{k,j},$$

where $(-\Delta)^{s/2}_{\mathbb{R}^{d+2k}}f$ is the fractional Laplacian on $\mathbb{R}^{d+2k}$ acting on f interpreted as a function on $\mathbb{R}^{d+2k}$ (that is, we define $f(x)$ for $x \in \mathbb{R}^{d+2k}$ to be $f(x')$ where x' is any point in $\mathbb{R}^d$ satisfying $|x|_{\mathbb{R}^{d+2k}} = |x'|_{\mathbb{R}^d}$).

Remark 11.4. Let $P_{k,j}f_1$ and $P_{k',j'}f_2 \in \dot{H}^s(\mathbb{R}^d)$. Then

$$\langle P_{k,j}f_1, P_{k',j'}f_2 \rangle_{\dot{H}^s(\mathbb{R}^d)} =$$
$$\begin{cases} \int_0^\infty r^{2s+2k+d-1}g_1(r)\overline{g_2(r)}\,dr & (k,j) = (k',j'), \\ 0 & (k,j) \neq (k',j'). \end{cases} \quad (11.5)$$

by orthonormality of $\phi_{k,j}$, where $g_i = \mathcal{F}_{d,k}[f_i] = i^k\mathcal{F}_{d+2k,0}[f_i]$ for $i \in \{1,2\}$.

We see that the right hand side of (11.5) (for $(k,j) = (k',j')$) can be rewritten as $\langle f_1, f_2 \rangle_{\dot{H}^s(\mathbb{R}^{d+2k})}$ (since $\phi_{0,1} = \Omega_d^{-1/2}$), where the radial functions f_i are treated as functions defined on $\mathbb{R}^{d+2k}$ (as described in the remark above). We thus have a unitary correspondence between elements $x \mapsto f(|x|_{\mathbb{R}^{d+2k}}) \in \dot{H}^s(\mathbb{R}^{d+2k})$ and elements $x \mapsto f(|x|_{\mathbb{R}^d})P_{k,j}(x) \in \dot{H}^s(\mathbb{R}^d)$.

For $k \in \mathbb{N}$ and $1 \leq j \leq \dim(H_k)$, we define the Hilbert space $\dot{H}^s_{k,j}(\mathbb{R}^d)$ to be the space of all functions of the form $P_{k,j}f$ where $x \mapsto f(|x|_{\mathbb{R}^{d+2k}}) \in \dot{H}^s(\mathbb{R}^{d+2k})$ is radial and $P_{k,j}f \in \dot{H}^s(\mathbb{R}^d)$. By 11.5, we see that $\dot{H}^s_{k,j}(\mathbb{R}^d)$ are orthogonal. In fact, they also span $\dot{H}^s(\mathbb{R}^d)$:

Lemma 11.5. $\dot{H}^s_{k,j}(\mathbb{R}^d)$ *are orthogonal subspaces spanning* $\dot{H}^s(\mathbb{R}^d)$.

Proof. We only need to check the spanning condition. Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $\dot{H}^s(\mathbb{R}^d)$, it suffices to show that all $g \in \mathcal{S}(\mathbb{R}^d)$ can be written as a linear combination of terms in $\dot{H}^s_{k,j}(\mathbb{R}^d)$. To do this, we use the stated fact that $\{\omega \mapsto \phi_{k,j}(\omega) : k \in \mathbb{N}, 1 \leq j \leq \dim H_k\}$ a basis for $L^2(S^{d-1})$. We compute for every sphere of radius $|x|$:

$$\langle \omega \mapsto g(|x|\omega), \phi_{k,j} \rangle_{L^2(S^{d-1})} = \int_{S^{d-1}} g(|x|\omega)\phi_{k,j}(\omega)\,d\omega =: \rho_{k,j}(|x|),$$

and see that $g(x) = \sum_{k,j} \rho_{k,j}(|x|)\phi_{k,j}(x/|x|) = \sum_{k,j} |x|^{-k}\rho_{k,j}(|x|)P_{k,j}(x)$. Define $g_{k,j}(x) = |x|^{-k}\rho_{k,j}(|x|)P_{k,j}(x)$ and let $\chi_R(x)$ be the characteristic function of an annulus of radii $1/R$ and R, where $R > 1$. It is clear that $g_{k,j}(x)\chi_R(x)$ is an element of $L^2(\mathbb{R}^d)$, since $\|g_{k,j}(x)\chi_R(x)\|_{L^2(\mathbb{R}^d)} \leq \|g\|_{L^2(\mathbb{R}^d)}$ (by orthogonality of $\phi_{k,j}(x/|x|)$), thus by Fatou's Lemma $g_{k,j} \in L^2(\mathbb{R}^d)$. Hence, it follows that the Fourier transform $\widehat{g}_{k,j}$ exists and is in $L^2(\mathbb{R}^d)$. Following the same reasoning as above with $\xi \mapsto |\xi|^{2s}\widehat{g}_{k,j}(\xi)$, we have that $x \mapsto \rho_{k,j}(|x|)P_{k,j}(x) \in \dot{H}^s_{k,j}(\mathbb{R}^d)$ as required. $\square$

11.3. Spherical decomposition of the FGF

We now study the spherical decomposition of the $\mathrm{FGF}_s(\mathbb{R}^d)$, which we denote by h^d. From the completeness and orthogonality of $\dot{H}^s_{k,j}(\mathbb{R}^d)$,

$$h^d = \sum_{k=0}^{\infty} \sum_{j=1}^{\dim H_k} h^d_{k,j}, \tag{11.6}$$

where the $h^d_{k,j}$ are independent standard Gaussians on the space of $\dot{H}^s_{k,j}(\mathbb{R}^d)$ (this follows from the same reasoning as in Section 5).

We note that $\dot{H}^s_{k,j}(\mathbb{R}^d)$ is unitarily isomorphic to the Hilbert space $\mathcal{R}^s_{d,k}$ consisting of radial functions $f_1, f_2 \in \dot{H}^s(\mathbb{R}^{d+2k})$ with inner product given in (11.5):

$$\langle f_1, f_2 \rangle_{\mathcal{R}^s_{d,k}} = \int_0^{\infty} r^{2s+2k+d-1} g_1(r) \overline{g_2(r)} \, dr,$$

where $g_i = \mathcal{F}_{d+2k,0}[f_i]$ for $i \in \{1, 2\}$. Thus, it follows that we can construct a standard Gaussian on $\mathcal{R}^s_{d,k}$, which we call $\widetilde{h}^d_{k,j}$ that corresponds to a standard Gaussian $h^d_{k,j}$ on $\dot{H}^s_{k,j}(\mathbb{R}^d)$.

The key observation is that the inner product on the Hilbert space $\mathcal{R}^s_{d,k}$ above only depends on $d + 2k$ (and s). This means that $\widetilde{h}^d_{k,j}$ has the same distribution as $\widetilde{h}^{d+2k}_{0,1}$ (equivalently, $\mathcal{R}^s_{d,k}$ is unitarily equivalent to $\mathcal{R}^s_{d+2k,0}$). Averaging both sides of (11.6) over $S^{d-1}_r := rS^{d-1}$, we have

$$h^d_{0,1} = \frac{1}{r^{d-1}\Omega_d} \int_{S^{d-1}_r} h^d(x) dx = \frac{1}{\Omega_d} \int_{S^d} h^d(r\theta) d\theta.$$

Note that we have used that $P_{0,1}(x) = \Omega_d^{-1/2}$, so that $\dot{H}^s_{0,1}(\mathbb{R}^d)$ is the set of radial functions $f \in \dot{H}^s(\mathbb{R}^d)$. This implies that $h^d_{0,1}$ averaged over a sphere is $h^d_{0,1}$. By the same observation, we have

$$\widetilde{h}^d_{0,1}(r) = \frac{1}{\sqrt{\Omega_d}} \int_{S^{d-1}} h^d(r\theta) d\theta,$$

a constant multiple of the spherical average of h^d. We collect these results in the following theorem.

Theorem 11.6. *In the decompostion of $h^d = \mathrm{FGF}_s(\mathbb{R}^d)$ in (11.6), the coefficient processes $\widetilde{h}^d_{k,j}$ with respect to the normalized harmonic polynomials $\{P_{k,j}\}$ are independent processes with the same distribution as*

$$r \mapsto \frac{1}{\sqrt{\Omega_{d+2k}}} \int_{S^{d+2k-1}} h^{d+2k}(r\theta) \, d\theta,$$

where h^{d+2k} is an $\mathrm{FGF}_s(\mathbb{R}^{d+2k})$.

Remark 11.7. We notice that since $\widetilde{h}^d_{k,j}$ is the average process of $\mathrm{FGF}_s(\mathbb{R}^{d+2k})$, it is defined modulo degree $\lfloor s - \frac{d}{2} - k \rfloor$ polynomials. Since $\widetilde{h}_{d,k,j}$ is the coefficient of $P_{k,j}$, which is a polynomial of degree k, this is consistent with the fact that h^d itself is defined up to polynomials of degree $\lfloor s - \frac{d}{2} \rfloor$.

Remark 11.8. From Theorem 11.6, one can analyze the average process in an arbitrary dimension by understanding the whole spherical decomposition of the FGF in dimensions 2 and 3 with the same index s. We remark that the distribution of the coefficient processes of $\mathrm{FGF}_{\frac{3}{2}}(\mathbb{R}^2)$ and $\mathrm{FGF}_2(\mathbb{R}^3)$ have been explicitly computed in [McK63]. Furthermore, [McK63] computes the coefficient processes for Lévy Brownian motion (FGF with Hurst parameter $H = 1/2$) in any dimension and gives the explicit covariance structure for $d \in \{2,3\}$. In principle, we can also represent the covariance kernel for other values of s with an integral involving a 2- or 3-dimensional harmonic polynomial and the covariance kernel of FGF. If $d = 2$, it involves trigonometric functions. If $d = 3$, it will further involve associated Legendre polynomials; see Chapter 14 of [Olv10].

When s is a positive integer, we have $(-\Delta_{\mathbb{R}^{d+2k}})^s f = (-L_{d,k})^s(f)$, where $L_{d,k}f = f'' + (d + 2k - 1)r^{-1}f'$. In this case, the inner product of $\mathcal{R}^s_{d,k}$ is given by $\int_0^\infty (-L_{d,k})^s(f)(r)g(r)\,dr$. Since this inner product is defined by a differential operator, $\widetilde{h}^d_{k,j}$ shares the same kind of Markov property as the FGF_s when s is an integer, which we described at the end of Section 5: given the values of $\widetilde{h}_{d,k,j}$ in the interval $[0, a]$, the conditional law of $\widetilde{h}_{d,k,j}$ on the interval (a, ∞) depends only on $\{\widetilde{h}_{d,k,j}(a), \widetilde{h}'_{d,k,j}(a), \cdots, \widetilde{h}^{(s-1)}_{d,k,j}(a)\}$.

12. The discrete fractional Gaussian field

12.1. Fractional gradient

Recall that if $f : \mathbb{R}^d \to \mathbb{R}$ is differentiable, then the gradient ∇f is a vector-valued function on $\mathbb{R}^d$ with the property that for all $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \nabla f(x) \cdot \nabla g(x)\,dx = \int_{\mathbb{R}^d} (-\Delta f(x))g(x)\,dx. \qquad (12.1)$$

For $0 < s < 1$, we will define the *fractional gradient* $\nabla^s f$ so that an analogue of (12.1) holds with the fractional Laplacian in place of the usual Laplacian. Rather than a vector-valued function, however, we define $\nabla^s f$ to be a *function*-valued function on $\mathbb{R}^d$. More precisely, if $f : \mathbb{R}^d \to \mathbb{R}$ is measurable, then we define

$$\nabla^s f(x) = \left(y \mapsto \frac{f(x+y) - f(x)}{|y|^{\frac{d}{2}+s}} \right), \qquad (12.2)$$

where the domain of the function on the right-hand side is $\mathbb{R}^d \setminus \{0\}$. We will establish the following analogue of the integration-by-parts formula (12.1) for the fractional gradient.

Proposition 12.1. *For all $d \geq 1$, $s \in (0,1)$, and $f, g \in \mathcal{S}(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} (\nabla^s f(x), \nabla^s g(x))_{L^2(\mathbb{R}^d)} \, dx = \int_{\mathbb{R}^d} ((-\Delta)^s f(x)) g(x) \, dx \qquad (12.3)$$

Note that we have replaced the gradient and Laplacian with their fractional counterparts, and we replaced the dot product with an $L^2(\mathbb{R}^d)$ inner product.

Proof. Since each side of (12.3) is a bilinear form in f and g, it suffices to show that the formula holds with $f = g$. We simplify the left-hand side of (12.3) to obtain

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(\nabla^s f(x))(y)|^2 \, dy \, dx$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x+y) - f(x)|^2}{|y|^{d+2s}} \, dy \, dx$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(x)^2 - 2f(x)f(x+y) + f(x+y)^2}{|y|^{d+2s}} \, dy \, dx$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{[2f(x)^2 - 2f(x)f(x+y)] + [f(x+y)^2 - f(x)]}{|y|^{d+2s}} \, dy \, dx.$$

Changing variables for $x + y$ in the second square-bracketed expression shows that the left-hand side of (12.3) is equal to

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) \frac{f(x+y) - 2f(x) + f(x-y)}{|y|^{d+2s}} \, dy \, dx,$$

which equals the right-hand side of (12.3) by Proposition 2.2. $\qquad\square$

12.2. The discrete fractional Gaussian field

In this section we define a sequence of discrete random distributions converging in law to the fractional Gaussian field $\mathrm{FGF}_s(D)$, where $s \in (0,1)$ and $D \subset \mathbb{R}^d$ is a sufficiently regular bounded domain. We follow the strategy of [Cap00] and prove convergence using a random walk representation of the field covariances. This method was introduced by Dynkin [Dyn80].

Suppose that $D \subset \mathbb{R}^d$ is a bounded domain and $s \in (0,1)$. For $\delta > 0$, define $V^\delta := \delta \mathbb{Z}^d \cap D$. Recall that the zero-boundary discrete Gaussian free field (DGFF) is defined to be the mean-zero Gaussian field with density at $f \in \mathbb{R}^{V^\delta}$ proportional to

$$\exp\left(-\frac{1}{2} \sum_{(x,y) \in (\delta\mathbb{Z}^d) \times (\delta\mathbb{Z}^d)} C_d \mathbf{1}_{|x-y|=\delta} |f(x) - f(y)|^2 \delta^d \right), \qquad (12.4)$$

where C_d is a constant and where we interpret the expression in parentheses as a quadratic form in the variables $\{f(x) : x \in \delta\mathbb{Z}^d \cap D\}$ by substituting zero

for each instance of the variable $f(x)$ for all $x \notin D$. Observing that the sum in (12.4) is a rescaled discretized version of the L^2 norm of the gradient of f, we define the zero-boundary discrete fractional Gaussian field $\mathrm{DFGF}_s(D)$ by replacing this expression with a rescaled discretized L^2 norm of the fractional gradient of f. More precisely, we let

$$C_{d,s} = \left(\int_{\mathbb{R}^d} (1 - \cos x_1) |x|^{-d-2s} \, dx \right)^{-1}, \quad \text{where } x = (x_1, \dots, x_d),$$

and define $h^\delta \sim \mathrm{DFGF}_s^\delta(D)$ to be a Gaussian function h^δ with density at $f \in \mathbb{R}^{V^\delta}$ proportional to

$$\exp\left(-\frac{1}{2} \sum_{(x,y) \in (\delta\mathbb{Z}^d)^2, \, x \neq y} C_{d,s} \frac{|f(x) - f(y)|^2}{|x - y|^{d+2s}} \delta^d \right),$$

where we interpret the expression in parentheses as a quadratic form in the variables $\{f(x) \,:\, x \in \delta\mathbb{Z}^d \cap D\}$ (as we did for the DGFF). Observe that this quadratic form includes long-range interactions, unlike the quadratic form for the GFF which includes only nearest-neighbor interactions. The constant $C_{d,s}$ is chosen so that the discrete FGF converges to the FGF with no further normalization–see (12.9) below to understand the role that this constant plays in the calculation.

We interpret h^δ as a linear functional on $C_c^\infty(D)$ by setting

$$(h^\delta, \phi) := \sum_{x \in V^\delta} h^\delta(x)\phi(x)\delta^d, \quad \text{for all } \phi \in C_c^\infty(D). \tag{12.5}$$

To motivate (12.5), we note that the right-hand side is approximately the same as the integral of an interpolation of h^δ against ϕ. The following theorem is a rigorous formulation of the idea that the DFGF converges to the FGF as $\delta \to 0$ when D is sufficiently regular. The idea of its proof is to compare a random walk describing the covariance structure of the DFGF to the $2s$-stable Lévy process describing FGF covariances. Recall that D is said to be $C^{1,1}$ if for every $z \in \partial D$, there exists $r > 0$ such that $B(z, r) \cap \partial D$ is the graph of a function whose first derivatives are Lipschitz [CS98].

Proposition 12.2. *Let $D \subset \mathbb{R}^d$ be a bounded $C^{1,1}$ domain, and let $s \in (0, 1)$. The discrete fractional Gaussian field $h^\delta \sim \mathrm{DFGF}_s(D)$ converges to the fractional Gaussian field $h \sim \mathrm{FGF}_s(D)$ in the sense that for any finite collection of test functions $\phi_1, \dots, \phi_n \in C_c^\infty(D)$, we have*

$$((h^\delta, \phi_1), \dots, (h^\delta, \phi_n)) \to ((h, \phi_1), \dots, (h, \phi_n)) \tag{12.6}$$

in distribution as $\delta \to 0$.

Proof. Because both sides of (12.6) are multivariate Gaussians and since h^δ and h are linear, it suffices to show that $\mathbb{E}[(h^\delta, \phi)^2] \to \mathbb{E}[(h, \phi)^2]$ for all $\phi \in C_c^\infty(D)$.

From (12.5) we calculate

$$\mathbb{E}\left[(h^\delta, \phi)^2\right] = \sum_{(x,y) \in V^\delta \times V^\delta} \mathbb{E}[h^\delta(x)h^\delta(y)]\phi(x)\phi(y)\, \delta^{2d}. \qquad (12.7)$$

Define an independent family of exponential clocks indexed by edges

$$\{(w, z) : w \in \delta\mathbb{Z}^d, z \in \delta\mathbb{Z}^d, \text{ and } w \neq z\}$$

such that the intensity of the clock corresponding to (w, z) is $C_{d,s}\delta^d|w-z|^{-d-2s}$. Define a continuous-time process $(X_t^\delta)_{t\geq 0}$ which starts at $x \in V^\delta$ and moves from its current vertex w to a new vertex $z \in \delta\mathbb{Z}^d$ whenever the clock associated with (w, z) rings. Then

$$\mathbb{E}[h^\delta(x)h^\delta(y)] = \mathbb{E}\left[\int_0^T \mathbf{1}_{\{X_t^\delta = y\}}\, dt\right], \qquad (12.8)$$

where T is the exit time from D [She07, Section 4.1].

We define a discrete-time version $(\widetilde{Y}_n^\delta)_{n\geq 0}$ of the process $(X_t^\delta)_{t\geq 0}$ which tracks the sequence of vertices visited by X^δ. That is, $\widetilde{Y}_n^\delta$ is the vertex at which X_t^δ is located after its nth jump. Let

$$\gamma_{d,s} := C_{d,s}^{-1} \sum_{z \in \mathbb{Z}^d \setminus \{0\}} |z|^{-d-2s}.$$

From $\widetilde{Y}^\delta$ we define the continuous-time process $(Y_t^\delta)_{t\geq 0}$ by $Y_t^\delta = \widetilde{Y}_{\lfloor \gamma_{d,s}^{-1}\delta^{-2s}t \rfloor}^\delta$. Since the minimum of a collection of exponential random variables with intensities $(\lambda_i)_{i\in I}$ is exponential random variable with intensity $\sum_{i\in I} \lambda_i$, (12.8) implies that

$$\begin{aligned}
\mathbb{E}[h^\delta(x)h^\delta(y)] &= \mathbb{E}^x[\#\{n : \widetilde{Y}_n^\delta = y\}] \left(\sum_{z \in \delta\mathbb{Z}^d} C_{d,s}\delta^d|y-z|^{-d-2s}\right)^{-1} \\
&= \mathbb{E}^x\left[\int_0^\infty \mathbf{1}_{\{Y_t^\delta = y\}}\, dt\right] \times \frac{\delta^{-2s}\gamma_{d,s}^{-1}C_{d,s}^{-1}\delta^{-d+d+2s}}{\sum_{z\in\mathbb{Z}^d\setminus\{0\}} |z|^{-d-2s}} \\
&= \mathbb{E}^x\left[\int_0^\infty \mathbf{1}_{\{Y_t^\delta = y\}}\, dt\right],
\end{aligned}$$

by our choice of $\gamma_{s,d}$ and $C_{s,d}$. If Z is a Markov process, we denote by $p_t(x, y)\, dy = p_t^Z(x, y)\, dy$ the density of the law of Z_t given $Z_0 = x$ (assuming that this law is absolutely continuous with respect to Lebesgue measure). Recall that the symmetric $2s$-stable process $(Y_{t\geq 0})$ is the Lévy process on $\mathbb{R}^d$ whose transition kernel density p_t has Fourier transform $\xi \mapsto \exp(-t|\xi|^{2s})$.

By calculating the characteristic function of the step distribution of $\widetilde{Y}^\delta$, (see Remark 5.1 in [Cap00] for details), we see that

$$Y_1^\delta \overset{\text{law}}{\to} Y_1 \qquad (12.9)$$

By Theorem 2.7 in [Sko57], this implies that $(Y_t^\delta)_{t \geq 0}$ converges in distribution to $(Y_t)_{t \geq 0}$ with respect to the Skorokhod J_1 metric [Sko56], which is defined as follows. For an interval $I \subset [0, \infty)$, we denote by $\mathcal{D}(I, \mathbb{R}^d)$ the set of functions from I to $\mathbb{R}^d$ which are right-continuous with left limits, and for $t > 0$ we denote by Λ_t the set of increasing homeomorphisms from $[0, t]$ to itself. For $f, g \in \mathcal{D}([0, t], \mathbb{R}^d)$, we define the metric $d_{J_1(t)}$ by

$$d_{J_1(t)}(f, g) = \inf_{\lambda \in \Lambda_t} \max \left(\|f \circ \lambda - g\|_\infty, \|\lambda - \mathrm{id}\|_\infty \right),$$

where $\mathrm{id}(s) := s$. Then we define the metric

$$d_{J_1}(f, g) = \int_0^\infty e^{-t} \min(1, d_{J_1(t)}(f, g)) \, dt$$

for $f, g \in \mathcal{D}([0, \infty), \mathbb{R}^d)$ [MZ13]. A different definition that is equivalent and is also called the J_1 metric is given in [Bil99], where it is also proved that $d_{J_1}(f_n, f) \to 0$ if and only if $d_{J_1(t)}(f_n|_{[0,t]}, f|_{[0,t]}) \to 0$ for every continuity point t of f.

Given a stochastic process X started in D, denote by T the exit time of the process from D. Denote by $\mu_{X,x}$ the occupation measure $\mu_{X,x}(A) := \mathbb{E}^x[\int_0^T \mathbf{1}_{X_t \in A} \, dt]$ for all Borel sets $A \subset D$. We have $T < \infty$ almost surely, and $\mu_{X,x}$ is a finite measure—see the proof of Lemma 12.3 where a stronger statement is proved. By Lemma 12.3, $\mu_{X_n,x} \to \mu_{X,x}$ weakly. Since weak convergence implies convergence of integrals against bounded continuous functions, we have

$$\sum_{y \in V^\delta} \mathbb{E}^x \left[\int_0^T \mathbf{1}_{\{Y_t^\delta = y\}} \, dt \right] \phi(y) \delta^d = \int_D \phi(y) \mu_{Y_t,x}(dy) + o(1), \qquad (12.10)$$

where the quantity denoted $o(1)$ is uniformly bounded as x varies over the support of ϕ and tends to 0 as $\delta \to 0$ for each fixed x. Substituting (12.10) into (12.7) and using the convergence of the Riemann integral (as well as dominated convergence to handle the $o(1)$ term), we obtain

$$\mathbb{E}\left[(h^\delta, \phi)^2\right] \to \int_{D \times D} \phi(x) \mu_{Y,x}(dy) \, dx = \int_{D \times D} G(x, y) \, dx \, dy \qquad (12.11)$$

as $\delta \to 0$, where G is the density of the occupation measure (that is, the Green's function) of Y. This Green's function is in turn equal to $G_D^s(x, y)$ (see (4.2)), the Green's function of the fractional Laplacian [CS98]. Therefore, the right-hand side of (12.11) is equal to $\mathbb{E}[(h, f)^2]$, as desired. $\qquad \square$

Lemma 12.3. *Let $(X_n)_{n \geq 1}$ be a sequence of processes in $\mathbb{R}^d$ converging in law with respect to the J_1 metric to a symmetric α-stable process X. Let $D \subset \mathbb{R}^d$ be a $C^{1,1}$ domain. If T is the hitting time of $\mathbb{R}^d \setminus D$, then the occupation measure of X_n^T converges weakly to the occupation measure of X^T.*

Proof. For $n \geq 1$, denote by μ_n the occupation measure of X_n^T:

$$\mu_n(A) := \mathbb{E}\left[\int_0^T \mathbf{1}_{X_n^T(t) \in A} \, dt\right],$$

Similarly, define μ to be the occupation measure of X^T.

Recall the following definition of the Lévy-Prohorov metric π on the set of finite measures on $\mathbb{R}^d$. For $A \subset \mathbb{R}^d$ a Borel set, denote by A^ϵ the ϵ-neighborhood of A, defined by

$$A^\epsilon := \{x \in \mathbb{R}^d \, : \, \exists \, y \in A \text{ such that } |x - y| < \epsilon\}.$$

Define for finite measures μ and ν

$$\pi(\mu, \nu) := \inf_{\epsilon > 0}\{\mu(A) \leq \nu(A^\epsilon) + \epsilon \text{ and } \nu(A) \leq \mu(A^\epsilon) + \epsilon \text{ for all } A \text{ Borel}\}.$$

Recall that for probability measures, convergence with respect to π is equivalent to weak convergence [Bil99]. Since weak convergence of a sequence of finite measures $(\mu_n)_{n \geq 1}$ to a nonzero measure μ is equivalent to weak convergence of the normalized measures $\mu_n / \mu_n(\mathbb{R}^d) \to \mu / \mu(\mathbb{R}^d)$ along with convergence of the total mass (that is, $\mu_n(\mathbb{R}^d) \to \mu(\mathbb{R}^d)$), we see that convergence with respect to π is equivalent to weak convergence for finite measures too. Therefore, it suffices to show that for all $\epsilon > 0$ and $A \subset \mathbb{R}^d$, we have $\mu_n(A) \leq \mu(A^\epsilon) + \epsilon$ and $\mu(A) \leq \mu_n(A^\epsilon) + \epsilon$. Since $\mu_n(\mathbb{R}^d \setminus D) = \mu(\mathbb{R}^d \setminus D) = 0$, it suffices to consider $A \subset D$. For $\eta > 0$, define $D_\eta = \{x \in D \, : \, \text{dist}(x, \partial D) > \eta\}$. For $\eta > 0$, define B_η to be the event that X stopped upon exiting D_η is contained in D^η. By integrating the upper bound in Theorem 1.5 in [CS98], we conclude that B_η has probability tending to 0 as $\eta \to 0$. Furthermore, for each positive integer n, the event E_n that $|X_{n+1} - X_n|$ is larger than the diameter of D has probability bounded below. Since the events $(E_n)_{n \geq 1}$ are independent, it follows the amount of time X spends in D has an exponential tail. Therefore, given $\epsilon > 0$ we may choose $\eta \in (0, \epsilon/2)$ such that

$$\mathbb{E}\left[\int_0^T \mathbf{1}_{\{X(t) \in A\}} \, dt \, \mathbf{1}_{B_\eta}\right] < \epsilon/2,$$

by the Cauchy-Schwarz inequality.

Since $(D[0, \infty), d_{J_1})$ is separable [Bil99, Theorem 16.3], we may use Skorokhod's representation theorem [Bil99, Theorem 6.7] to couple $(X_n)_{n \geq 1}$ and X in such a way that $d_{J_1}(X_n, X) \to 0$ as $n \to \infty$. Choosing n_0 large enough that $d_{J_1}(X_n, X) < \eta/2$ whenever $n \geq n_0$, we have for all $n \geq n_0$,

$$\mathbb{E}\left[\int_0^T \mathbf{1}_{\{X_n(t) \in A\}} \, dt\right] = \mathbb{E}\left[\int_0^T \mathbf{1}_{\{X_n(t) \in A\}} \, dt(\mathbf{1}_{B_\eta^c} + \mathbf{1}_{B_\eta})\right]$$
$$< \mathbb{E}\left[\left(\int_0^T \mathbf{1}_{\{X(t) \in A\}} \, dt\right) \mathbf{1}_{B_\eta^c}\right] + \frac{\epsilon}{2}.$$

By the definition of the J_1 metric, the first term is bounded above by

$$\mathbb{E}\left[\int_0^T \mathbf{1}_{\{X(t) \in A^\epsilon\}} \, dt + \epsilon/2\right],$$

which gives $\mu_n(A) \leq \mu(A^\epsilon) + \epsilon$. We conclude by applying the same argument with the roles of X_n and X reversed. $\square$

13. Open questions

In this section, we will ask some questions regarding the FGF. Section 13.1 presents several questions on level lines, and Section 13.2 contains other FGF questions.

13.1. Questions on level sets

1. In dimension 2, $\mathrm{FGF}_{1+\epsilon}$ is a function for all $\epsilon > 0$. Do the level sets of $\mathrm{FGF}_{1+\epsilon}$ converge to the level sets of the Gaussian free field, as defined in [SS10]? One may interpret the mode of convergence to be in probability, with the coupling of Proposition 6.3, or in law.
   The Hausdorff dimension of the level sets of $\mathrm{FGF}_{1+\epsilon}(\mathbb{R}^2)$ is $2 - \epsilon$ [Xia13], while the Hausdorff dimension of SLE_4 is $\frac{3}{2}$. Thus if the level sets of $\mathrm{FGF}_{1+\epsilon}$ do converge to the level sets of the Gaussian free field, then the Hausdorff dimension of these sets is not continuous in ϵ.
2. Let h be an instance of any FGF_s that is defined as a distribution, but not as a function. One can mollify h with a bump function supported on an ϵ-ball in order to obtain a smooth function. Under what circumstances do the level sets of these mollified functions converge to a continuum limit as $\epsilon \to 0$?
3. Instead of mollifying, one could instead try to project h onto some subspace of piecewise-polynomial functions, like the projection of the two-dimensional GFF in [SS10] onto the space of functions piecewise affine on the triangles of a triangular lattice with side length ϵ. It was shown in [SS10] that in the case of the two-dimensional GFF, the level sets of these approximations do converge to a continuum limit as $\epsilon \to 0$. Can anything similar be obtained for any other dimension or any other value of s?
4. In d dimensions, can one consider a $(d-1)$-tuple of independent FGFs (understood as a map from $\mathbb{R}^d$ to $\mathbb{R}^{d-1}$) and make sense of the scaling limit of the zero level set as a random curve? Can one understand any discrete analogs of this problem? For the fractal properties of this curve when the corresponding FGF is a fractional Brownian motion, we refer to [Xia13].

13.2. Other questions

1. Are there any non-trivial local set explorations for FGF fields that are not defined as functions, as in the Gaussian free field case ([MS12a, MS12b, MS, MS13])?
2. If we restrict an LGF in $\mathbb{R}^3$ to a curved 2D surface, and conformally map that curved surface to a flat surface, can we pull back the restricted LGF to the flat surface and obtain a distribution whose law is locally absolutely continuous with respect to that of an ordinary LGF restricted to the flat surface?

Notation

We fix the relation $H = s - d/2$ for the definitions of the following spaces. We refer the reader to the referenced page numbers for the spaces' topologies.

Space	Description	Page
$\mathcal{S}(\mathbb{R}^d)$	The Schwartz space of real-valued functions on $\mathbb{R}^d$ whose derivatives of all orders exist and decay faster than any polynomial at infinity.	9
$\mathcal{S}'(\mathbb{R}^d)$	The space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$. Elements of $\mathcal{S}'(\mathbb{R}^d)$ are called tempered distributions.	9
$\mathcal{S}_k(\mathbb{R}^d)$	For $k \in \{-1, 0, 1, \ldots\}$, denotes the space of Schwartz functions ϕ such that $(\partial^\alpha \widehat{\phi})(0) = 0$ for all multi-indices α such that $|\alpha| \le k$. Equivalently, $\mathcal{S}_k(\mathbb{R}^d)$ is the space of Schwartz functions ϕ such that $\int_{\mathbb{R}^d} x^\alpha \phi(x)\, dx = 0$ whenever $|\alpha| \le k$.	10
$\mathcal{S}_r(\mathbb{R}^d)$	For $r \in \mathbb{R}$, denotes $\mathcal{S}_{\max(-1, \lfloor r \rfloor)}(\mathbb{R}^d)$	10
$\mathcal{S}'_k(\mathbb{R}^d)$	For $k \in \{-1, 0, 1, \ldots\}$, denotes the space of continuous linear functionals on $\mathcal{S}_k(\mathbb{R}^d)$. Equivalently, $\mathcal{S}'_k(\mathbb{R}^d)$ may defined to be the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions modulo polynomials of degree less than or equal to k.	10
$\dot{H}^s(\mathbb{R}^d)$	The subspace of $\mathcal{S}'_H(\mathbb{R}^d)$ consisting of functions whose Fourier transform $\xi \mapsto \widehat{f}(\xi)$ is in $L^2(|\xi|^{2s}\, d\xi)$	10
$\mathcal{U}_s(\mathbb{R}^d)$	The space of all functions $\phi \in C^\infty(\mathbb{R}^d)$ such that $x \mapsto (1 + |x|^{d+2s})(\partial^\alpha f)(x)$ is bounded for all multi-indices α	12
$(-\Delta)^s \mathcal{S}_k(\mathbb{R}^d)$	For $k \in \{-1, 0, 1, 2, \ldots\}$ and $s > -\frac{1}{2}(d+k+1)$, this space is the range of the injective operator $(-\Delta)^s : \mathcal{S}_k(\mathbb{R}^d) \to \mathcal{U}_{s+(k+1)/2}$.	12
$T_s(\mathbb{R}^d)$	The closure of $\mathcal{S}_H(\mathbb{R}^d)$ in $\dot{H}^{-s}(\mathbb{R}^d)$. This space serves as a test function space for $\mathrm{FGF}_s(\mathbb{R}^d)$.	18
$C_c^\infty(D)$	The space of smooth functions supported on a compact subset of a domain $D \subset \mathbb{R}^d$.	21
$\dot{H}_0^s(D)$	The closure of $C_c^\infty(D)$ in $\dot{H}^s(\mathbb{R}^d)$.	21
$T_s(D)$	The closure of the space of restrictions to D of Schwartz functions under the metric $d(\phi, \psi) = \|\phi - \psi\|_{\dot{H}^{-s}(D)}$. This space serves as a test function space for $\mathrm{FGF}_s(\mathbb{R}^d)$.	22
$C^{k,\alpha}(D)$	For $k \in \{0, 1, 2, \ldots\}$, $\alpha \in (0, 1)$, and $D \subset \mathbb{R}^d$, denotes the space of functions f on $\mathbb{R}^d$ such that $\partial^\beta f$ is α-Hölder continuous for all multi-indices β such that $|\beta| \le k$.	27

Acknowledgements

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