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ON MEASURE DIFFERENTIAL INCLUSIONS IN OPTIMAL CONTROL THEORY

1. Introduction

Differential inclusions are a fundamental tool in optimal control theory. In fact an optimal control problem

$$\min_{(x,u)\in\Omega} J[x,u]$$

can be reduced (via a deparameterization process) to a problem of Calculus of Variation whose solutions can be deduced by suitable closure theorems for differential inclusions.

More precisely, if the cost functional is of the type

$$J[x,u]=\int_I f_0(t,x(t),u(t))\,d\lambda \tag{1}$$

and Ω is a class of admissible pairs subjected to differential and state constraints

$$(t,x(t))\in A \qquad x'(t)=f(t,x(t),u(t)), \qquad u(t)\in U(t,x(t)) \qquad t\in I \tag{2}$$

the corresponding differential inclusion is

$$(t,x(t))\in A \qquad x'(t)\in \tilde{Q}(t,x(t)) \qquad t\in I \tag{3}$$

where multifunction $\tilde{Q}$ is related to the epigraph of the integrand i.e.

$$\tilde{Q}(t,x)=\{(z,v): z\geq f_0(t,x,u),\ u=f(t,x,v),\ v\in U(t,x)\}\,.$$

We refer to Cesari's book [8] where the theory is developed in Sobolev spaces widely.

The extension of this theory to BV setting, motivated by the applications to variational models for plasticity [2, 3, 6, 13], allowed the authors to prove new existence results of discontinuous optimal solutions [4, 5, 9, 10, 11, 12].

This generalized formulation involved differential inclusions of the type

$$(t,x(t))\in A \qquad x'(t)\in \tilde{Q}(t,x(t)) \qquad \text{a.e. in } I \tag{3*}$$

where u' represents the "essential gradient" of the BV function x, i.e. the density of the absolutely continuous part of the distributional derivative with respect to Lebesgue measure; moreover the Lagrangian functional (1) is replaced by the Serrin-type relaxed functional

$$J[x,u]=\inf_{(x_k,u_k)\to(x,u)}\ \liminf_{k\to\infty} I[x_k,u_k]\,. \tag{1*}$$

A further extension of this theory was given in [4] where we discussed the existence of L^1 solutions for the abstract evolution equation

$$(3^{**}) \qquad (t, u(t)) \in A \qquad v(t) \in Q(t, u(t)) \qquad \text{a.e. in } I$$

where u and v are two surfaces not necessarely connected. This generalization allowed us to deal with a more general class of optimization problems in BV setting, also including differential elements of higher order or non linear operators (see [4] for the details).

Note that the cost functional J takes into account of the whole distributional gradient of the BV function u, while the constraints control only the "essential" derivative.

To avoid this inconsistency a new class of inclusions involving the measure distributional derivative should be taken into consideration. This is the aim of the research we developed in the present note.

At our knowledge, the first differential inclusion involving the distributional derivative of a BV function was taken into consideration by M. Monteiro Marques [18, 19] who discussed the existence of right continuous and BV solutions for the inclusion

$$(4) \qquad u(t) \in C(t) \qquad -\frac{du}{|du|}(t) \in N_{C(t)}(u(t)) \qquad |du|\text{–a.e. in } I$$

where $C(t)$ is a closed convex set and $N_{C(t)}(a)$ is the normal cone at $C(t)$ in the point $a \in C(t)$.

These inclusions model the so called sweepping process introduced by J.J. Moreau to deal with some mechanical problems.

In [21, 22] J.J. Moreau generalized this formulation to describe general rigid body mechanics with Coulomb friction and introduced the so called measure differential inclusions

$$(4^{*}) \qquad \frac{d\mu}{d\lambda}(t) \in K(t) \qquad \lambda_\mu\text{–a.e. in } I$$

where $\lambda_\mu = \lambda + |\mu|$, with λ is the Lebesgue measure and μ is a Borel measure, and where $K(t)$ is a cone.

Both the inclusions (4) and (4*) are not suitable for our purpose since they can not be applied to multifunction $\tilde{Q}(t, u) = \text{epi}\, F(t, u, \cdot)$ whose values are not cones, in general.

Recently S.E. Stewart [23] extended this theory to the case of a closed convex set $K(t)$, not necessarely a cone. Inspired by Stewart's research we consider here the following measure differential inclusion

$$(4^{**}) \qquad \begin{aligned} \frac{d\mu_a}{d\lambda}(t) &\in Q(t, u(t)) && \lambda\text{–a.e. in } I \\ \frac{d\mu_s}{d|\mu_s|}(t) &\in [Q(t, u(t))]_\infty && \mu_s\text{–a.e. in } I \end{aligned}$$

where $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of the Borel measure μ and $[Q(t, a)]_\infty$ is the asymptotic cone of the non empty, closed, convex set $Q(t, a)$.

Note that measure μ and BV function u are not necessarely correlated, analogously to inclusion (3**). In particular, if μ coincides with the distributional derivative of u, i.e. $\frac{d\mu_a}{d\lambda} = u'$, the first inclusion is exactly (3*), while the second one involves the singular part of the measure derivative.

In other words formulation (4^{**}) is the generalization of (3^*) in the spirit of (3^{**}).

The closure theorem we prove here for inclusion (4^{**}) represents a natural extension of that given in [9, 10, 4, 5] for evolution equations of types (3^*) and (3^{**}). In particular we adopt the same assumption on multifunction Q, which fits very well for the applications to $\tilde{Q}$ and hence to optimal control problems.

Moreover, we wish to remark that our results improve those given by Stewart under stronger assumptions on multifunction Q (see Section 6).

2. Preliminaries

We list here the main notations and some preliminary results.

2.1. On asymptotic cone

DEFINITION 1. *The* asymptotic cone *of a convex set $C \subset \mathbb{R}^n$ is given by*

$$[C]_\infty = \{\lim_{k\to\infty} a_k x_k : a_k \searrow 0,\ x_k \in C,\ k \in \mathbb{N}\}\,.$$

A discussion of the properties of the asymptotic cone can be found in [16] and [23]. We recall here only the results that will be useful in the following.

P_1. *If C is non empty, closed and convex, then $[C]_\infty$ is a closed convex cone.*

P_2. *If C is a closed convex cone, then $C = [C]_\infty$.*

P_3. *If C is non empty, closed and convex, then $[C]_\infty$ is the largest cone K such that $x + K \subset C$, with $x \in C$.*

Let $(C_j)_{j\in J}$ be a family of nonempty closed convex values. Then the following results hold.

P_4. $\displaystyle \mathrm{cl\ co} \bigcup_{j\in J} [C_j]_\infty \subset \left[\mathrm{cl\ co} \bigcup_{j\in J} C_j\right]_\infty$

P_5. *if* $\displaystyle \bigcap_{j\in J} C_j \neq \phi$, *then* $\displaystyle \left[\bigcap_{j\in J} C_j\right]_\infty = \bigcap_{j\in J} [C_j]_\infty$.

2.2. On property (Q)

Let E be a given subset of a Banach space and let $Q : E \to \mathbb{R}^m$ be a given multifunction. Fixed a point $t_0 \in E$, and a number $h > 0$, we denote by $B_h = B(t_0, h) = \{t \in E : |t - t_0| \leq h\}$.

DEFINITION 2. *Multifunction Q is said to satisfy Kuratowski* property (K) *at a point $t_0 \in E$, provided*

$$Q(t_0) = \bigcap_{h>0} \mathrm{cl} \bigcup_{t\in B_h} Q(t)\,. \tag{K}$$

The graph of multifunction Q is the set graph $Q := \{(t, v) : v \in Q(t),\ t \in E\}$.

It is well known that (see e.g. [8])

P_6. graph Q *is closed in* $E \times \mathbb{R}^m \Longleftrightarrow Q$ *satisfies condition* (K) *at every point.*

Cesari [8] introduced the following strengthening of Kuratowski condition which is suitable for the differential inclusions involved in optimal control problems in BV setting.

DEFINITION 3. *Multifunction* Q *is said to satisfy Cesari's* property (Q) *at a point* $t_0 \in E$, *provided*

$$Q(t_0) = \bigcap_{h>0} \text{cl co} \bigcup_{t \in B_h} Q(t)\,. \tag{Q}$$

Note that if (Q) holds, then the set $Q(t_0)$ *is necessarily closed and convex.*

We will denote by $\mathcal{C}(\mathbb{R}^m)$ the class of non empty, closed, convex subsets of $\mathbb{R}^m$.

Property (Q) is an intermediate condition between Kuratowski condition (K) and upper semicontinuity [8] which is suitable for the applications to optimal control theory. In fact the multifunction defined by

$$\widetilde{Q}(x, u) = \text{epi}\, F(x, u, \cdot)$$

satisfies the following results (see [8]).

P_7. $\widetilde{Q}$ *has closed and convex values iff* $F(x, u, \cdot)$ *is lower semicontinuous and convex.*

P_8. $\widetilde{Q}$ *satisfies property* (Q) *iff* F *is seminormal.*

We wish to recall that seminormality is a classical Tonelli's assumption in problems of calculus of variations (see e.g. [8] for more details).

Given a multifunction $Q : E \to \mathcal{C}(\mathbb{R}^m)$, we denote by $Q_\infty : E \to \mathcal{C}(\mathbb{R}^m)$ the multifunction defined by

$$Q_\infty(t) = [Q(t)]_\infty \qquad t \in E\,.$$

PROPOSITION 1. *If* Q *satisfies property* (Q) *at a point* t_0, *then also multifunction* Q_∞ *does.*

Proof. Since

$$\phi \neq Q(t_0) = \bigcap_{h>0} \text{cl co} \bigcup_{t \in B_h} Q(t)$$

from P_4 and P_5 we deduce that

$$Q_\infty(t_0) = \bigcap_{h>0} \left[\text{cl co} \bigcup_{t \in B_h} Q(t) \right]_\infty \subset \bigcap_{h>0} \text{cl co} \bigcup_{t \in B_h} Q_\infty(t)\,.$$

The converse inclusion is trivial and the assertion follows.

□

3. On measure differential inclusions, weak and strong formulations

Let $Q : I \to \mathbb{R}^n$, with $I \subset \mathbb{R}$ closed interval, be a given multifunction with nonempty closed convex values and let μ be a Borel measure on I, of bounded variation.

In [23] Stewart considered the two formulations of measure differential inclusions.

Strong formulation.

$$\text{(S)} \qquad \begin{cases} \frac{d\mu_a}{d\lambda}(t) \in Q(t) & \lambda\text{–a.e. in } I \\ \frac{d\mu_s}{d|\mu_s|}(t) \in Q_\infty(t) & \mu_s\text{–a.e. in } I \end{cases}$$

where $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of measure μ.

Weak formulation.

$$\text{(W)} \qquad \frac{\int_I \phi\, d\mu}{\int_I \phi\, d\lambda} \in \mathrm{cl\,co} \bigcup_{t \in I \cap \mathrm{Supp}\,\phi} Q(t)$$

for every $\phi \in \mathcal{C}_0$, where $\mathcal{C}_0$ denotes the set of all continuous functions $\phi : \mathbb{R} \to \mathbb{R}_0^+$, with compact support, such that $\int_I \phi\, d\lambda \neq 0$.

Stewart proved that the two formulations are equivalent, under suitable assumptions on Q (see Theorem 2), by means of a transfinite induction process.

We provide here a direct proof of the equivalence, under weaker assumption.

Moreover, for our convenience, we introduce also the following local version of weak formulation.

Local-weak formulation.
Let $t_0 \in I$ be fixed. There exists $\overline{h} = \overline{h}(t_0) > 0$ such that for every $0 < h < \overline{h}$,

$$\text{(LW)} \qquad \frac{\int_{B_h} \phi\, d\mu}{\int_{B_h} \phi\, d\lambda} \in \mathrm{cl\,co} \bigcup_{t \in B_h} Q(t)$$

for every $\phi \in \mathcal{C}_0$ such that $\mathrm{Supp}\,\phi \subset B_h$.

Of course, *if μ satisfies* (W), *then* (LW) *holds for every $t_0 \in I$.*

Rather surprising also the convers hold, as we shall show in the following (Theorem 3).

In other words, also this last formulation proves to be equivalent to the previous ones.

THEOREM 1. *Every solution of* (S) *is also a solution of* (W).

Proof. Let $\phi \in \mathcal{C}_0$ be given. Note that $\int_I \phi\, d\mu = \int_I \phi\, d\mu_a + \int_I \phi\, d\mu_s$ moreover

$$\text{(5)} \qquad \int_I \phi\, d\mu_a = \int_I \frac{d\mu_a}{d\lambda} \phi\, d\lambda = \int_{I \cap \mathrm{Supp}\,\phi} \frac{d\mu_a}{d\lambda}\, d\lambda_\phi$$

$$\text{(6)} \qquad \int_I \phi\, d\mu_s = \int_I \frac{d\mu_s}{d|\mu_s|} \phi\, d|\mu_s| = \int_{I \cap \mathrm{Supp}\,\phi} \frac{d\mu_s}{d|\mu_s|}\, d\mu_{s,\phi}$$

where λ_ϕ and $\mu_{s,\phi}$ are the Borel measures defined respectively by

$$\lambda_\phi(E) = \int_E \phi\, d\lambda \qquad \mu_{s,\phi}(E) = \int_E \phi\, d|\mu_s| \qquad E \subset I\,.$$

From (5), in force of the assumption and taking Theorem 1.3 in [1] into account, we get

$$\phi_a := \frac{\int_I \phi \, d\mu_a}{\int_I \phi \, d\lambda} = \frac{\int_{I\cap \operatorname{Supp}\phi} \frac{d\mu_a}{d\lambda}\, d\lambda_\phi}{\lambda_\phi(I\cap \operatorname{Supp}\phi)} \in \operatorname{cl}\operatorname{co} \bigcup_{t\in I\cap \operatorname{Supp}\phi} Q(t)\,. \tag{7}$$

In the case $\int_I \phi \, d|\mu_s| = 0$, then $\int_I \phi \, d\mu_s = 0$ and the assertion is an immediate consequence of (7).

Let us put

$$Q_\phi := \operatorname{cl}\operatorname{co} \bigcup_{t\in I\cap \operatorname{Supp}\phi} Q(t)\,. \tag{7'}$$

Let us assume now that $\int_I \phi \, d|\mu_s| \neq 0$. Then from (6), in force of the assumption we get, as before

$$\begin{aligned} \frac{\int_I \phi \, d\mu_s}{\int_I \phi \, d|\mu_s|} &= \frac{\int_{I\cap \operatorname{Supp}\phi} \frac{d\mu_s}{d|\mu_s|}\, d\mu_{s,\phi}}{\mu_{s,\phi}(I\cap \operatorname{Supp}\phi)} \\ &\in \operatorname{cl}\operatorname{co} \bigcup_{t\in I\cap \operatorname{Supp}\phi} Q_\infty(t) \subset \left[\operatorname{cl}\operatorname{co} \bigcup_{t\in I\cap \operatorname{Supp}\phi} Q(t)\right]_\infty = [Q_\phi]_\infty \end{aligned}$$

and since the right-hand side is a cone, we deduce

$$\phi_s := \frac{\int_I \phi \, d\mu_s}{\int_I \phi \, d\lambda} = \frac{\int_I \phi \, d\mu_s}{\int_I \phi \, d|\mu_s|} \cdot \frac{\int_I \phi \, d|\mu_s|}{\int_I \phi \, d\lambda} \in [Q_\phi]_\infty\,. \tag{8}$$

From (7) and (8) we have that

$$\frac{\int_I \phi \, d\mu}{\int_I \phi \, d\lambda} = \phi_a + \phi_s \text{ with } \phi_a \in Q_\phi \quad \phi_s \in [Q_\phi]_\infty$$

and, by virtue of P_3, we conclude that

$$\frac{\int_I \phi \, d\mu}{\int_I \phi \, d\lambda} \in Q_\phi = \operatorname{cl}\operatorname{co} \bigcup_{t\in \operatorname{Supp}\phi} Q(t)$$

which proves the assertion.

□

THEOREM 2. *Let μ be a solution of* (LW) *in $t_0 \in I$.*

(*a*) *If Q has properties* (Q) *at t_0 and the derivative $\frac{d\mu_a}{d\lambda}(t_0)$ exists, then*

$$\frac{d\mu_a}{d\lambda}(t_0) \in Q(t_0)\,.$$

(*b*) *If Q_∞ has properties* (Q) *at t_0 and the derivative $\frac{d\mu_s}{d|\mu_s|}(t_0)$ exists, then*

$$\frac{d\mu_s}{d|\mu_s|}(t_0) \in Q_\infty(t_0)\,.$$

Proof. Let S_μ denote the set where measure μ_s is concentrated, i.e. $S_\mu = \{t \in I : \mu_s\{t\} \neq 0\}$. Since μ_s is of bounded variation, then S_μ is denumerable; let us put

$$S_\mu = \{s_n,\ n \in \mathbb{N}\}\,.$$

Let us fix a point $t_0 \in I^0$. The case where t_0 is an end-point for I is analogous.

The proof will proceed into steps.

Step 1. Let us prove first that for every $B_h = B(t_0, h) \subset I$ with $0 < h < \overline{h}(t_0)$ and such that $\partial B_h \cap S_\mu = \phi$, we have

$$\frac{\mu(B_h - S_\mu)}{2h} = \frac{\mu_a(B_h)}{2h} \in \text{cl co} \bigcup_{t \in B_h} Q(t)\,. \tag{9}$$

Let $\overline{n} \in \mathbb{N}$ be fixed. For every $1 \leq i \leq \overline{n}$, we consider a constant $0 < r_i = r_i(\overline{n}) \leq \frac{1}{\overline{n}2^i}$ such that $B(s_i, r_i) \cap B(s_j, r_j) = \phi$, $i \neq j$, $1 \leq i, j \leq \overline{n}$.

Moreover, we put $I_{\overline{n}} = \bigcup_{i=1}^{\overline{n}} B^0(s_i, r_i)$.

Fixed a constant $0 < \eta < \min\{h,\ r_i,\ 1 \leq i \leq \overline{n}\}$, we denote by $I_{\overline{n},\eta} = \bigcup_{i=1}^{\overline{n}} B^0(s_i, r_i - \eta)$ and consider the function

$$\phi_{\overline{n},\eta}(t) = \begin{cases} 0 & t \in I - B_h \cup I_{\overline{n},\eta} \\ 1 & t \in B_{h-\eta} - I_{\overline{n}} \\ \text{linear} & \text{otherwise} \end{cases}$$

Of course $\phi_{\overline{n},\eta} \in \mathcal{C}_0$ thus, by virtue of the assumption, we have

$$R_{\overline{n},\eta} := \frac{\int_I \phi_{\overline{n},\eta}\, d\mu}{\int_I \phi_{\overline{n},\eta}\, d\lambda} \in \text{cl co} \bigcup_{t \in B_h} Q(t)\,. \tag{10}$$

Note that, put $C_{\overline{n},\eta} = B_h - \left[I_{\overline{n},\eta} \cup (B_{h-\eta} - I_{\overline{n}})\right]$, we have

$$R_{\overline{n},\eta} = \frac{\int_{B_h - I_{\overline{n},\eta}} \phi_{\overline{n},\eta}\, d\mu}{\int_{B_h - I_{\overline{n},\eta}} \phi_{\overline{n},\eta}\, d\lambda} = \frac{\mu\left(B_{h-\eta} - I_{\overline{n}}\right) + \int_{C_{\overline{n},\eta}} \phi_{\overline{n},\eta}\, d\mu}{\lambda\left(B_{h-\eta} - I_{\overline{n}}\right) + \int_{C_{\overline{n},\eta}} \phi_{\overline{n},\eta}\, d\lambda}\,. \tag{11}$$

If we let $\eta \to 0$, we get

$$B_{h-\eta} - I_{\overline{n}} \nearrow B_h^0 - I_{\overline{n}} \qquad I_{\overline{n},\eta} \nearrow I_{\overline{n}}$$

and hence

$$C_{\overline{n},\eta} \searrow \partial B_h = \{t_0 - h, t_0 + h\}\,.$$

As a consequence, we have (see e.g. [14])

$$\begin{aligned} &\lim_{\eta \to 0} \mu(B_{h-\eta} - I_{\overline{n}}) = \mu(B_h - I_{\overline{n}}) \\ &\lim_{\eta \to 0} \lambda(B_{h-\eta} - I_{\overline{n}}) = \lambda(B_h - I_{\overline{n}}) \\ &\lim_{\eta \to 0} |\mu|(C_{\overline{n},\eta}) = \lim_{\eta \to 0} \lambda(C_{\overline{n},\eta}) = 0 \end{aligned} \tag{12}$$

and hence

$$\lim_{\eta\to 0}\int_{C_{\overline{n},\eta}} \phi_{\overline{n},\eta}\, d\mu = \lim_{\eta\to 0}\int_{C_{\overline{n},\eta}} \phi_{\overline{n},\eta}\, d\lambda = 0\,. \tag{12$'$}$$

From (11), (12) and (12′), we obtain

$$\lim_{\eta\to 0} R_{\overline{n},\eta} = \frac{\mu(B_h - I_{\overline{n}})}{\lambda(B_h - I_{\overline{n}})} = \frac{\mu_a(B_h - I_{\overline{n}}) + \mu_s(B_h - I_{\overline{n}})}{\lambda(B_h - I_{\overline{n}})}\,. \tag{13}$$

Note that since

$$\lambda(I_{\overline{n}}) = \sum_{i=1}^{\overline{n}} 2r_i \le \frac{2}{\overline{n}}\sum_{i=1}^{\overline{n}}\frac{1}{2^i} < \frac{2}{\overline{n}}$$

we have

$$\lim_{\overline{n}\to+\infty}\lambda(I_{\overline{n}}) = \lim_{\overline{n}\to+\infty}\mu_a(I_{\overline{n}}) = 0\,. \tag{14}$$

Moreover

$$|\mu_s(B_h - I_{\overline{n}})| \le |\mu_s|(B_h - I_{\overline{n}}) \le |\mu_s|(S_\mu - I_{\overline{n}}) = \sum_{n>\overline{n}} |\mu_s|(\{s_n\})$$

and, recalling that μ has bounded variation

$$\lim_{\overline{n}\to+\infty} |\mu_s(B_h - I_{\overline{n}})| \le \lim_{\overline{n}\to+\infty}\sum_{n>\overline{n}} |\mu_s|(\{s_n\}) = 0\,. \tag{14$'$}$$

Finally, from (13), (14) and (14′) we conclude that

$$\lim_{\overline{n}\to+\infty}\lim_{\eta\to 0} R_{\overline{n},\eta} = \frac{\mu_a(B_h)}{2h}$$

that, by virtue of (10), proves (9).

Step 2. Let us prove now part (a). We recall that

$$\frac{d\mu_a}{d\lambda}(t_0) = \lim_{h\to 0}\frac{\mu_a(B_h)}{2h} \tag{15}$$

By virtue of step 1, for every fixed $\overline{h} > 0$ such that $B_{\overline{h}} \subset I$, we have

$$\frac{\mu_a(B_h)}{2h} \in \text{cl co}\bigcup_{t\in B_h} Q(t) \subset \text{cl co}\bigcup_{t\in B_{\overline{h}}} Q(t) \qquad \lambda\text{–a.e.}\quad 0 < h < \overline{h}$$

and hence, by letting $h \to 0$, and taking (15) into account, we get

$$\frac{d\mu_a}{d\lambda}(t_0) \in \text{cl co}\bigcup_{t\in B_{\overline{h}}} Q(t)\,.$$

By virtue of the arbitrariness of $\overline{h} > 0$ and in force of assumption (Q), we conclude that

$$\frac{d\mu_a}{d\lambda}(t_0) \in \bigcap_{\overline{h}>0}\text{cl co}\bigcup_{t\in B_{\overline{h}}} Q(t) = Q(t_0)\,.$$

Step 3. For the proof of part (b) let us note that

$$\frac{d\mu_s}{d|\mu_s|}(t_0) = \frac{\mu_s(\{t_0\})}{|\mu_s|(\{t_0\})} \tag{16}$$

since $\mu_s(\{t_0\}) = \int_{\{t_0\}} d\mu_s = \int_{\{t_0\}} \frac{d\mu_s}{d|\mu_s|}\, d|\mu_s| = \frac{d\mu_s}{d|\mu_s|}(t_0)\, |\mu_s|(\{t_0\})$.
Let $h > 0$ be fixed in such a way that $B_h = B(t_0, h) \subset I$. For every $0 < \eta < h$ we consider the continuous function defined by

$$\phi_\eta(t) = \begin{cases} 1 & t \in B_{\frac{\eta}{2}} \\ 0 & t \in I - B_\eta \\ \text{linear} & \text{otherwise.} \end{cases}$$

Note that (see e.g. [14])

$$\mu_s(\{t_0\}) = \lim_{\eta\to 0} \mu(B_{\frac{\eta}{2}}) = \lim_{\eta\to 0} \mu(B_\eta). \tag{17}$$

Moreover we have

$$\begin{aligned} \mu(B_{\frac{\eta}{2}}) &= \int_{B_\eta} \phi_\eta\, d\mu = \int_I \phi_\eta\, d\mu - \int_{B_\eta - B_{\frac{\eta}{2}}} \phi_\eta\, d\mu \\ &= \frac{\int_I \phi_\eta\, d\mu}{\int_I \phi_\eta\, d\lambda} \cdot \int_I \phi_\eta\, d\lambda - \int_{B_\eta - B_{\frac{\eta}{2}}} \phi_\eta\, d\mu\, . \end{aligned} \tag{18}$$

By assumption we know that

$$\frac{\int_I \phi_\eta\, d\mu}{\int_I \phi_\eta\, d\lambda} \in \text{cl co} \bigcup_{t\in B_\eta} Q(t) \subset \text{cl co} \bigcup_{t\in B_h} Q(t)$$

let us put

$$Q_h := \text{cl co} \bigcup_{t\in B_h} Q(t)\, .$$

Since $\lim_{\eta\to 0} \int_I \phi_\eta\, d\mu = 0$, by virtue of P_4 we get

$$\lim_{\eta\to 0} \frac{\int_I \phi_\eta\, d\mu}{\int_I \phi_\eta\, d\lambda} \cdot \int_I \phi_\eta\, d\lambda \in [Q_h]_\infty\, . \tag{19}$$

Furthermore, by virtue of (17) we have

$$\left| \int_{B_\eta - B_{\frac{\eta}{2}}} \phi_\eta\, d\mu \right| \leq |\mu|(B_\eta) - |\mu|(B_{\frac{\eta}{2}}) \overset{\eta\to 0}{longrightarrow} 0 \tag{20}$$

thus, from (18) and taking (17), (19) and (20) into account, we obtain

$$\mu(\{t_0\}) \in [Q_h]_\infty \qquad \text{for every } h > 0 \text{such that } B_h = B(t_0, h) \subset I\, .$$

Finally, recalling P_5 we deduce that

$$\mu(\{t_0\}) \in \bigcap_{h>0} [Q_h]_\infty = \left[\bigcap_{h>0} Q_h \right]_\infty = Q_\infty(t_0)$$

and taking (16) into account, since $Q_\infty(t_0)$ is a cone, the assertion follows.

□

DEFINITION 4. *Let μ be a given measure. We will say that a* property P holds (λ, μ_s)–*a.e. if property P is satisfied for every point t with the exception perhaps of a set N with $\lambda(N) + \mu_s(N) = 0$.*

From Theorem 2 the following result can be deduced.

THEOREM 3. *Assume that*

(i) *Q has properties* (Q) λ–*a.e.*

(ii) *Q_∞ has properties* (Q) μ_s–*a.e.*

Then every measure μ which is a solution of (LW) (λ, μ_s)–*a.e. is also a solution of* (S).

As we will observe in Section 6, the present equivalence result [among the three formulations (S), (W), (LW)] improves the equivalence between strong and weak formulation proved by Stewart, by means of a transfinite process in [23].

It is easy to see that Theorem 3 admits the following generalization.

THEOREM 4. *Let $Q_h : I \to \mathcal{C}(\mathbb{R}^m)$, $h \geq 0$ be a net of multifunctions and let μ be a Borel measure. Assume that*

(i) $Q_0(t_0) = \bigcap_{h>0} Q_h(t_0)$ λ–*a.e.;*

(ii) $[Q_0]_\infty(t_0) = \bigcap_{h>0} [Q_h]_\infty(t_0)$ μ_s–*a.e.;*

(iii) *for* (λ, μ_s)–*a.e.* t_0 *there exists* $\overline{h} = \overline{h}(t_0) > 0$ *such that for every* $0 < h < \overline{h}$

$$\frac{\int_{B_h} \phi \, d\mu}{\int_{B_h} \phi \, d\lambda} \in Q_h(t_0)$$

for every $\phi \in \mathcal{C}_0$ such that $\operatorname{Supp} \phi \subset B_h$.

Then μ is a solution of (S).

Proof. Let $t_0 \in I$ be fixed in such a way that all the assumptions hold.
Following the proof of step 1 in Theorem 3, from assumption (iii) we deduce that

$$\frac{\mu_a(B_h)}{2h} \in Q_h(t_0)$$

and hence from assumption (i) (as in step 2) we get

$$\frac{d\mu_a}{d\lambda}(t_0) \in \bigcap_{h>0} Q_h(t_0) = Q_0(t_0) \, .$$

Finally, analogously to the proof of step 3, from asumptions (iii) and (ii) we obtain

$$\mu(\{t_0\}) \in \bigcap_{h>0} [Q_h]_\infty(t_0) = Q_\infty(t_0)$$

and since $Q_\infty(t_0)$ is a cone, we get

$$\frac{d\mu_s}{d|\mu_s|}(t_0) = \frac{\mu(\{t_0\})}{|\mu|(\{t_0\})} \in Q_\infty(t_0)\,.$$

□

4. The main closure theorem

Let $I \subset \mathbb{R}$ be a closed interval and let $Q_k : I \to \mathcal{C}(\mathbb{R}^m)$, $k \geq 0$, be a sequence of multifunctions.

We introduce first the following definition.

DEFINITION 5. *We will say that* $(Q_k)_{k\geq 0}$ satisfies condition (QK) at a point $t_0 \in E$ *provided*

$$Q_0(t_0) = \bigcap_{h>0} \bigcap_{n\in N} \mathrm{cl} \bigcup_{k\geq n} \mathrm{cl\,co} \bigcup_{t\in B_h} Q_k(t)\,. \tag{QK}$$

We are able now to state and prove our main closure result.

THEOREM 5. *Let* $Q_k : I \to \mathcal{C}(\mathbb{R}^m)$, $k \geq 0$ *be a sequence of multifunctions and let* $(\mu_k)_{k\geq 0}$ *be a sequence of Borel measures such that*

(i) $(Q_k)_{k\geq 0}$ *satisfies* (QK) *condition* $(\lambda, \mu_{0,s})$*–a.e.;*

(ii) μ_k w^**–converges to* μ_0*;*

(iii) $\begin{cases} \frac{d\mu_{k,a}}{d\lambda}(t) \in Q_k(t) & \lambda\text{–a.e.} \\ \frac{d\mu_{k,s}}{d|\mu_{k,s}|}(t) \in [Q_k]_\infty(t) & \mu_{k,s}\text{–a.e.} \end{cases}$

Then the following inclusion holds

$$\begin{cases} \frac{d\mu_{0,a}}{d\lambda}(t) \in Q_0(t) & \lambda\text{–a.e.} \\ \frac{d\mu_{0,s}}{d|\mu_{0,s}|}(t) \in [Q_0]_\infty(t) & \mu_{0,s}\text{–a.e.} \end{cases}$$

Proof. We prove this result as an application of Theorem 4 to the net

$$Q_h(t) = \bigcap_{n\in N} \mathrm{cl} \bigcup_{k\geq n} \mathrm{cl\,co} \bigcap_{\tau\in B(t,h)} Q_k(\tau)\,.$$

By virtue of P_5 assumption (i) assures that both assumptions (i) and (ii) in Theorem 4 hold.

Now, let $t_0 \in I$ be fixed in such a way that assumption (iii) holds and let $\phi \in \mathcal{C}_0$ be given with $\mathrm{Supp}\,\phi \subset B_h \cap I$.

From Theorem 1 we deduce

$$\frac{\int_{\mathrm{Supp}\,\phi} \phi\, d\mu_k}{\int_{\mathrm{Supp}\,\phi} \phi\, d\lambda} \in \mathrm{cl\,co} \bigcup_{t\in\mathrm{Supp}\,\phi} Q_k(t) \qquad k \in \mathbb{N} \tag{21}$$

and from assumption (ii) we get

$$\frac{\int_{\mathrm{Supp}\,\phi} \phi\, d\mu_0}{\int_{\mathrm{Supp}\,\phi} \phi\, d\lambda} = \lim_{k\to+\infty} \frac{\int_{\mathrm{Supp}\,\phi} \phi\, d\mu_k}{\int_{\mathrm{Supp}\,\phi} \phi\, d\lambda} \in Q_h(t_0) \tag{22}$$

which gives assumption (iii) in Theorem 4.

□

5. Further closure theorems for measure differential inclusions

We present here some applications of the main result to remarkable classes of measure differential inclusions.

According to standard notations, we denote by L^1 the space of summable functions $u : I \to \mathbb{R}^m$ and by BV the space of the functions $u \in L^1$ which are of bounded variation in the sense of Cesari [7], i.e. $V(u) < +\infty$.

Let $u_k : I \to \mathbb{R}^m$, $k \geq 0$, be a given sequence in L^1 and let $Q : I \times A \subset \mathbb{R}^{n+1} \to \mathcal{C}(\mathbb{R}^m)$ be a given multifunction.

DEFINITION 6. *We say that the sequence $(u_k)_{k\geq 0}$ satisfies the property of* local equi-oscillation *at a point $t_0 \in I$ provided*

$$\text{(LEO)} \qquad \lim_{h\to 0} \limsup_{k\to\infty} \sup_{t\in B_h} |u_k(t) - u_0(t_0)| = 0\,.$$

It is easy to see that the following result holds.

PROPOSITION 2. *If u_k converges uniformly to a continuous function u_0, then condition* (LEO) *holds everywhere in I.*

In [10] an other sufficient condition for property (LEO) can be found (see the proof of Theorem 1).

PROPOSITION 3. *If $(u_k)_{k\geq 0}$ is a sequence of BV functions such that*

(i) *u_k converges to u_0 λ–a.e. in I;*

(ii) $\sup\limits_{k\in\mathbb{N}} V(u_k) < +\infty$.

Then a subsequence $(u_{s_k})_{k\geq 0}$ satisfies condition (LEO) *λ–a.e. in I.*

Let us prove now a sufficient condition for property (QK).

THEOREM 6. *Assume that the following conditions are satisfied at a point $t_0 \in I$*

(i) *Q satisfies property* (Q)*;*

(ii) *$(u_k)_{k\geq 0}$ satisfies condition* (LEO).

Then the sequence of multifunctions $Q_k : I \to \mathcal{C}(\mathbb{R}^m)$, $k \geq 0$, defined by

$$Q_k(t) = Q(t, u_k(t)) \qquad k \geq 0$$

satisfies property (QK) *at t_0.*

Proof. By virtue of assumption (ii), fixed $\varepsilon > 0$ a number $0 < h_\varepsilon < \varepsilon$ exists such that for every $0 < h < h_\varepsilon$ an integer k_h exists with the property that for every $k \geq k_h$

$$t \in B_h(t_0) \Longrightarrow |u_0(t_0) - u_k(t)| < \varepsilon\,.$$

Then for every $k \geq k_h$

$$\operatorname{cl\,co} \bigcup_{t\in B_h} Q(t, u_k(t)) \subset \operatorname{cl\,co} \bigcup_{|t-t_0|\leq\varepsilon,\, |x-u_0(t_0)|\leq\varepsilon} Q(t,x) = Q_\varepsilon\,.$$

Fixed $n \geq k_h$
$$\operatorname{cl} \bigcup_{k \geq n} \operatorname{cl} \operatorname{co} \bigcup_{t \in B_h} Q(t, u_k(t)) \subset Q_\varepsilon$$
and hence
$$\bigcap_{n \in \mathbb{N}} \operatorname{cl} \bigcup_{k \geq n} \operatorname{cl} \operatorname{co} \bigcup_{t \in B_h} Q(t, u_k(t)) \subset Q_\varepsilon \,.$$
Finally, by virtue of assumption (i), we have
$$\bigcap_{\varepsilon > 0} \bigcap_{n \in \mathbb{N}} \operatorname{cl} \bigcup_{k \geq n} \operatorname{cl} \operatorname{co} \bigcup_{t \in B_h} Q(t, u_k(t)) \subset \bigcap_{\varepsilon > 0} Q_\varepsilon = Q(t_0, u_0(t_0))$$
which proves the assertion.

□

In force of this result, the following closure Theorem 5 can be deduced as an application of the main theorem.

THEOREM 7. *Let $Q : I \times A \subset \mathbb{R}^{n+1} \to \mathcal{C}(\mathbb{R}^m)$ be a multifunction, let $(\mu_k)_{k \geq 0}$ be a sequence of Borel measures of bounded variations and let $u_k : I \to A$, $k \geq 0$ be a sequence of BV functions which satisfy the conditions*

(i) *Q has properties* (Q) *at every point (t, x) with the exception of a set of points whose t-coordinate lie on a set of $(\lambda, \mu_{0,s})$–null measure;*

(ii) $\begin{cases} \frac{d\mu_{k,a}}{d\lambda}(t) \in Q(t, u_k(t)) & \lambda\text{–a.e.} \\ \frac{d\mu_{k,s}}{d|\mu_{k,s}|}(t) \in Q_\infty(t, u_k(t)) & \mu_{k,s}\text{–a.e.} \end{cases}$

(iii) *μ_k w^*–converges to μ_0;*

(iv) $\sup_{k \in \mathbb{N}} V(u_k) < +\infty$;

(v) *u_k converges to u_0 pointwise λ–a.e. and satisfies condition* (LEO) *at $\mu_{0,s}$–a.e.*

Then the following inclusion holds
$$\begin{cases} \frac{d\mu_{0,a}}{d\lambda}(t) \in Q(t, u_0(t)) & \lambda\text{–a.e.} \\ \frac{d\mu_{0,s}}{d|\mu_{0,s}|}(t) \in Q_\infty(t, u_0(t)) & \mu_{0,s}\text{–a.e.} \end{cases}$$

REMARK 1. We recall that the distributional derivative of a BV function u is a Borel measure of bounded variation [17] that we will denote by μ_u.

Moreover u admits an "essential derivative" u' (i.e. computed by usual incremental quotients disregarding the values taken by u on a suitable Lebesgue null set) which coincides with $\frac{d\mu_{u,a}}{d\lambda}$ [25].

Note that Theorem 7 is an extension and a generalization of the main closure theorem in [10] (Theorem 1) given for a differential inclusion of the type
$$u'(t) \in Q(t, u(t)) \qquad \lambda\text{–a.e. in } I \,.$$
To this purpose, we recall that if $(u_k)_{k \geq 0}$, is a sequence of equi–BV functions, then a subsequence of distributional derivatives w^*–converges.

The following closure theorem can be considered as a particular case of Theorem 7.

THEOREM 8. *Let $Q : I \times E \to \mathcal{C}(\mathbb{R}^m)$, with E subset of a Banach space, be a multifunction, let $(\mu_k)_{k\geq 0}$ be a sequence of Borel measures of bounded variations and let $(a_k)_{k\geq 0}$ be a sequence in E. Assume that the following conditions are satisfied*

(i) *Q has properties (Q) at every point (t, x) with the exception of a set of points whose t-coordinate lie on a set of $(\lambda, \mu_{0,s})$–null measure;*

(ii) $\begin{cases} \frac{d\mu_{k,a}}{d\lambda}(t) \in Q(t, a_k) & \lambda\text{–a.e.} \\ \frac{d\mu_{k,s}}{d|\mu_{k,s}|}(t) \in Q_\infty(t, a_k) & \mu_{k,s}\text{–a.e.} \end{cases}$

(iii) *μ_k w^*–converges to μ_0;*

(iv) *$(a_k)_k$ converges to a_0.*

Then the following inclusion holds

$$\begin{cases} \frac{d\mu_{0,a}}{d\lambda}(t) \in Q(t, a_0) & \lambda\text{–a.e.} \\ \frac{d\mu_{0,s}}{d|\mu_{0,s}|}(t) \in Q_\infty(t, a_0) & \mu_{0,s}\text{–a.e.} \end{cases}$$

As we will prove in Section 6, this last result is an extension of closure Theorem 3 in [10].

As an application of Theorem 7 also the following result can be proved.

THEOREM 9. *Let $Q : I \times \mathbb{R}^n \times \mathbb{R}^p \to \mathcal{C}(\mathbb{R}^m)$, be a multifunction, let $f : I \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ be a function and let $(u_k, v_k) : I \to \mathbb{R}^n \times \mathbb{R}^q$, $k \geq 0$, be a sequence of functions.*
Assume that

(i) *Q satisfies property (Q) at every point (t, x, y) with the exception of a set of points whose t-coordinate lie on a set of $(\lambda, \mu_{v_0,s})$–null measure;*

(ii) *f is a Carathéodory function and*
$|f(t,u,v)| \leq \psi_1(t) + \psi_2(t)\,|u| + \psi_3(t)\,|v|$ *with* $\psi_i \in L^1$ $i = 1, 2, 3$*;*

(iii) $\begin{cases} v_k'(t) \in Q(t, u_k(t)) - f(t, u_k(t), v_k(t)) & \lambda\text{–a.e.} \\ \frac{d\mu_{v_k,s}}{d|\mu_{v_k,s}|}(t) \in Q_\infty(t, u_k(t)) & \mu_{v_k,s}\text{–a.e.} \end{cases}$

(iv) $\sup_{k\in\mathbb{N}} V(v_k) < +\infty$ *and $(v_k)_k$ converges to v_0 λ–a.e.;*

(v) *$(u_k)_k$ converges uniformly to a continuous function u_0.*

Then the following inclusion holds

$$\begin{cases} v_0'(t) \in Q(t, u_0(t)) - f(t, u_0(t), v_0(t)) & \lambda\text{–a.e.} \\ \frac{d\mu_{v_0,s}}{d|\mu_{v_0,s}|}(t) \in Q_\infty(t, u_0) & \mu_{v_0,s}\text{–a.e.} \end{cases}$$

Proof. If we consider the sequence of Borel measures defined by

$$\nu_k([a,b]) = \int_a^b [v_k'(t) + f(t, u_k(t), v_k(t))]\,d\lambda \qquad [a,b] \subset I \qquad k \geq 0$$

it is easy to see that

$$d\nu_{k,s} = d\mu_{v_k,s} \qquad \frac{d\nu_{k,a}}{d\lambda}(t) = v_k'(t) + f(t, u_k(t), v_k(t)) \qquad \lambda\text{–a.e.}$$

It is easy to verify that assumptions assure that $(\nu_k)_{k\geq 0}$ is a sequence of BV measure which w^*–converges and the result is an immediate application of Theorem 7.

□

REMARK 2. Differential incusions of this type are adopted as a model for rigid body dynamics (see [20] for details). As we will observe in Section 6 the previous result improves the analogous theorem proved in [23] (Theorem 4).

6. On comparison with Stewart's assumptions

This section is dedicated to a discussion on the comparison between our assumptions and that adopted by Stewart in [23].

Let $Q : E \to \mathcal{C}(\mathbb{R}^n)$ be a given multifunction where E is a subset of a Banach space.

The main hypotheses adopted by Stewart in [23] on muntifunction Q are the closure of the graph (i.e. property (K)) and the following condition:

$$\text{for every } t_0 \in E \text{ there exist } \sigma_0 > 0 \text{ and } R_0 > 0 \text{ such that} \quad \sup_{t \in B_\sigma} \inf_{x \in Q(t)} \|x\| \leq R_0 \,. \tag{23}$$

We will prove here that these assumptions are stricly stronger than property (Q). As a consequence, the results of the present paper improve that given in [23].

PROPOSITION 4. *Let Q be a multifunction with closed graph and let $t_0 \in E$ be fixed. Assume that*
for a given $t_0 \in E$ there exist $\sigma_0 > 0$ and $R_0 > 0$ such that

$$\sup_{t \in B_\sigma} \inf_{x \in Q(t)} \|x\| \leq R_0$$

then multifunction Q satisfies property (Q) *at t_0.*

Proof. By virtue of Lemma 5.1 in [23], fixed a number $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$t \in B_\delta \Longrightarrow Q(t) \subset Q(t_0) + \varepsilon\, B(0,1) + (Q_\infty(t_0))_\varepsilon$$

where $(Q_\infty(t_0))_\varepsilon$ denotes the ε–enlargement of the set $Q_\infty(t_0)$.
Since the right-hand side is closed and convex

$$\text{cl co} \bigcup_{t \in B_\delta} Q(t) \subset Q(t_0) + \varepsilon\, B(0,1) + (Q_\infty(t_0))_\varepsilon$$

then

$$Q^*(t_0) := \bigcap_{\delta > 0} \text{cl co} \bigcup_{t \in B_\delta} Q(t) \subset Q(t_0) + \varepsilon\, B(0,1) + (Q_\infty(t_0))_\varepsilon \,.$$

Now, fixed an integer $n \in \mathbb{N}$ and $0 < \varepsilon < \frac{1}{n}$, we get

$$Q^*(t_0) \subset Q(t_0) + \varepsilon\, B(0,1) + (Q_\infty(t_0))_\varepsilon \subset Q(t_0) + \varepsilon\, B(0,1) + (Q_\infty(t_0))_{\frac{1}{n}}$$

and letting $\varepsilon \to 0$, we obtain

$$Q^*(t_0) \subset Q(t_0) + (Q_\infty(t_0))_{\frac{1}{n}} \,. \tag{24}$$

Recalling that (see P_3)

$$Q(t_0) + Q_\infty(t_0) \subset Q(t_0)$$

we have

$$Q(t_0) + (Q_\infty(t_0))_{\frac{1}{n}} \subset (Q(t_0))_{\frac{1}{n}} + (Q_\infty(t_0))_{\frac{1}{n}} \subset (Q(t_0) + Q_\infty(t_0))_{\frac{1}{n}} \subset (Q(t_0))_{\frac{2}{n}}$$

and from (24) letting $n \to +\infty$ we get

$$Q^*(t_0) \subset Q(t_0)$$

which proves the assertion.

□

This result proves that even if Kuratowski condition (K) is weaker than Cesari's property (Q) (see Section 2), together with hypothesis (23) it becomes a stronger assumption. The following example will show that assumption (23) and (K) are strictly stronger than property (Q).
Finally, we recall that in BV setting property (Q) can not be replaced by condition (K), as it occurs in Sobolev's setting (see [10], Remark 1).

EXAMPLE 1. Let us consider the function $F : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(t, v) = \begin{cases} \frac{1}{t}\sin^2\frac{1}{t} + |v| & t \neq 0 \\ |v| & t = 0 \end{cases}$$

and the multifunction

$$\tilde{Q}(t, \cdot) = \operatorname{epi} F(t, \cdot)\,.$$

Of course assumption (ii) in Proposition 2 does not holds for $\tilde{Q}$ at the point $t_0 = 0$.
Moreover, in force of the Corollary to Theorem 3 in A.W.J. Stoddart [24], it can be easily proved that F is seminormal. Thus $\tilde{Q}$ satisfies condition (Q) at every point $t \in \mathbb{R}_0^+$ (see P_8).

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