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THE NON-STANDARD LQR PROBLEM FOR BOUNDARY CONTROL SYSTEMS[†]

Abstract.
An overview of recent results concerning the non-standard, finite horizon Linear Quadratic Regulator problem for a class of boundary control systems is provided.

1. Introduction

In the present paper we give an account of recent results concerning the regulator problem with non-coercive, quadratic cost functionals over a finite time interval, for a class of abstract linear systems in a Hilbert space X, of the form

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & 0 \le \tau < t < T \\ x(\tau) = x_0 \in X\,. \end{cases} \tag{1}$$

Here, A (free dynamics operator) is at least the generator of a strongly continuous semigroup on X, and B (input operator) is a linear operator subject to a suitable regularity assumption. The control function u is L^2 in time, with values in a Hilbert space U. Through the abstract assumptions on the operators A and B, a class of partial differential equations, with boundary/point control, is identified. We shall mostly focus our attention on systems which satisfy condition $(H2) = (8)$, see §1.2 below. It is known ([13]) that this condition amounts to a trace regularity property which is fulfilled by the solutions to a variety of hyperbolic (hyperbolic-like) partial differential equations.

With system (1), we associate the following cost functional

$$J_{\tau,T}(x_0; u) = \int_\tau^T F(x(t), u(t))\,\mathrm{d}t + \langle P_T x(T), x(T)\rangle\,, \tag{2}$$

where F is a continuous quadratic form on $X \times U$,

$$F(x,u) = \langle Qx, x\rangle + \langle Su, x\rangle + \langle x, Su\rangle + \langle Ru, u\rangle\,, \tag{3}$$

and $x(t) = x(t; \tau, x_0, u)$ is the solution to system (1) due to $u(\cdot) \in L^2(\tau, T; U)$. It is asked to provide conditions under which, for each $x_0 \in X$, a constant $c_{\tau,T}(x_0)$ exists such that

$$\inf_{u \in L^2(\tau,T;U)} J_{\tau,T}(x_0; u) \ge c_{\tau,T}(x_0)\,. \tag{4}$$

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The special, important case, where

$$S = 0, \quad Q,\ P_T \geq 0, \quad R \geq \gamma > 0\,, \tag{5}$$

is now referred to as the classical (or, *standard*) LQR problem. We can say that this problem is now pretty well understood even for boundary control systems: the corresponding Riccati operator yields the synthesis of the optimal control (see [13]).

Functionals which do not display property (5) arise in different fields of systems/control theory. To name a few, the study of dissipative systems ([25]), where typical cases are

$$F(x,u) = |u|^2 - |x|^2\,, \qquad F(x,u) = \langle x, u\rangle\,;$$

the analysis of second variations of nonlinear control problems; H_∞ theory. It is worth recalling that the theory of infinite horizon linear quadratic control developed in [17], including the case of *singular* functionals, with $R = 0$, has more recently provided new insight in the study of the standard LQR problem for special classes of boundary control systems, see [21, 12].

In conclusion, the characterization of property (4), in a more general framework than the one defined by (5), is the object of the *non-standard* Linear Quadratic Regulator (LQR) problem.

Most results of the theory of the non-standard LQR problem for finite dimensional systems have been extended to the boundary control setting. We shall see that in particular, necessary conditions or sufficient conditions in order that (4) is satisfied can be provided, in term of non-negativity of suitable functionals. Unlike the infinite time horizon case, a gap still remains between necessary (non-negativity) conditions and sufficient (non-negativity) conditions, even when system (1) is exactly controllable. We shall examine this issue more in detail in §3.

The infinite dimensional problem reveals however new distinctive features. It is well known that in the finite dimensional case, the condition $R \geq 0$ has long been recognized as necessary in order that (4) is fulfilled; this applies even to time-dependent systems, see [7]. This property extends to infinite dimensional systems, when $P_T = 0$ (see [14, 6]). In contrast, in [6] an example is provided where, in spite of the fact that R is negative definite, the cost functional is coercive in $L^2(0,T;U)$, so that (4) is obviously satisfied. Crucially in that example $P_T \neq 0$, while the dynamics is given by a first order hyperbolic equation in one dimension, with control acted on the boundary.

Finally, we note that over an infinite horizon, the non-negativity condition which is necessary (and sufficient, under controllability of system (1)) for boundedness from below of the cost, is in fact equivalent to a suitable frequency domain inequality, (15) in §2, whose validity can be easily checked. In contrast, when T is finite, there is a lack of a frequency domain interpretation of the conditions provided.

The plan of the paper is the following. In §1.1 we provide a brief outline of the literature concerning the non-standard, finite horizon LQR problem for infinite dimensional systems. In §1.2 we introduce the abstract assumptions which characterize the class of dynamics of interest. In §2 we derive necessary conditions in order that (4) is satisfied, whereas §3 contains the statement of sufficient conditions. Most results of §2 and §3 are extracted from [6].

1.1. Literature

In this section we would like to provide a broad outline of contributions to the non-standard, *finite* horizon LQR problem for infinite dimensional systems. For a review of the richest literature on the same problem over an *infinite* horizon, we refer to [20]. We just recall that most recent extensions to the boundary control setting are given in [11], [14, Ch. 9], [18, 22, 23, 24].

Application to stability of holomorphic semigroup systems with boundary input is obtained, e.g., in [4].

The LQR problem with non-coercive functionals over a finite time interval has been the object of research starting around the 1970s. The most noticeable contribution to the study of this problem has been given, in our opinion, in [19]. For a comprehensive account of the theory developed in a finite dimensional context, and an extensive list of references, we refer to the monography [7].

The first paper which deals with the non-standard LQR problem over a finite time interval in infinite dimensions is, to our knowledge, [27]. The author considers dynamics of the form (1), which model distributed systems, with distributed control. Partial results are provided in order to characterize (4), without constraints on the form (3), except for $S = 0$. Moreover, the issue of the existence (and uniqueness) of an optimal control is considered, under the additional assumption that R is coercive.

A paper which deals with minimization of possible non-convex and non-coercive functionals, in a context which is more general than ours, is [1]. Necessary conditions or sufficient conditions for the existence of minimizers are stated therein, which involve a suitable 'recession functional' associated with the original functional.

In [9], the analysis is again restricted to cost functionals for which $R = I$, $S = 0$ (Q, P_T are allowed indefinite). Since R is coercive, the issue of the existence of solutions to the Riccati equation associated with the control problem is investigated. A new feature of the non-standard problem is pointed out, that the existence of an optimal control is not equivalent to the existence of a solution to the Riccati equation on $[0, T]$.

The study of the LQR problem with general cost functionals, still in the case of distributed systems with distributed control, has been carried out in [5]. Extensions of most finite dimensional results of [19] are provided. The application of the Bellman optimality principle to the infimization problem leads to introduce a crucial integral operator inequality, the so called 'Dissipation Inequality',

$$\langle P(a)x(a), x(a)\rangle \le \langle P(b)x(b), x(b)\rangle + \int_a^b F(x(s), u(s))\,\mathrm{d}s\,, \quad \tau \le a < b \le T\,,$$

whose solvability is equivalent to (4). Moreover, in [5] the regularity properties of the value function

$$V(\tau; x_0) = \inf_{u\in L^2(\tau,T;U)} J_{\tau,T}(x_0; u) \tag{6}$$

of the infimization problem are investigated, and new results are provided in this direction. In particular, it is showed that – unlike the standard case – the function $\tau \to V(\tau; x_0)$ is in general only upper semicontinuous on $[0, T]$, and that lack of continuity in the interior of $[0, T]$ may occurr, for instance, in the case of delay systems.

We remark that in all the aforementioned papers [27, 9, 5], among necessary conditions for finiteness of (6), a basic non-negativity condition is provided, namely (13) below, which in turn implies $R \ge 0$. On the other hand, sufficient conditions are so far given in a form which requires coercivity of the operator R.

Finally, more recently, extensions to the boundary control setting have been provided for a class of holomorphic semigroup systems ([14, Ch. 9], [26]), and for a class of 'hyperbolic-like' dynamics ([6]), respectively. We note that in [14] and [26] a greater emphasis is still placed on the *non-singular* case, since R is assumed coercive.

1.2. Notations, basic assumptions and abstract classes of dynamics

As explained in the introduction, we consider systems of the form (1) in abstract spaces of infinite dimension. A familiarity with the representation of controlled infinite dimensional systems is assumed, compatible with, e.g., [2].

Most notation used in the paper is standard. We just point out that inner products in any Hilbert space are denoted by $\langle\cdot,\cdot\rangle$; norms and operator norms are denoted by the symbols $|\cdot|$ and $\|\cdot\|$, respectively. The linear space of linear, bounded operators from X to Y is denoted by $\mathcal{L}(X,Y)$ ($\mathcal{L}(X)$, if $X=Y$).

Throughout the paper we shall make the following standing assumptions on the state equation (1) and the cost functional (2):

(i) *$A : D(A) \subset X \to X$ is the generator of a strongly continuous (s.c.) semigroup e^{At} on X, $t > 0$;*

(ii) *$B \in \mathcal{L}(U, (D(A^*))')$; equivalently,*

$$A^{-\gamma}B \in \mathcal{L}(U,X) \text{ for some constant } \gamma \in [0,1]\,. \tag{7}$$

(iii) *$Q, P_T \in \mathcal{L}(X)$, $S \in \mathcal{L}(U,X)$, $R \in \mathcal{L}(U)$; Q, P_T, R are selfadjoint.*

REMARK 1. Assumptions (i)-(ii) identify dynamics which model distributed systems with distributed/boundary/point control. More specifically, the case of distributed control leads to a bounded input operator B, namely $\gamma = 0$ in (ii), whereas $\gamma > 0$ refers to the more challenging case of boundary/point control.

In order to characterize two main classes of partial differential equations problems of interest, roughly the 'parabolic' class and the 'hyperbolic' class, we follow [13] and introduce two distinct abstract conditions:

($H1$) *the s.c. semigroup e^{At} is analytic on X, $t > 0$, and the constant γ appearing in (7) is strictly < 1;*

($H2$) *there exists a positive constant k_T such that*

$$\int_0^T |B^* e^{A^*t}x|^2\,dt \le k_T|x|^2 \qquad \forall x \in D(A^*)\,. \tag{8}$$

It is well known that under either ($H1$) or ($H2$), the (input-solution) operator

$$L_\tau : u \to (L_\tau u)(t) := \int_\tau^t e^{A(t-s)}Bu(s)\,rmds\,, \tag{9}$$

is continuous from $L^2(\tau,T;U)$ to $L^2(\tau,T;X)$. Consequently, system (1) admits a unique *mild* solution on (τ,T) given by

$$x(t) = e^{A(t-\tau)}x_0 + (L_\tau u)(t)\,, \tag{10}$$

which is (at least) L^2 in time. For a detailed analysis of examples of partial differential equations with boundary/point control which fall into either class, we refer to [13].

Let us recall that ($H2$) is in fact equivalent to ([8])

$$L_\tau \text{ continuous } : L^2(\tau,T;U) \to C(\tau,T;X)\,, \tag{11}$$

and that the following estimate holds true, for a positive constant $C_{\tau,T}$ and for any $u(\cdot)$ in $L^2(\tau, T; U)$:

$$|(L_\tau u)(t)| \leq C_{\tau,T}\, |u|_{L^2(\tau,T;U)} \quad \forall t \in [\tau, T] \tag{12}$$

Therefore, for any initial datum $x_0 \in X$, the unique mild solution $x(\cdot; \tau, x_0, u)$ to equation (1), given by (10), is continuous on $[\tau, T]$, in particular at $t = T$. Thus, the term $\langle x(T), P_T\, x(T)\rangle$ makes sense for every control $u(\cdot) \in L^2(\tau, T; U)$.

REMARK 2. We note that $(H2)$, hence (11), follows as well from $(H1)$, when $\gamma \in [0, 1/2[$. Instead, when $(H1)$ holds with $\gamma \in [1/2, 1[$, counterexamples can be given to continuity of solutions at $t = T$, see [15, p. 202]. In that case, unless smoothing properties of P_T are required, the class of admissible controls need to be restricted. Comprehensive surveys of the theory of the standard LQR problem for systems subject to $(H1)$ are provided in [13] and [3]. Partial results for the corresponding non-standard regulator can be found in [14, Ch. 9].

In the present paper we shall mainly consider systems of the form (1) which satisfy assumption $(H2)$. This model covers many partial differential equations with boundary/point control, including, e.g., second order hyperbolic equations, Euler–Bernoulli and Kirchoff equations, the Schrödinger equation (see [13]).

2. Necessary conditions

In this section we are concerned with necessary conditions in order that (4) is satisfied, with special regard to the role of condition $R \geq 0$.

We begin with the statement of two basic necessary conditions, in the case of distributed systems with *distributed* control. For the sake of completeness, an outline of the proof is given; we refer to [5] for details. Condition (13) below is often referred to as the *non-negativity condition*.

THEOREM 1. *Assume that $B \in \mathcal{L}(U, X)$ (equivalently, $(H2)$ holds, with $\gamma = 0$). If there exist a $0 \leq \tau < T$ and an $x_0 \in X$ such that (4) is satisfied, then*

$$J_{\tau,T}(0; u) \geq 0 \qquad \forall u \in L^2(\tau, T; U)\,, \tag{13}$$

which in turn implies

$$R \geq 0\,. \tag{14}$$

Sketch of the proof. For simplicity of exposition we assume that (4) is satisfied, with $\tau = 0$. In order to show that this implies (13), one first derives a representation of the cost $J_{0,T}(x_0; u)$ as a quadratic functional on $L^2(0, T; U)$, when x_0 is fixed, namely

$$J_{0,T}(x_0; u) = \langle \mathcal{M}x_0, x_0\rangle_X + 2\, Re\, \langle \mathcal{N}x_0, u\rangle_{L^2(0,T;U)} + \langle \mathcal{R}u, u\rangle_{L^2(0,T;U)}\,,$$

with $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{R}$ suitable bounded operators. Readily $\langle \mathcal{R}u, u\rangle = J_{0,T}(0; u)$, and condition (13) follows from general results pertaining to infimization of quadratic functionals (see [5]).

Next, we use the actual expression of the operator $\mathcal{R}$ and the regularity of the input-solution operator L_0 defined by (9). Boundedness of the input operator B has here a crucial role. Proceeding by contradiction, (14) follows as a consequence of (13).

□

REMARK 3. A counterpart of Theorem 1 can be stated in infinite horizon, namely when $T = +\infty$ in (2) (set $P_T = 0$). In this case, if the semigroup e^{At} is not exponentially stable, the cost is not necessarily finite for an arbitrary control $u(\cdot) \in L^2(0, \infty; U)$. Consequently, the class of admissible controls need to be restricted. However, under stabilizability of the system (1), a non-negativity condition and (14) follow as well from (4). Even more, as remarked in the introduction, the non-negativity condition has a frequency domain counterpart ([17]), which in the stable case reads as

$$\begin{aligned}\Pi(i\omega) :=& B^*(-i\omega I - A^*)^{-1} Q (i\omega I - A)^{-1} B + S^*(i\omega I - A)^{-1} B \\ &+ B^*(-i\omega I - A^*)^{-1} S + R \geq 0 \quad \forall \omega \in \mathbb{R}\,.\end{aligned} \tag{15}$$

Theorem 1 can be extended to boundary control systems only in part.

THEOREM 2 ([6]). *Assume* $(H2)$. *Then the following statements hold true:*

(i) *if there exists an* $x_0 \in X$ *such that (4) is satisfied, then (13) holds;*

(ii) *if* $P_T = 0$, *then (13) implies (14); hence (14) is a necessary condition in order that (4) is satisfied.*

(iii) *if* $P_T \neq 0$, *then (14) is not necessary in order that (13) is satisfied.*

Sketch of the proof. Item (i) can be shown by using essentially the same arguments as in the proof of Theorem 1, which still apply to the present case, due to assumption $(H2)$. Similarly, when $P_T = 0$, (ii) follows as well.

The following example ([6, Ex. 4.4]) illustrates the third item. Let us consider, for a fixed $T \in (0, 1)$ and $\epsilon > 0$, the cost functional

$$J_{0,T}(x_0(\cdot); u) = \int_0^T \left\{ \int_T^1 |x(t,\xi)|^2 \,\mathrm{d}\xi - \epsilon |u(t)|^2 \right\} \mathrm{d}t + \int_0^T |x(T,\xi)|^2 \,\mathrm{d}\xi\,,$$

where $x(t, \xi)$ solves the boundary value problem

$$\begin{cases} x_t(t,\xi) = -x_\xi(t,\xi) & \\ x(0,\xi) = x_0(\xi) & 0 < \xi < 1 \\ x(t,0) = u(t) & 0 < t < T\,. \end{cases} \tag{16}$$

Note that here $R = -\epsilon I$, $P_T = I$.

The solution to (16), corresponding to $x_0 \equiv 0$, is given by

$$x(t,\xi) = \begin{cases} 0 & t < \xi \\ u(t-\xi) & t > \xi\,, \end{cases} \tag{17}$$

so that

$$\begin{aligned} J_{0,T}(x_0 \equiv 0; u) &= -\epsilon \int_0^T |u(t)|^2 \,\mathrm{d}t + \int_0^T |u(T-\xi)|^2 \,\mathrm{d}\xi \\ &= (1-\epsilon) \int_0^T |u(t)|^2 \,\mathrm{d}t\,. \end{aligned}$$

Therefore, if $0 < \epsilon < 1$, $J_{0,T}(0; u)$ is not only positive but even coercive in L^2, which implies (4). Nevertheless, $R < 0$.

□

A better result can be provided in the case of holomorphic semigroup systems, by using the smoothing properties of the operator L_τ. Somehow the ‘analytic case’ parallels the case when the input operator is bounded. See [14, Ch. 9, Theorem 3.1] for the proof.

THEOREM 3. *Assume that $(H1)$ holds, with $\gamma < 1/2$. Then (14) is a necessary condition in order that (4) is satisfied, even when $P_T \neq 0$.*

3. Sufficient conditions

In this section we provide sufficient conditions in order that (4) is satisfied. Let us go back to the non-negativity conditions of Theorem 1. It can be easily shown that neither (14), nor (13), are, by themselves, sufficient to guarantee that the cost functional is bounded from below.

EXAMPLE 1. Let $X = U = \mathbb{R}$, and set $\tau = 0$, $A = -1$, $B = 0$ in (1); moreover, let $F(x, u) = xu$. Note that here $R = 0$. For any x_0, the solution to (1) is given by $x(t) = x_0 e^{-t}$, so that

$$J_{0,T}(x_0; u) = x_0 \int_0^T e^{-t} u(t)\, \mathrm{d}t$$

for any admissible control u. Therefore $J_{0,T}(0; u) \equiv 0$, and (13) holds true, whereas it is readily verified that when $x_0 \neq 0$, $\inf_u J_{0,T}(x_0; u) = -\infty$ (if $x_0 > 0$ take, for instance, the sequence $u_k(t) = -k$ on $[0, T]$).

If $T = +\infty$, the same example shows that Theorem 1 cannot be reversed without further assumptions. However it turns out that, over an infinite horizon, the necessary non-negativity condition (13) is also sufficient in order that (4) is satisfied, if system (1) is completely controllable. This property is well known in the finite dimensional case, since the early work [10].

Recently, the aforementioned result has been extended to boundary control systems, under the following assumptions:

(i') *$A : D(A) \subset X \to X$ is the generator of a s.c. group e^{At} on X, $t \in \mathbb{R}$;*

$(H2')$ *there exists a $T > 0$ and a constant $k_T > 0$ such that*

(18) $$\int_0^T |B^* e^{A^* t} x|^2\, \mathrm{d}t \le k_T |x|^2 \qquad \forall x \in D(A^*)\,;$$

$(H3')$ *system (1) is completely controllable, namely for each pair $x_0, x_1 \in X$ there is a T and an admissibile control $v(\cdot)$ such that $x(T; 0, x_0, v) = x_1$.*

For simplicity of exposition, we state the theorem below under the additional condition that e^{At} is exponentially stable.

THEOREM 4 ([22]). *Assume (i')–$(H2')$–$(H3')$. If*

$$J_\infty(0; u) \ge 0 \qquad \forall u \in L^2(0, \infty; U)\,,$$

then for each $x_0 \in X$ there exists a constant $c_\infty(x_0)$ such that

$$\inf_{L^2(0,\infty;U)} J_\infty(x_0; u) \ge c_\infty(x_0) \qquad \forall u \in L^2(0, \infty; U)\,.$$

We now return to the finite time interval $[0, T]$ and introduce the assumption that the system is exactly controllable on a certain interval $[0, r]$ (see, e.g., [3]):

$(H3)$ *there is an* $r > 0$ *such that, for each pair* $x_0, x_1 \in X$, *there exists an admissibile control* $v(\cdot) \in L^2(0, r; U)$ *yielding* $x(r; 0, x_0, v) = x_1$. *Equivalently,*

$$\exists\, r,\ c_r > 0 : \quad \int_0^r |B^* e^{A^* t} x|^2 \, dt \geq c_r |x|^2 \qquad \forall x \in D(A^*)\,. \tag{19}$$

On the basis of Theorem 4, one would be tempted to formulate the following claim.

CLAIM 5. Assume (i')–$(H2)$–$(H3)$. If

$$J_{0,T}(0; u) \geq 0 \qquad \forall u \in L^2(0, T; U)\,,$$

then (4) is satisfied for $0 \leq \tau \leq T$.

It turns out that this claim is false, as it can been shown by means of examples: see [7, 6]. A correct counterpart of Theorem 4 over a finite time interval has been given in [6].

THEOREM 6 ([6]). *Assume* (i')–$(H2)$–$(H3)$. *If*

$$J_{0,T+r}(0; u) \geq 0 \qquad \forall u \in L^2(0, T + r; U)\,, \tag{20}$$

then (4) is satisfied for $0 \leq \tau \leq T$.

We point out that in fact a proof of Theorem 6 can be provided which does not make *explicit* use of assumption $(i)'$, see Theorem 7 below. Let us recall however that, when the input operator B is bounded, controllability of the pair (A, B) on $[0, r]$, namely assumption $(H3)$ above, implies that the semigroup e^{At} is *right* invertible, [16]. Therefore, the actual need of some kind of 'group property' in Theorem 6 is an issue which is left for further investigation.

THEOREM 7. *Assume* $(H2)$–$(H3)$. *If (20) holds, then (4) is satisfied for* $0 \leq \tau \leq T$.

Proof. Let $x_1 \in X$ be given. By $(H3)$ there exists a control $v(\cdot) \in L^2(0, r; U)$ steering the solution of (1) from $x_0 = 0$ to x_1 in time r, namely $x(r; 0, 0, v) = x_1$. Obviously, v depends on x_1: more precisely, it can be shown that, as a consequence of assumptions $(H2)$ and $(H3)$, a constant K exists such that

$$|v|_{L^2(0,r;U)} \leq K|x_1|\,,$$

see [6]. For arbitrary $u \in L^2(r, T + r; U)$, set now

$$u_v(t) = \begin{cases} v(t) & 0 \leq t \leq r \\ u(t) & r < t \leq T + r\,. \end{cases}$$

Readily $u_v(\cdot) \in L^2(0, T + r; U)$, and $J_{0,T+r}(0; u_v) \geq 0$ due to (20). On the other hand,

$$J_{0,T+r}(0; u_v) = \int_0^r F(x(s; 0, 0, v), v(s))\, ds + J_{r,T+r}(x_1; u)\,,$$

where the first summand is a constant which depends only on x_1. A straightforward computation shows that the second summand equals $J_{0,T}(x_1; u_r)$, with $u_r(t) = u(t+r)$ an arbitrary admissible control on $[0, T]$. In conclusion,

$$J_{0,T}(x_1; u_r) \geq -\int_0^r F(x(s; 0, 0, v), v(s))\, \mathrm{d}s =: c(x_1)\,,$$

and (4) holds for $\tau = 0$. The case $\tau > 0$ can be treated by using similar arguments.

□

REMARK 4. In conclusion, we have provided the sufficiency counterpart of item (i) of Theorem 2, under the additional condition that system (1) is exactly controllable in time $r > 0$. Apparently, in order that (4) is satisfied, the non-negativity condition need to be required on a larger interval than $[0, T]$, precisely on an interval of lenght $T + r$. This produces a gap between necessary conditions and sufficient conditions, which was already pointed out in finite dimensions ([7]).

Finally, we stress that the exact controllability assumption cannot be weakened to null controllability, as pointed out in [6, Ex. 4.5].

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