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ON THE GEVREY ANALYTICITY OF SOLUTIONS OF SEMILINEAR PERTURBATIONS OF POWERS OF THE MIZOHATA OPERATOR

Abstract. We consider the Gevrey hypoellipticity of nonlinear equations with multiple characteristics, consisting of powers of the Mizohata operator and a Gevrey analytic nonlinear perturbation. The main theorem states that under some conditions on the perturbation every H^s solution (with $s \geq s_0$) of the equation is a Gevrey function. To prove the result we do not use the technique of cut-off functions, instead we have to control the behavior of the solutions through a fundamental solution. We combine some ideas of Grushin and Friedman.

1. Introduction

Since a Hilbert's conjecture the analyticity of solutions of partial differential equations has attracted considerable interest of specialists. The conjecture stated that every solution of an analytic nonlinear elliptic equation is analytic (see [1]). This conjecture caused considerable research by several authors. In 1904 Bernstein solved the problem for second order nonlinear elliptic equations in two variables [2]. For equations with an arbitrary number of variables the problem was established by Hopf [3], Giraud [4]. For a general elliptic system it was proved by Petrowskii [5], Morrey [6], and Friedman [7]. The last author also obtained results on the Gevrey analyticity of solutions. At present the results on the analyticity and Gevrey analyticity of solutions of linear partial differential equations have gone beyond the frame of the elliptic theory. A linear partial differential equation is called Gevrey hypoelliptic if every its solution is Gevrey analytic (in this context we can assume a solution in a very large space such as the space of distributions), provided the data is Gevrey analytic. Recently the class of linear Gevrey hypoelliptic equations with multiple characteristics has drawn much interest of mathematicians (see [8], and therein references). In this paper we consider the Gevrey analyticity of solutions in the Sobolev spaces of the nonlinear equation with multiple characteristics:

$$M_{2k}^h u + \varphi\left(x_1, x_2, u, \dots, \frac{\partial^{l_1+l_2} u}{\partial_{x_1}^{l_1} \partial_{x_2}^{l_2}}\right)_{l_1+l_2 \leq (h-1)} = 0\,,$$

where $u = u(x_1, x_2)$, $M_{2k} = \frac{\partial}{\partial x_1} + i x_1^{2k} \frac{\partial}{\partial x_2}$, the Mizohata operator in $\mathbb{R}^2$. The Gevrey hypoellipticity for the equation was obtained in [9] in the linear case. We will prove a theorem for a so-called Grushin type perturbation φ (see the notations in §2 below). The analyticity of

*Dedicated to my mother on her sixtieth birthday.

C^∞ solutions was considered in [10]. The paper is organized as follows. In §2 we introduce notations used in the paper and state some auxiliary lemmas. In §3 we state the main theorem and prove a theorem on C^∞ regularity of the solutions. In §4 we establish the Gevrey analyticity of the solutions.

2. Some notations and lemmas

First let us define the function

$$F_{2k}^h(x_1, x_2, y_1, y_2) = \frac{1}{2\pi(h-1)!} \frac{(x_1-y_1)^{h-1}}{\frac{x_1^{2k+1}-y_1^{2k+1}}{2k+1} + i(x_2-y_2)} .$$

For $j = 1, \ldots, h-1$ we have (see [11])

$$M_{2k}^j F_{2k}^h = \frac{1}{2\pi(h-j-1)!} \frac{(x_1-y_1)^{h-j-1}}{\frac{x_1^{2k+1}-y_1^{2k+1}}{2k+1} + i(x_2-y_2)} \quad \text{and} \quad M_{2k}^h F_{2k}^h = \delta(x-y) ,$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $\delta(\cdot)$ is the Dirac function. We will denote by Ω a bounded domain in $\mathbb{R}^2$ with piece-wise smooth boundary. We now state a lemma on the representation formula. Its proof can be found in [10].

LEMMA 4.1. *Assume that* $u \in C^h(\bar{\Omega})$ *then we have*

$$\begin{aligned} u(x) &= \int_\Omega (-1)^h F_{2k}^h(x,y) M_{2k}^h u(y)\, dy_1 dy_2 \\ &\quad + \int_{\partial\Omega} \left(\sum_{j=0}^{h-1} (-1)^j M_{2k}^j u M_{2k}^{h-j-1} F_{2k}^h(x,y) \right) (n_1 + i y_1^{2k} n_2)\, ds . \end{aligned} \tag{1}$$

For reason of convenience we shall use the following Heaviside function

$$\theta(z) = \begin{cases} 1 & \text{if} \quad z \geq 0 , \\ 0 & \text{if} \quad z < 0 . \end{cases}$$

We then have the following formula $\frac{d^m}{dt^m}(z^n) = n \ldots (n-m+1)\theta(n-m+1) z^{n-m}$.
We will use ∂_1^α, ∂_2^β, ${}_\gamma\partial_{\alpha,\beta}$, ∂_{1y}^α, ∂_{2y}^β and ${}_\gamma\partial_{\alpha,\beta}^y$ instead of $\frac{\partial^\alpha}{\partial x_1^\alpha}$, $\frac{\partial^\beta}{\partial x_2^\beta}$, $x_1^\gamma \frac{\partial^{\alpha+\beta}}{\partial x_1^\alpha \partial x_2^\beta}$, $\frac{\partial^\alpha}{\partial y_1^\alpha}$, $\frac{\partial^\beta}{\partial y_2^\beta}$ and $y_1^\gamma \frac{\partial^{\alpha+\beta}}{\partial y_1^\alpha \partial y_2^\beta}$, respectively. Throughout the paper we use the following notation

$$n! = \begin{cases} n! & \text{if} \quad n \geq 1 , \\ 1 & \text{if} \quad n \leq 0 , \end{cases} \quad \text{and} \quad C^n = \begin{cases} C^n & \text{if} \quad n \geq 1 , \\ 1 & \text{if} \quad n \leq 0 . \end{cases}$$

Furthermore all constants C_i which appear in the paper are taken such that they are greater than 1. Put $h(2k+1) = r_0$. For any integer $r \geq 0$ let Γ_r denote the set of pair of multi-indices (α, β) such that $\Gamma_r = \Gamma_r^1 \cup \Gamma_r^2$ where

$$\Gamma_r^1 = \{(\alpha, \beta) : \alpha \leq r_0,\ 2\alpha + \beta \leq r\}, \quad \Gamma_r^2 = \{(\alpha, \beta) : \alpha \geq r_0,\ \alpha + \beta \leq r - r_0\} .$$

For a pair (α,β) we denote by $(\alpha,\beta)^*$ the minimum of r such that $(\alpha,\beta)\in\Gamma_r$. For any nonnegative integer t we define the following sieves

$$\begin{aligned}\Xi_t &= \{(\alpha,\beta,\gamma):\alpha+\beta\le t,\ 2kt\ge\gamma\ge\alpha+(2k+1)\beta-t\}\,,\\ \Xi_t^0 &= \{(\alpha,\beta,\gamma):(\alpha,\beta,\gamma)\in\Xi_t,\ \gamma=\alpha+(2k+1)\beta-t\}\,.\end{aligned}$$

Later on we will use the following properties of Ξ_t:

Ξ_t contains a finite number of elements (less than $2k(t+1)^3$ elements).

If $(\alpha,\beta,\gamma)\in\Xi_t$, $\beta\ge1$, $\gamma\ge1$, then $(\alpha,\beta-1,\gamma-1)\in\Xi_{t-1}$.

For every $(\alpha,\beta,\gamma)\in\Xi_t$, $(\alpha_0,\beta_0,\gamma_0)\in\Xi_{t_0}$ we can express ${}_\gamma\partial_{\alpha,\beta}({}_{\gamma_0}\partial_{\alpha_0,\beta_0})$ as a linear combination of ${}_{\gamma'}\partial_{\alpha',\beta'}$, where $(\alpha',\beta',\gamma')\in\Xi_{t+t_0}$.

For every $(\alpha,\beta,\gamma)\in\Xi_t$, $(\alpha_1,\beta_1)\in\Gamma_r$ and a nonnegative integer m we can rewrite ${}_{\gamma-m}\partial_{\alpha+\alpha_1-m,\beta+\beta_1}$ as ${}_{\gamma-m}\partial_{\alpha_2,\beta_2}(\partial_1^{\alpha_3}\partial_2^{\beta_3})$ where $(\alpha_2,\beta_2,\gamma-m)\in\Xi_t$ and $(\alpha_3,\beta_3)\in\Gamma_{r-m}$ (see [12]).

For any nonnegative integer r let us define the norm

$$|u,\Omega|_r=\max_{(\alpha_1,\beta_1)\in\Gamma_r}\left|\partial_1^{\alpha_1}\partial_2^{\beta_1}u,\Omega\right|+\max_{\substack{(\alpha_1,\beta_1)\in\Gamma_r\\ \alpha_1\ge1,\beta_1\ge1}}\max_{x\in\bar\Omega}\left|\partial_1^h\left(\partial_1^{\alpha_1}\partial_2^{\beta_1}u(x)\right)\right|,$$

where $|w,\Omega|=\sum_{(\alpha,\beta,\gamma)\in\Xi_{h-1}}\max_{x\in\bar\Omega}|{}_\gamma\partial_{\alpha,\beta}w(x)|$.

For any nonnegative integer l let $\mathbb{H}^l_{\mathrm{loc}}(\Omega)$ denote the space of all u such that for any compact K in Ω we have $\sum_{(\alpha,\beta,\gamma)\in\Xi_l}\|{}_\gamma\partial_{\alpha,\beta}u\|_{L^2(K)}<\infty$. We note the following properties of $\mathbb{H}^l_{\mathrm{loc}}(\Omega)$:

$H^l_{\mathrm{loc}}(\Omega)\subset\mathbb{H}^l_{\mathrm{loc}}(\Omega)$ where $H^l_{\mathrm{loc}}(\Omega)$ stands for the standard Sobolev spaces.

$\mathbb{H}^{4k+2}_{\mathrm{loc}}(\Omega)\subset H^2_{\mathrm{loc}}(\Omega)\subset C(\Omega)$.

If $u\in\mathbb{H}^l_{\mathrm{loc}}(\Omega)$ and $(\alpha,\beta,\gamma)\in\Xi_t$, $t\le l$ then ${}_\gamma\partial_{\alpha,\beta}u\in\mathbb{H}^{l-t}_{\mathrm{loc}}(\Omega)$.

The following lemma is due to Grushin (see [12])

LEMMA 4.2. *Assume that* $u\in D'(\Omega)$ *and* $M^h_{2k}u\in\mathbb{H}^l_{\mathrm{loc}}(\Omega)$ *then* $u\in\mathbb{H}^{l+h}_{\mathrm{loc}}(\Omega)$.

Next we define a space generalizing the space of analytic functions (see for example [7]). Let L_n and $\bar L_n$ be two sequences of positive numbers, satisfying the monotonicity condition $\binom{i}{n}L_iL_{n-i}\le AL_n$ $(i=1,2\ldots;\ n=1,2\ldots)$, where A is a positive constant. A function $F(x,v)$, defined for $x=(x_1,x_2)$ and for $v=(v_1,\ldots,v_\mu)$ in a μ-dimensional open set E, is said to belong to the class $C\{L_{n-a};\Omega|\bar L_{n-a};E\}$ (a is an integer) if and only if $F(x,v)$ is infinitely differentiable and to every pair of compact subsets $\Omega_0\subset\Omega$ and $E_0\subset E$ there correspond constants A_1 and A_2 such that for $x\in\Omega_0$ and $v\in E_0$

$$\left|\frac{\partial^{j+k}F(x,v)}{\partial x_1^{j_1}\partial x_2^{j_2}\partial v_1^{k_1}\ldots\partial v_\mu^{k_\mu}}\right|\le A_1A_2^{j+k}L_{j-a}\bar L_{k-a}$$
$$\Big(j_1+j_2=j,\ \sum k_i=k;\ j,k=0,1,2\ldots\Big).$$

We use the notation $L_{-i}=1$ $(i=1,2,\ldots)$. If $F(x,v)=f(x)$, we simply write $f(x)\in C\{L_{n-a};\Omega\}$. Note that $C\{n!;\Omega\}$, $(C\{n!^s;\Omega\})$ is the space of all analytic functions (s-Gevrey functions), respectively, in Ω. Extensive treatments of non-quasi analytic functions (in particular, the Gevrey functions) can be found in [13, 14]. Finally we would like to mention the following lemma of Friedman [7].

LEMMA 4.3. *There exists a constant C_1 such that if $g(z)$ is a positive monotone decreasing function, defined in the interval $0 \le z \le 1$ and satisfying*

$$g(z) \le \frac{1}{8^{12^k}} g\left(z\left(1-\frac{6^k}{n}\right)\right) + \frac{C}{z^{n-r_0-1}} \quad (n \ge r_0+2,\ C > 0)\,,$$

then $g(z) < CC_1/z^{n-r_0-1}$.

3. The main theorem

We will consider the following problem

$$M_{2k}^h u + \varphi(x_1, x_2, u, \ldots, {}_\gamma\, \partial_{\alpha,\beta} u)_{(\alpha,\beta,\gamma)\in\Xi_{h-1}} = 0 \quad \text{in} \quad \Omega\,. \tag{2}$$

We now state our main

THEOREM 4.1. *Let $s \ge 4k^2+6k+h+1$. Assume that u is a $\mathbb{H}^s_{\rm loc}(\Omega)$ solution of the equation (2) and $\varphi \in C\{L_{n-a-2}; \Omega | L_{n-a-2}; E\}$ for every $a \in [0, r_0]$. Then $u \in C\{L_{n-r_0-2}; \Omega\}$. In particular, if φ is a s-Gevrey function then so is u.*

The proof of this theorem consists of Theorem 4.2 and Theorem 4.3.

THEOREM 4.2. *Let $s \ge 4k^2 + 6k + h + 1$. Assume that u is a $\mathbb{H}^s_{\rm loc}(\Omega)$ solution of the equation (2) and $\varphi \in C^\infty$. Then u is a $C^\infty(\Omega)$ function.*

Proof. Note that M^h_{2k} is elliptic on $\mathbb{R}^2$, except the line $(x_1, x_2) = (0, x_2)$. Therefore from the elliptic theory we have already $u \in C^\infty(\Omega \cap \mathbb{R}^2 \setminus (0, x_2))$.

LEMMA 4.4. *Let $s \ge 4k^2 + 6k + h + 1$. Assume that $u \in \mathbb{H}^s_{\rm loc}(\Omega)$ and $\varphi \in C^\infty$ then $\varphi(x_1, x_2, u, \ldots, {}_\gamma\, \partial_{\alpha,\beta} u)_{(\alpha,\beta,\gamma)\in\Xi_{h-1}} \in \mathbb{H}^{s-h+1}_{\rm loc}(\Omega)$.*

Proof. It is sufficient to prove that ${}_{\gamma_1}\partial_{\alpha_1,\beta_1}\varphi(x_1, x_2, u, \ldots, {}_\gamma\, \partial_{\alpha,\beta} u) \in L^2_{\rm loc}(\Omega)$ for every $(\alpha_1, \beta_1, \gamma_1) \in \Xi_{s-h+1}$. Let us denote $(u, \ldots, {}_\gamma\, \partial_{\alpha,\beta} u)_{(\alpha,\beta,\gamma)\in\Xi_{h-1}}$ by $(w_1, w_2, \ldots, w_\mu)$ with $\mu \le 2kh^3$. Since $s \ge 4k^2 + 6k + h + 1$ it follows that $w_1, \ldots, w_\mu \in C(\Omega)$. It is easy to verify that $\partial_1^{\alpha_1}\partial_2^{\beta_1}\varphi(x_1, x_2, u, \ldots, {}_\gamma\, \partial_{\alpha,\beta} u)$ is a linear combination with positive coefficients of terms of the form

$$\frac{\partial^k \varphi}{\partial x_1^{k_1} \partial x_2^{k_2} \partial w_1^{k_3} \ldots \partial w_\mu^{k_{\mu+2}}} \prod_{j=1}^{\mu} \prod_{(\alpha_{1,j},\beta_{1,j})} \left(\partial_1^{\alpha_{1,j}} \partial_2^{\beta_{1,j}} w_j\right)^{\zeta(\alpha_{1,j},\beta_{1,j})}, \tag{3}$$

where $k = k_1 + k_2 + \cdots + k_{\mu+2} \le \alpha_1 + \beta_1$; $\zeta(\alpha_{1,j}, \beta_{1,j})$ may be a multivalued function of $\alpha_{1,j}, \beta_{1,j}$; $\alpha_{1,j}, \beta_{1,j}$ may be a multivalued function of j, and

$$\begin{aligned} \sum_j \alpha_{1,j} \cdot \zeta(\alpha_{1,j}, \beta_{1,j}) &\le \alpha_1\,, \\ \sum_j \beta_{1,j} \cdot \zeta(\alpha_{1,j}, \beta_{1,j}) &\le \beta_1\,. \end{aligned}$$

Hence $x_1^{\gamma_1}\partial_1^{\alpha_1}\partial_2^{\beta_1}\varphi(x_1,x_2,u,\ldots,_\gamma\,\partial_{\alpha,\beta}u)$ is a linear combination with positive coefficients of terms of the form

$$\frac{\partial^k\varphi}{\partial x_1^{k_1}\partial x_2^{k_2}\partial w_1^{k_3}\ldots\partial w_\mu^{k_{\mu+2}}}x_1^{\gamma_1}\prod_{j=1}^{\mu}\prod_{(\alpha_{1,j},\beta_{1,j})}\left(\partial_1^{\alpha_{1,j}}\partial_2^{\beta_{1,j}}w_j\right)^{\zeta(\alpha_{1,j},\beta_{1,j})}.$$

Therefore the theorem is proved if we can show this general terms are in $L^2_{\text{loc}}(\Omega)$. If all $\zeta(\alpha_{1,j},\beta_{1,j})$ vanish then it is immediate that $\partial^k\varphi/\partial x_1^{k_1}\partial x_2^{k_2}\partial w_1^{k_3}\ldots\partial w_\mu^{k_{\mu+2}}\in C$, since $\varphi\in C^\infty$, $w_1,\ldots,w_\mu\in C(\Omega)$. Then we can assume that there exists at least one of $\zeta(\alpha_{1,j},\beta_{1,j})$ that differs from 0. Choose j_0 such that there exists $\alpha_{1,j_0},\beta_{1,j_0}$ with $\zeta(\alpha_{1,j_0},\beta_{1,j_0})\geq 1$ and

$$\alpha_{1,j_0}+(2k+1)\beta_{1,j_0}=\max_{\substack{j=1,\ldots,\mu\\ \zeta(\alpha_{1,j},\beta_{1,j})\geq 1}}\alpha_{1,j}+(2k+1)\beta_{1,j}\,.$$

Consider the following possibilities

I) $\zeta(\alpha_{1,j_0},\beta_{1,j_0})\geq 2$. We then have $\alpha_{1,j}+\beta_{1,j}\leq l-(h-1)-(4k+2)$. Indeed, if $j\neq j_0$ and $\alpha_{1,j}+\beta_{1,j}>l-(h-1)-(4k+2)$ then $\alpha_{1,j_0}+\beta_{1,j_0}\geq 2k$. Therefore

$$\begin{aligned}l-(h-1)-(4k+2)&<\alpha_{1,j}+\beta_{1,j}\leq\alpha_{1,j}+(2k+1)\beta_{1,j}\\&\leq\alpha_{1,j_0}+(2k+1)\beta_{1,j_0}\leq(2k+1)(\alpha_{1,j_0}+\beta_{1,j_0})\\&\leq 2k(2k+1)\,.\end{aligned}$$

Thus $l<(2k+2)(2k+1)+(h-1)$, a contradiction.
If $j=j_0$ and $\alpha_{1,j_0}+\beta_{1,j_0}>l-(h-1)-(4k+2)$ then we have

$$l-(h-1)\geq\alpha_1+\beta_1\geq 2(\alpha_{1,j_0}+\beta_{1,j_0})>2(l-(h-1)-(4k+2))\,.$$

Therefore $l<(h+1)+4(2k+1)$, a contradiction.
Next define

$$\gamma(\alpha_{1,j},\beta_{1,j})=\max\{0,\alpha_{1,j}+(2k+1)\beta_{1,j}+(h-1)+(4k+2)-l\}\,.$$

We claim that $\gamma(\alpha_{1,j},\beta_{1,j})\leq 2k(l-(h-1)-(4k+2))$. Indeed, if $j\neq j_0$ and $\gamma(\alpha_{1,j},\beta_{1,j})>2k(l-(h-1)-(4k+2))$ then

$$\begin{aligned}(2k+1)(l-(h-1))&\geq\alpha_1+(2k+1)\beta_1\\&\geq(\alpha_{1,j}+2\alpha_{1,j_0})+(2k+1)(\beta_{1,j_0}+2\beta_{1,j_0})\\&>3(2k+1)(l-(h-1)-(4k+2))\,.\end{aligned}$$

Thus $l<(h-1)+3(2k+1)$, a contradiction.
If $j=j_0$ and $\gamma(\alpha_{1,j_0},\beta_{1,j_0})>2k(l-(h-1)-(4k+2))$ then it follows that

$$\begin{aligned}(2k+1)(l-(h-1))&\geq\alpha_1+(2k+1)\beta_1\geq 2(\alpha_{1,j_0}+(2k+1)\beta_{1,j_0})\\&>2(2k+1)(l-(h-1)-(4k+2))\,.\end{aligned}$$

Thus $l<(h-1)+4(2k+1)$, a contradiction.

From all above arguments we deduce that $(\alpha_{1,j}, \beta_{1,j}, \gamma(\alpha_{1,j}, \beta_{1,j})) \in \Xi_{l-(h-1)-(4k+2)}$. Next we claim that $\sum \gamma(\alpha_{1,j}, \beta_{1,j})\zeta(\alpha_{1,j}, \beta_{1,j}) \le \gamma_1$. Indeed, if $\sum \gamma(\alpha_{1,j}, \beta_{1,j})\zeta(\alpha_{1,j}, \beta_{1,j}) > \gamma_1$ then we deduce that

$$\alpha_1 + (2k+1)\beta_1 - 2(l-(h-1)-(4k+2)) \ge \sum \gamma(\alpha_{1,j}, \beta_{1,j})\zeta(\alpha_{1,j}, \beta_{1,j}) > \gamma_1 \\ \ge \alpha_1 + (2k+1)\beta_1 - (l-(h-1))\,.$$

Therefore $l < (h-1) + 4(2k+1)$, a contradiction.
Now we have

$$x_1^{\gamma_1} \prod_{j=1}^{\mu} \prod_{(\alpha_{1,j},\beta_{1,j})} \left(\partial_1^{\alpha_{1,j}} \partial_2^{\beta_{1,j}} w_j\right)^{\zeta(\alpha_{1,j},\beta_{1,j})} \\ = x_1^{\gamma_1} \prod_{j=1}^{\mu} \prod_{(\alpha_{1,j},\beta_{1,j})} \left(x_1^{\gamma(\alpha_{1,j},\beta_{1,j})} \partial_1^{\alpha_{1,j}} \partial_2^{\beta_{1,j}} w_j\right)^{\zeta(\alpha_{1,j},\beta_{1,j})} \in C(\Omega)$$

since $x_1^{\gamma(\alpha_{1,j},\beta_{1,j})} \partial_1^{\alpha_{1,j}} \partial_2^{\beta_{1,j}} w_j \in \mathbb{H}^{4k+2}_{\text{loc}}(\Omega) \subset C(\Omega)$.
II) $\zeta(\alpha_{1,j_0}, \beta_{1,j_0}) = 1$ and $\zeta(\alpha_{1,j}, \beta_{1,j}) = 0$ for $j \ne j_0$. We have

$$x_1^{\gamma_1} \prod_{j=1}^{\mu} \prod_{(\alpha_{1,j},\beta_{1,j})} \left(\partial_1^{\alpha_{1,j}} \partial_2^{\beta_{1,j}} w_j\right)^{\zeta(\alpha_{1,j},\beta_{1,j})} = x_1^{\gamma_1} \partial_1^{\alpha_{1,j_0}} \partial_2^{\beta_{1,j_0}} w_{j_0} \in L^2_{\text{loc}}(\Omega)\,.$$

III) $\zeta(\alpha_{1,j_0}, \beta_{1,j_0}) = 1$ and there exists $j_1 \ne j_0$ such that $\zeta(\alpha_{1,j_1}, \beta_{1,j_1}) \ne 0$. Define

$$\bar\gamma(\alpha_{1,j_0}, \beta_{1,j_0}) = \max\{0, \alpha_{1,j_0} + (2k+1)\beta_{1,j_0} + (h-1) - l\}\,.$$

As in part I) we can prove $(\alpha_{1,j}, \beta_{1,j}, \gamma(\alpha_{1,j}, \beta_{1,j})) \in \Xi_{l-(h-1)-(4k+2)}$ for $j \ne j_0$ and $(\alpha_{1,j_0}, \beta_{1,j_0}, \bar\gamma(\alpha_{1,j_0}, \beta_{1,j_0})) \in \Xi_{l-(h-1)}$. Therefore $x_1^{\gamma(\alpha_{1,j},\beta_{1,j})} \partial_1^{\alpha_{1,j}} \partial_2^{\beta_{1,j}} w_j \in \mathbb{H}^{4k+2}_{\text{loc}}(\Omega) \subset C(\Omega)$ for $j \ne j_0$ and $x_1^{\bar\gamma(\alpha_{1,j_0},\beta_{1,j_0})} \partial_1^{\alpha_{1,j_0}} \partial_2^{\beta_{1,j_0}} w_{j_0} \in L^2_{\text{loc}}(\Omega)$. We also have $\sum_{j \ne j_0} \gamma(\alpha_{1,j}, \beta_{1,j})\zeta(\alpha_{1,j}, \beta_{1,j}) + \bar\gamma(\alpha_{1,j_0}, \beta_{1,j_0}) \le \gamma_1$ as in part I). Now the desired result follows from the decomposition of the general terms. □

(continuing the proof of Theorem 4.2) $u \in H^l_{\text{loc}}(\Omega)$, $l \ge 4k^2 + 6k + h + 1 \Longrightarrow u \in \mathbb{H}^l_{\text{loc}}(\Omega)$, $l \ge 4k^2 + 6k + h + 1 \Longrightarrow \varphi(x_1, x_2, u, \dots,_\gamma \partial_{\alpha,\beta} u) \in \mathbb{H}^{l-h+1}_{\text{loc}}(\Omega)$ (by Lemma 4.4). Therefore by Lemma 4.2 we deduce that $u \in \mathbb{H}^{l+1}_{\text{loc}}(\Omega)$. Repeat the argument again and again we finally arrive at $u \in \mathbb{H}^{l+t}_{\text{loc}}(\Omega)$ for every positive t, i.e.

$$u \in \cap_l \mathbb{H}^l_{\text{loc}}(\Omega) = C^\infty(\Omega)\,.$$

□

REMARK 4.1. Theorem 4.2 can be extended to the so-called Grushin type operators (non-linear version).

4. Gevrey analyticity of the solutions

PROPOSITION 4.1. *Assume that*

$$\varphi(x_1, x_2, u, \ldots, {}_\gamma\,\partial_{\alpha,\beta} u)_{(\alpha,\beta,\gamma)\in\Xi_{h-1}} \in C\{L_{n-a-2}; \Omega | L_{n-a-2}; E\}$$

for every $a \in [0, r_0]$*. Then there exist constants* $\widetilde{H}_0, \widetilde{H}_1, C_2, C_3$ *such that for every* $H_0 \geq \widetilde{H}_0$*,* $H_1 \geq \widetilde{H}_1$*,* $H_1 \geq C_2 H_0^{2r_0+3}$ *if*

$$|u, \Omega|_q \leq H_0 H_1^{(q-r_0-2)} L_{q-r_0-2}, \qquad 0 \leq q \leq N+1,\ r_0+2 \leq N$$

then

$$\max_{x\in\bar{\Omega}} \left| \partial_1^{\alpha_1} \partial_2^{\beta_1} \varphi(x_1, x_2, u, \ldots, {}_\gamma\,\partial_{\alpha,\beta} u) \right| \leq C_3 H_0 H_1^{N-r_0-1} L_{N-r_0-1}$$

for every $(\alpha_1, \beta_1) \in \Gamma_{N+1}$.

Proof. $\partial_1^{\alpha_1} \partial_2^{\beta_1} \varphi(x_1, x_2, u, \ldots, {}_\gamma\,\partial_{\alpha,\beta} u)$ as in Lemma 4.4 is a linear combination with positive coefficients of terms of the form

$$\frac{\partial^k \varphi}{\partial x_1^{k_1} \partial x_2^{k_2} \partial w_1^{k_3} \ldots \partial w_\mu^{k_{\mu+2}}} \prod_{j=1}^{\mu} \prod_{(\alpha_{1,j},\beta_{1,j})} \left(\partial_1^{\alpha_{1,j}} \partial_2^{\beta_{1,j}} w_j \right)^{\zeta(\alpha_{1,j},\beta_{1,j})}.$$

Substituting w_j by one of the terms ${}_\gamma\,\partial_{\alpha,\beta} u$ with $(\alpha, \beta, \gamma) \in \Xi_{h-1}$ we obtain

$$\partial_1^{\alpha_{1,j}} \partial_2^{\beta_{1,j}} ({}_\gamma\,\partial_{\alpha,\beta} u) = \sum_{m=0}^{\alpha_{1,j}} \binom{m}{\alpha_{1,j}} \gamma \cdots (\gamma - m + 1) \theta(\gamma - m + 1)\ {}_{(\gamma-m)}\partial_{\alpha+\alpha_{1,j}-m, \beta+\beta_{1,j}} u\,.$$

We can decompose ${}_{(\gamma-m)}\partial_{\alpha+\alpha_{1,j}-m,\beta+\beta_{1,j}} u$ into ${}_{(\gamma-m)}\partial_{\alpha_2,\beta_2}\left(\partial_1^{\alpha_3}\partial_2^{\beta_3} u\right)$ with $(\alpha_2, \beta_2, \gamma - m) \in \Xi_{h-1}$ and $(\alpha_3, \beta_3) \in \Gamma_{(\alpha_{1,j},\beta_{1,j})^*-m}$. Put $S = N + 1 - \alpha_1 - \beta_1$. Define $R = r_0 - S$. It is easy to see that R is a nonnegative integer and less than r_0. Since $\alpha_{1,j} \leq \alpha_1$ we deduce that $(\alpha_{1,j}, \beta_{1,j}) \in \Gamma_{(\alpha_{1,j}+\beta_{1,j}+S)}$. Using the monotonicity condition on L_n and the inductive assumption we have

$$\begin{aligned}
&\left| \binom{m}{\alpha_{1,j}} \gamma \cdots (\gamma - m + 1) \theta(\gamma - m + 1)\ {}_{(\gamma-m)}\partial_{\alpha+\alpha_{1,j}-m,\beta+\beta_{1,j}} u \right| \\
&\quad \leq \binom{m}{\alpha_{1,j}} \gamma \cdots (\gamma - m + 1) \theta(\gamma - m + 1) H_0 H_1^{\alpha_{1,j}+\beta_{1,j}-m-R-2} L_{\alpha_{1,j}+\beta_{1,j}-m-R-2} \\
&\quad \leq C_4 H_0 H_1^{\alpha_{1,j}+\beta_{1,j}-m-R-2} L_{\alpha_{1,j}+\beta_{1,j}-R-2}\,.
\end{aligned}$$

Therefore we deduce that

$$\begin{aligned}
&\left| \partial_1^{\alpha_{1,j}} \partial_2^{\beta_{1,j}} ({}_\gamma\,\partial_{\alpha,\beta} u) \right| \\
&\quad \leq \left| \sum_{m=0}^{\alpha_{1,j}} \binom{m}{\alpha_{1,j}} \gamma \cdots (\gamma - m + 1) \theta(\gamma - m + 1)\ {}_{(\gamma-m)}\partial_{\alpha+\alpha_{1,j}-m,\beta+\beta_{1,j}} u \right| \\
&\quad \leq C_5 H_0 H_1^{\alpha_{1,j}+\beta_{1,j}-R-2} L_{\alpha_{1,j}+\beta_{1,j}-R-2}\,.
\end{aligned}$$

and the general terms can be estimated by

$$\prod_{j=1}^{\mu} C_5 H_0 H_1^{\alpha_{1,j}+\beta_{1,j}-R-2} L_{\alpha_{1,j}+\beta_{1,j}-R-2} \,.$$

Since $\varphi \in C\{L_{n-a};\, \Omega | L_{n-a};\, E\}$, there exist constants C_6, C_7 such that

$$\left| \frac{\partial^k \varphi}{\partial x_1^{q_1} \partial x_2^{q_2} \partial w_1^{r_1} \ldots \partial w_\mu^{r_\mu}} \right| \le C_6 C_7^{q-R} L_{q-R-2} C_7^r L_{r-R-2} \quad (q = q_1 + q_2, r = r_1 + \cdots + r_\mu)\,.$$

Now we take $X(\xi, v) = X_1(v) \cdot X_2(\xi)$ where

$$X_1(v) = C_6 \sum_{i=0}^{p} \frac{C_8^i L_{i-R-2} v^i}{i!}, \quad X_2(\xi) = \sum_{i=0}^{p} \frac{C_7^{i-R} L_{i-R-2} \xi^i}{i!}\,.$$

and

$$v(\xi) = C_5 H_0 \sum_{i=1}^{p} \frac{H_1^{i-R-2} L_{i-R-2} \xi^i}{i!}\,.$$

By comparing terms of the form (3) with the corresponding terms in $\frac{d^p}{d\xi^p} X(\xi, v)$ it follows that

$$\left| \partial_1^{\alpha_1} \partial_2^{\beta_1} \varphi(x_1, x_2, u, \ldots,_\gamma \partial_{\alpha,\beta} u) \right| \Big|_{x=x_0} \le \left. \frac{d^p}{d\xi^p} X(\xi, v) \right|_{\xi=0}\,.$$

Next we introduce the following notation: $v(\xi) \ll h(\xi)$ if and only if $v^{(j)}(0) \le h^{(j)}(0)$ for $1 \le j \le p$. It is not difficult to check that there exists a constant C_9 (independent of p) such that

$$v^2(\xi) \ll (C_5 H_0)^2 C_9 \sum_{i=2}^{p} \frac{H_1^{i-R-3} L_{i-R-3} \xi^i}{(i-1)!}\,.$$

And by induction we have

$$v^j(\xi) \ll (C_5 H_0)^j C_9^{j-1} \sum_{i=j}^{p} H_1^{i-j-R-1} \frac{L_{i-j-R-1} \xi^i}{(i-j+1)!}\,.$$

Next, it is easy to verify that $X_1(0) = 0$, $\left.\frac{dX_1(\xi)}{d\xi}\right|_{\xi=0} \le C_5 C_6 C_8 H_0$, and $\left.\frac{d^j X_2(\xi)}{d\xi^j}\right|_{\xi=0} = C_7^{j-R} L_{j-R-2}$.

We now compute $\left.\frac{d^j X_1(\xi)}{d\xi^j}\right|_{\xi=0}$ when $2 \le j \le p$. If $2 \le j \le 2R+4$ we can always choose a constant C_{10} such that $\left.\frac{d^j X_1(\xi)}{d\xi^j}\right|_{\xi=0} \le C_{10} H_0 H_1^{j-R-2} L_{j-R-2}$, provided $H_1 \ge (C_5 C_8 C_9 H_0)^{2r_0+3}$.

If $j \geq 2R+5$, we have

$$\left.\frac{d^j X_1(\xi)}{d\xi^j}\right|_{\xi=0} \leq C_6 \left(\sum_{i=1}^{R+2} \frac{C_8^i (C_5 H_0)^i C_9^{i-1} H_1^{j-i-R-1} L_{j-i-R-1} j!}{i!(j-i+1)!} \right.$$

(4)
$$+ \sum_{i=R+3}^{j-R-2} \frac{C_8^i (C_5 H_0)^i C_9^{i-1} H_1^{j-i-R-1} L_{i-R-2} L_{j-i-R-1} j!}{i!(j-i+1)!}$$

$$\left. + \sum_{i=j-R-1}^{j} \frac{C_8^i (C_5 H_0)^i C_9^{i-1} L_{i-R-2} j!}{i!(j-i+1)!} \right).$$

The first sum in (4) is estimated by

(5)
$$C_{10} H_0 H_1^{j-R-2} L_{j-R-2}$$

provided $H_1 \geq (C_5 C_8 C_9 H_0)^{2r_0+3}$ as for $2 \leq j \leq 2R+4$.
By using the monotonicity condition on L_n the second sum in (4) is estimated by
(6)

$$C_6 \sum_{i=R+3}^{j-R-2} \frac{C_8^i (C_5 H_0)^i C_9^{i-1} H_1^{j-i-R-1} L_{i-R-2} L_{j-i-R-1} j!}{i!(j-i+1)!}$$

$$\leq \frac{C_{11} H_0 H_1^{j-R-2} j! L_{j-2R-3}}{(j-2R-3)!} \sum_{i=R+3}^{j-R-2} \frac{1}{i \cdots (i-R-1)} \frac{1}{(j-i+1)\cdots(j-i-R)}$$

$$\leq C_{12} H_0 H_1^{j-R-2} L_{j-R-2},$$

provided $H_1 \geq C_5 C_8 C_9 H_0$.
For the third sum we see that

$$C_6 \sum_{i=j-R-1}^{j} \frac{C_8^i (C_5 H_0)^i C_9^{i-1} L_{i-R-2} j!}{i!(j-i+1)!}$$

(7)
$$\leq C_5 C_6 C_8 H_0 H_1^{j-R-2} j! \sum_{i=j-R-1}^{j} \frac{L_{i-R-2}}{i!(j-i+1)!}$$

$$\leq C_{13} H_0 H_1^{j-R-2} L_{j-R-2},$$

if $H_1 \geq (C_5 C_8 C_9 H_0)^2$.
By (4), (5), (6), (7) and taking $H_1 \geq (C_5 C_7 C_8 C_9 H_0)^{2r_0+3} = C_2 H_0^{2r_0+3}$ we obtain

$$\left.\frac{d^p X(\xi, v)}{d\xi^p}\right|_{\xi=0} \leq C_{14} \sum_{j=0}^{p} \binom{j}{p} H_0 H_1^{j-R-2} L_{j-R-2} C_7^{p-j-R} L_{p-j-R-2}$$

$$\leq C_{15} H_0 H_1^{p-R-2} L_{p-R-2}.$$

Hence

$$\max_{x \in \bar\Omega} \left| \partial_1^{\alpha_1} \partial_2^{\beta_1} \varphi(x_1, x_2, u, \ldots, {}_{\gamma}\, \partial_{\alpha,\beta} u) \right| \leq C_{15} H_0 H_1^{p-R-2} L_{p-R-2}$$

$$\leq C_3 H_0 H_1^{N-r_0-1} L_{N-r_0-1}.$$

□

REMARK 4.2. The constants $\widetilde{H}_0, \widetilde{H}_1, C_2, C_3$ depend on Ω increasingly in the sense that with the same $\widetilde{H}_0, \widetilde{H}_1, C_2, C_3$ Proposition 4.1 remains valid if we substitute Ω by any $\Omega' \subset \Omega$.

COROLLARY 4.1. *Under the same hypotheses of Proposition 4.1 with $q \leq N+1$ replaced by $q \leq N$, then*

$$\max_{x\in\bar{\Omega}} \left|\partial_1^{\alpha_1}\partial_2^{\beta_1}\varphi(x_1, x_2, u, \ldots, {}_{\gamma}\,\partial_{\alpha,\beta}u)\right| \leq C_3\left(|u, \Omega|_{N+1} + H_0 H_1^{N-r_0-1} L_{N-r_0-1}\right)$$

for every $(\alpha_1, \beta_1) \in \Gamma_{N+1}$.

Proof. Indeed, as in the proof of Proposition 4.1 all typical terms, except $\frac{\partial\varphi}{\partial w_{j_0}}\partial_1^{\alpha_1}\partial_2^{\beta_1}w_{j_0}$ can be estimated by $|\cdot|_N$. Replacing w_{j_0} by one of ${}_{\gamma}\,\partial_{\alpha,\beta}u$, we have

$$\frac{\partial\varphi}{\partial w_{j_0}}\partial_1^{\alpha_1}\partial_2^{\beta_1}w_{j_0} = \frac{\partial\varphi}{\partial w_{j_0}}\ {}_{\gamma}\,\partial_{\alpha+\alpha_1,\beta+\beta_1}u$$
$$+ \frac{\partial\varphi}{\partial w_{j_0}}\sum_{m=1}^{\alpha_1}\binom{m}{\alpha_1}\gamma\cdots(\gamma-m+1)\theta(\gamma-m+1)\ {}_{(\gamma-m)}\partial_{\alpha+\alpha_1-m,\beta+\beta_1}u\,.$$

The first summand is estimated by $C_3|u, \Omega|_{N+1}$. The second sum is majorized as in Proposition 4.1. □

THEOREM 4.3. *Let u be a C^∞ solution of the equation (2) and $\varphi \in C\{L_{n-a-2}; \Omega|L_{n-a-2}; E\}$ for every $a \in [0, r_0]$. Then $u \in C\{L_{n-r_0-2}; \Omega\}$. In particular, if φ is a s-Gevrey function then so is u.*

Proof. It suffices to consider the case $(0,0) \in \Omega$. Let us define a distance $\rho((y_1, y_2), (x_1, x_2)) = \max\left(\frac{|x_1^{2k+1} - y_1^{2k+1}|}{2k+1}, |x_2 - y_2|\right)$. For two sets S_1, S_2 the distance between them is defined as $\rho(S_1, S_2) = \inf_{x\in S_1, y\in S_2}\rho(x, y)$. Let V^T be the cube with edges of size (in the ρ metric) $2T$, which are parallel to the coordinate axes and centered at $(0,0)$. Denote by V_δ^T the subcube which is homothetic with V^T and such that the distance between its boundary and the boundary of V^T is δ. We shall prove by induction that if T is small enough then there exist constants H_0, H_1 with $H_1 \geq C_2 H_0^{2r_0+3}$ such that

$$\left|u, V_\delta^T\right|_n \leq H_0 \ \text{ for } \ 0 \leq n \leq \max\{r_0+2, 6^k+1\} \tag{8}$$

and

$$\left|u, V_\delta^T\right|_n \leq H_0\left(\frac{H_1}{\delta}\right)^{n-r_0-2} L_{n-r_0-2} \ \text{ for } \ n \geq \max\{r_0+2, 6^k+1\} \tag{9}$$

and δ sufficiently small. Hence the Gevrey analyticity of u follows. (8) follows easily from the C^∞ smoothness assumption on u. Assume that (9) holds for $n = N$. We shall prove it for $n = N+1$. Put $\delta' = \delta(1 - 1/N)$. Fix $x \in V_\delta^T$ and then define $\sigma = \rho(x, \partial V^T)$ and $\tilde{\sigma} = \sigma/N$. Let $V_{\tilde{\sigma}}(x)$ denote the cube with center at (x) and edges of length $2\tilde{\sigma}$ which are parallel to the coordinate axes, and $S_{\tilde{\sigma}}(x)$ the boundary of $V_{\tilde{\sigma}}(x)$. Let E_1, E_3 (E_2, E_4) be edges of $S_{\tilde{\sigma}}(x)$ which are parallel to Ox_1 (Ox_2) respectively. We have to estimate $\max_{x\in V_\delta^T}\left|{}_{\gamma}\,\partial_{\alpha,\beta}\left(\partial_1^{\alpha_1}\partial_2^{\beta_1}u(x)\right)\right|$

for all $(\alpha,\beta,\gamma)\in\Xi_{h-1}$, $(\alpha_1,\beta_1)\in\Gamma_{N+1}$, and $\max_{x\in V_\delta^T}\left|\partial_1^h\left(\partial_1^{\alpha_1}\partial_2^{\beta_1}u(x)\right)\right|$ for all $(\alpha_1,\beta_1)\in\Gamma_{N+1}$, $\alpha_1\ge 1$, $\beta_1\ge 1$. But when $(\alpha_1,\beta_1)\in\Gamma_N$ we have already the desired estimate. Hence it suffices to obtain the estimate only for $(\alpha_1,\beta_1)\in\Gamma_{N+1}\setminus\Gamma_N$.

LEMMA 4.5. *Assume that $(\alpha,\beta,\gamma)\in\Xi_{h-1}$ and $(\alpha_1,\beta_1)\in\Gamma_{N+1}$. Then if $\alpha_1\ge 1$, $\beta_1\ge 1$ there exists a constant C_{16} such that*

$$\begin{aligned}&\max_{x\in V_\delta^T}\left|{}_\gamma\partial_{\alpha,\beta}\left(\partial_1^{\alpha_1}\partial_2^{\beta_1}u(x)\right)\right|\\&\qquad\le C_{16}\left(T^{\frac{1}{2k+1}}\left|u,V_{\delta'}^T\right|_{N+1}+H_0\left(\frac{H_1}{\delta}\right)^{N-r_0-1}L_{N-r_0-1}\left(T^{\frac{1}{2k+1}}+\frac{1}{H_1}\right)\right).\end{aligned}$$

Proof. First we observe that $M_{2k}^h u=\left(\partial_{x_1}+ix_1^{2k}\partial_{x_2}\right)^h u=\sum_{(\tilde\alpha,\tilde\beta,\tilde\gamma)\in\Xi_h^0}C^1_{\tilde\alpha\tilde\beta\tilde\gamma}\ {}_{\tilde\gamma}\partial_{\tilde\alpha,\tilde\beta}u$, where $C^1_{\tilde\alpha\tilde\beta\tilde\gamma}$ are some complex constants. Differentiating the equation (2) α_1 times in x_1 and β_1 times in x_2 then applying the representation formula (1) for $\Omega=V_{\tilde\sigma}(x)$ we have

$$\begin{aligned}\partial_1^{\alpha_1}\partial_2^{\beta_1}u(x)=&\int_{V_{\tilde\sigma}(x)}(-1)^hF_{2k}^h(y,x)\,(A(y)+B(y))\,dy_1dy_2\\&+\int_{S_{\tilde\sigma}(x)}\left(\sum_{j=0}^{h-1}(-1)^jM_{2k}^j\left(\partial_{1y}^{\alpha_1}\partial_{2y}^{\beta_1}u(y)\right)M_{2k}^{h-j-1}F_{2k}^h\right)(n_1+iy_1^{2k}n_2)\,ds\,,\end{aligned}$$

where

$$A(y)=-\sum_{\substack{(\tilde\alpha,\tilde\beta,\tilde\gamma)\in\Xi_h^0\\ \tilde\beta\ge1,\tilde\gamma\ge1}}\ \sum_{m=1}^{\min\{2kh,\alpha_1\}}C^1_{\tilde\alpha\tilde\beta\tilde\gamma m}\binom{m}{\alpha_1}\ {}_{(\tilde\gamma-m)}\partial_{\alpha_1+\tilde\alpha-m,\beta_1+\tilde\beta}u\,,\tag{10}$$

and

$$B(y)=-\partial_{1y}^{\alpha_1}\partial_{2y}^{\beta_1}\varphi(y_1,y_2,u,\ldots,{}_\gamma\,\partial^{(y)}_{\alpha,\beta}u)\,.$$

Therefore differentiating ${}_\gamma\partial_{\alpha,\beta}$ gives

$$\begin{aligned}{}_\gamma\partial_{\alpha,\beta}\left(\partial_1^{\alpha_1}\partial_2^{\beta_1}u(x)\right)=&\int_{V_{\tilde\sigma}(x)}(-1)^h\ {}_\gamma\partial_{\alpha,\beta}F_{2k}^h(y,x)\,(A(y)+B(y))\,dy_1dy_2\\&+\int_{S_{\tilde\sigma}(x)}\left(\sum_{j=0}^{h-1}(-1)^jM_{2k}^j\left(\partial_{1y}^{\alpha_1}\partial_{2y}^{\beta_1}u(y)\right)\ {}_\gamma\partial_{\alpha,\beta}M_{2k}^{h-j-1}F_{2k}^h\right)(n_1+iy_1^{2k}n_2)\,ds\,.\end{aligned}\tag{11}$$

It is easy to verify that

$$\left|{}_\gamma\partial_{\alpha,\beta}F_{2k}^h\right|\le\frac{C_{17}}{\left|\frac{x_1^{2k+1}-y_1^{2k+1}}{2k+1}\right|+|x_2-y_2|}\,.\tag{12}$$

Next we estimate $A(y)$. Consider two cases

I) $\alpha_1 \leq r_0\ (= (2k+1)h)$. The typical terms ${}_{(\tilde{\gamma}-m)}\partial^y_{\alpha_1+\tilde{\alpha}-m,\beta_1+\tilde{\beta}}u(y)$ in (10) can be rewritten as ${}_{(\gamma-m)}\partial^y_{\alpha_2,\beta_2}\left(\partial^{\alpha_3}_{1y}\partial^{\beta_3}_{2y}u(y)\right)$ with $(\alpha_2,\beta_2,\gamma-m)\in\Xi_{h-1}$ and $(\alpha_3,\beta_3)\in\Gamma_{N+1-m}$. Hence we have

$$
\begin{aligned}
|A(y)||_{V_{\tilde{\sigma}}(x)} &\leq C_{18}\sum_{m=1}^{\min\{2kh,\alpha_1\}} H_0\left(\frac{H_1}{\delta'}\right)^{N-r_0-m-1} L_{N-r_0-m-1}\\
&\leq C_{19}H_0\left(\frac{H_1}{\delta'}\right)^{N-r_0-2} L_{N-r_0-1}\,.
\end{aligned}
$$

II) If $\alpha_1 \geq r_0+1$ then we can decompose ${}_{(\tilde{\gamma}-m)}\partial^y_{\alpha_1+\tilde{\alpha}-m,\beta_1+\tilde{\beta}}u(y)$ into

$$
{}_{\tilde{\gamma}-m}\partial^y_{\tilde{\alpha}+\tilde{\beta},0}\left(\partial^{\alpha_1-\tilde{\beta}-m}_{1y}\partial^{\beta_1+\tilde{\beta}}_{2y}u(y)\right),
$$

where $(\alpha_1-\tilde{\beta}-m,\beta_1+\tilde{\beta})\in\Gamma_{N+1-m}$ and $\alpha_1-\tilde{\beta}-m\geq 1$, $\beta_1+\tilde{\beta}\geq 1$. Therefore by inductive assumptions we see that

$$
\begin{aligned}
|A(y)||_{V_{\tilde{\sigma}}(x)} &\leq C_{20}\sum_{m=1}^{\min\{2kh,\alpha_1\}}\binom{m}{\alpha_1} H_0\left(\frac{H_1}{\delta'}\right)^{N-r_0-m-1} L_{N-r_0-m-1}\\
&\leq C_{21}H_0\left(\frac{H_1}{\delta'}\right)^{N-r_0-2} L_{N-r_0-1}\,.
\end{aligned}
$$

Hence in both cases we have

$$
|A(y)||_{V_{\tilde{\sigma}}(x)} \leq C_{22}H_0\left(\frac{H_1}{\delta'}\right)^{N-r_0-2} L_{N-r_0-1}\,. \tag{13}
$$

On the other hand, from Proposition 4.1 and the inductive assumption we have

$$
|B(y)| \leq C_3\left(|u,V^T_{\delta'}|_{N+1} + H_0\left(\frac{H_1}{\delta'}\right)^{N-r_0-1} L_{N-r_0-1}\right). \tag{14}
$$

Combining (12), (13), (14) we obtain

$$
\begin{aligned}
&\left|\int_{V_{\tilde{\sigma}}(x)} {}_{\gamma}\partial_{\alpha,\beta}F^h_{2k}(y,x)\,(A(y)+B(y))\,dy_1dy_2\right|\\
&\quad\leq\quad C_{23}\left(|u,V^T_{\delta'}|_{N+1} + H_0\left(\frac{H_1}{\delta'}\right)^{N-r_0-1} L_{N-r_0-1}\right)\\
&\qquad\times\int_{V_{\tilde{\sigma}}(x)}\frac{1}{\left|\frac{x_1^{2k+1}-y_1^{2k+1}}{2k+1}\right| + |x_2-y_2|}dy_1dy_2\\
&\quad\leq\quad C_{24}T^{\frac{1}{2k+1}}\left(|u,V^T_{\delta'}|_{N+1} + H_0\left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}\right).
\end{aligned}
$$

To estimate the second integral in (11) we first note that

$$
{}_{\gamma}\partial_{\alpha,\beta}M^{h-j-1}_{2k}F^h_{2k} = \sum_{\lambda=0}^{\min\{j,\alpha\}}\ \sum_{\tilde{\lambda}\geq\frac{\alpha-\lambda}{2k+1}}^{\alpha-\lambda} C^3_{\alpha\beta\lambda\tilde{\lambda}}\frac{(x_1-y_1)^{j-\lambda}x_1^{(2k+1)\tilde{\lambda}+\lambda-\alpha+\gamma}}{\left(\frac{x_1^{2k+1}-y_1^{2k+1}}{2k+1}+i(x_2-y_2)\right)^{\beta+\tilde{\lambda}+1}}.
$$

Let us consider two cases:

I) $0 \le j \le h-2$. We then split into sub cases:

A) $|x_1| \le 2\tilde{\sigma}^{\frac{1}{2k+1}}$. We have the following estimate

$$\left| {}_{\gamma}\partial_{\alpha,\beta} M_{2k}^{h-j-1} F_{2k}^{h} \right|\Big|_{S_{\tilde{\sigma}}(x)} \le \frac{C_{25}}{\tilde{\sigma}^{\frac{h-j-1}{2k+1}+1}} .$$

We note that $M_{2k}^{j}\left(\partial_{1y}^{\alpha_1}\partial_{2y}^{\beta_1}\right)$ can be expressed as a linear combination of ${}_{\gamma'}\partial^{y}_{\alpha',\beta'}\left(\partial_{1y}^{\alpha_2}\partial_{2y}^{\beta_2}\right)$ with $(\alpha',\beta',\gamma') \in \Xi_{h-1}$, $(\alpha_2,\beta_2) \in \Gamma_{N+1-\frac{h-j-1}{2k+1}}$ if $\frac{h-j-1}{2k+1}$ is an integer and $(\alpha_2,\beta_2) \in \Gamma_{N-\left[\frac{h-j-1}{2k+1}\right]}$ if $\frac{h-j-1}{2k+1}$ is not an integer. Here $[\cdot]$ denotes the integer part of the argument. Therefore in both cases the integral can be estimated in the following way

$$\begin{aligned}
&\left| \int_{S_{\tilde{\sigma}}(x)} \left(\sum_{j=0}^{h-2} (-1)^j M_{2k}^{j}\left(\partial_{1y}^{\alpha_1}\partial_{2y}^{\beta_1} u(y)\right) {}_{\gamma}\partial_{\alpha,\beta} M_{2k}^{h-j-1} F_{2k}^{h} \right) (n_1 + i y_1^{2k} n_2)\, ds \right| \\
&\le C_{26} \sum_{j=0}^{h-2} \frac{N^{\frac{h-j-1}{2k+1}} H_0 \left(\frac{H_1}{\delta'}\right)^{N-r_0-\theta\left(\left\{\frac{h-j-1}{2k+1}\right\}\right)-\left[\frac{h-j-1}{2k+1}\right]-1}}{\delta^{\frac{h-j-1}{2k+1}}} L_{N-r_0-\theta\left(\left\{\frac{h-j-1}{2k+1}\right\}\right)-\left[\frac{h-j-1}{2k+1}\right]-1} \\
&\le \frac{C_{27} H_0}{H_1} \left(\frac{H_1}{\delta}\right)^{N-1-r_0} L_{N-r_0-1} ,
\end{aligned}$$

where $\{\cdot\}$ denotes the fractional part of the argument.

B) $|x_1| \ge 2\tilde{\sigma}^{\frac{1}{2k+1}}$. We then have on $S_{\tilde{\sigma}}(x)$

$$\left| y_1^{2k+1} - x_1^{2k+1} \right|^{\beta+\tilde{\lambda}-(h-j-1)} |x_1|^{2k(h-j-1)} \ge C_{28} |y_1 - x_1|^{j-\lambda} |x_1|^{(2k+1)\tilde{\lambda}-(\alpha-\lambda)+\gamma}$$

because $(\alpha,\beta,\gamma) \in \Xi_{h-1}$. For every $(\alpha',\beta',\gamma') \in \Xi_j$ we now write

$$\begin{aligned}
&\left| {}_{\gamma'}\partial^{y}_{\alpha',\beta'}\left(\partial_{1y}^{\alpha_1}\partial_{2y}^{\beta_1}\right) {}_{\gamma}\partial_{\alpha,\beta} M_{2k}^{h-j-1} F_{2k}^{h} \Big|_{S_{\tilde{\sigma}}(x)} \right| \\
&= \left| {}_{\gamma'+2k(h-j-1)}\partial^{y}_{\alpha',\beta'}\left(\partial_{1y}^{\alpha_1}\partial_{2y}^{\beta_1}\right) \frac{{}_{\gamma}\partial_{\alpha,\beta} M_{2k}^{h-j-1} F_{2k}^{h}}{y_1^{2k(h-j-1)}} \Bigg|_{S_{\tilde{\sigma}}(x)} \right| \\
&\le \frac{C_{29}}{\tilde{\sigma}^{1+(h-j-1)}} \times \left| {}_{\gamma'+2k(h-j-1)}\partial^{y}_{\alpha',\beta'}\left(\partial_{1y}^{\alpha_1}\partial_{2y}^{\beta_1}\right) \right| .
\end{aligned}$$

On the other hand ${}_{\gamma'+2k(h-j-1)}\partial^{y}_{\alpha',\beta'}\left(\partial_{1y}^{\alpha_1}\partial_{2y}^{\beta_1}\right)$ can be decomposed into

$${}_{\gamma'+2k(h-j-1)}\partial^{y}_{\alpha'+\alpha_2,\beta'+\beta_2}\left(\partial_{1y}^{\alpha_3}\partial_{2y}^{\beta_3}\right)$$

where $(\alpha'+\alpha_2, \beta'+\beta_2, \gamma'+2k(h-j-1)) \in \Xi_{h-1}$ and $(\alpha_3,\beta_3) \in \Gamma_{N+1-(h-j-1)}$. It follows that

$$\left|\int_{S_{\tilde\sigma}(x)}\left(\sum_{j=0}^{h-2}(-1)^j M_{2k}^j\left(\partial_{1y}^{\alpha_1}\partial_{2y}^{\beta_1}u(y)\right)\ \gamma\partial_{\alpha,\beta}M_{2k}^{h-j-1}F_{2k}^h\right)(n_1+iy_1^{2k}n_2)\,ds\right|$$
$$\leq\ C_{30}\sum_{j=0}^{h-2}\frac{\left|u,V_{\delta'}^T\right|_{N+1-(h-j-1)}}{\tilde\sigma^{1+(h-j-1)}}\left|\int_{S_{\tilde\sigma}(x)}(n_1+iy_1^{2k}n_2)\,ds\right|$$
$$\leq\ \frac{C_{31}H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{N-1-r_0}L_{N-r_0-1}\,.$$

II) $j=h-1$.

A) We first estimate the integrals along the edges E_2 and E_4. Integrating by parts gives

$$\left|\int_{E_2\cup E_4}M_{2k}^{h-1}\left(\partial_{1y}^{\alpha_1}\partial_{2y}^{\beta_1}u(y)\right)\ \gamma\partial_{\alpha,\beta}F_{2k}^h\,dy_2\right|$$
$$\leq\ C_{32}\left(\sum_{(\alpha',\beta',\gamma')\in\Xi_{h-1}}\left|\int_{E_2\cup E_4}\ \gamma'\partial_{\alpha',\beta'}^y\left(\partial_{1y}^{\alpha_1}\partial_{2y}^{\beta_1-1}\right)\partial_{2y}\left(\ \gamma\partial_{\alpha,\beta}F_{2k}^h\right)dy_2\right|\right.$$
$$\left.+\sum_{(\alpha',\beta',\gamma')\in\Xi_{h-1}}\left|\ \gamma'\partial_{\alpha',\beta'}^y\left(\partial_{1y}^{\alpha_1}\partial_{2y}^{\beta_1-1}\right)\ \gamma\partial_{\alpha,\beta}F_{2k}^h\Big|_{\partial E_2\cup\partial E_4}\right|\right)$$
$$\leq\ C_{33}\frac{\left|u,V_{\delta'}^T\right|_N}{(\delta-\delta')}$$
$$\leq\ \frac{C_{34}H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{N-r_0-1}L_{N-r_0-1}\,.$$

B) We now estimate the integrals along the edges E_1 and E_3. Integrating by parts gives

$$\left|\int_{E_1\cup E_3}M_{2k}^{h-1}\left(\partial_{1y}^{\alpha_1}\partial_{2y}^{\beta_1}u(y)\right)\ \gamma\partial_{\alpha,\beta}F_{2k}^h y_1^{2k}\,dy_1\right|$$
$$\leq\ C_{35}\left(\sum_{(\alpha',\beta',\gamma')\in\Xi_{h-1}}\left|\int_{E_1\cup E_3}\ \gamma'\partial_{\alpha',\beta'}^y\left(\partial_{1y}^{\alpha_1-1}\partial_{2y}^{\beta_1}\right)\ \gamma\partial_{\alpha,\beta}F_{2k}^h y_1^{2k-1}dy_1\right|\right.$$
$$+\sum_{(\alpha',\beta',\gamma')\in\Xi_{h-1}}\left|\int_{E_1\cup E_3}\ \gamma'\partial_{\alpha',\beta'}^y\left(\partial_{1y}^{\alpha_1-1}\partial_{2y}^{\beta_1}\right)\frac{\partial}{\partial y_1}\left(\ \gamma\partial_{\alpha,\beta}F_{2k}^h\right)y_1^{2k}\,dy_1\right|$$
$$\left.+\sum_{(\alpha',\beta',\gamma')\in\Xi_{h-1}}\left|\ \gamma'\partial_{\alpha',\beta'}\left(\partial_{1y}^{\alpha_1-1}\partial_{2y}^{\beta_1}\right)\ \gamma\partial_{\alpha,\beta}F_{2k}^h y_1^{2k}\Big|_{\partial E_1\cup\partial E_3}\right|\right)$$
$$\leq\ C_{36}\frac{\left|u,V_{\delta'}^T\right|_N}{(\delta-\delta')}$$
$$\leq\ \frac{C_{37}H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{N-r_0-1}L_{N-r_0-1}\,.$$

□

LEMMA 4.6. *Assume that* $(\alpha,\beta,\gamma)\in\Xi_{h-1}$. *Then there exists a constant* C_{38} *such that*

$$\begin{aligned}\max_{x\in V_\delta^T}\left|{}_\gamma\partial_{\alpha,\beta}\left(\partial_2^{N+1}u(x)\right)\right| &\le C_{38}\Bigg(T^{\frac{1}{2k+1}}\left|u,V^T_{\delta(1-6^k/N)}\right|_{N+1}\\ &\quad+H_0\left(\frac{H_1}{\delta}\right)^{N-r_0-1}L_{N-r_0-1}\left(T^{\frac{1}{2k+1}}+\frac{1}{H_1}\right)\Bigg).\end{aligned}$$

Proof. I) $|x_1|\le 2\tilde\sigma^{\frac{1}{2k+1}}$. Instead of $V_{\tilde\sigma}(x)$ we take the cube $V_{6^k\tilde\sigma}(x)$. As in Lemma 4.5 the following formula holds

$$\begin{aligned}{}_\gamma\partial_{\alpha,\beta}\left(\partial_2^{N+1}u(x)\right) &= \int_{V_{6^k\tilde\sigma}(x)}(-1)^h\ {}_\gamma\partial_{\alpha,\beta}F^h_{2k}(y,x)B(y)\,dy_1dy_2\\ &\quad+\int_{S_{6^k\tilde\sigma}(x)}\left(\sum_{j=0}^{h-1}(-1)^jM^j_{2k}\left(\partial_{2y}^{N+1}u(y)\right)\ {}_\gamma\partial_{\alpha,\beta}M^{h-j-1}_{2k}F^h_{2k}\right)(n_1+iy_1^{2k}n_2)\,ds\,,\end{aligned}\tag{15}$$

where

$$B(y)=-\frac{\partial^{N+1}}{\partial y_2^{N+1}}\varphi(y_1,y_2,u,\dots,{}_\gamma\partial^{(y)}_{\alpha,\beta}u)\,.$$

Therefore the first integral in (15) is estimated by

$$C_{39}T^{\frac{1}{2k+1}}\left(\left|u,V^T_{\delta(1-6^k/N)}\right|_{N+1}+H_0\left(\frac{H_1}{\delta}\right)^{N-r_0-1}L_{N-r_0-1}\right).$$

We now estimate the second integral in (15).

A) First consider the integrals along $\widetilde E_2$ and $\widetilde E_4$, where $\widetilde E_2,\widetilde E_4$ denote the edges of $S_{6^k\tilde\sigma}$, which are parallel to Ox_2. We note that $|y_1|\ge|x_1|$ on $\widetilde E_2,\widetilde E_4$.

1) $0\le j\le h-2$. As in Lemma 4.5, for every $(\alpha',\beta',\gamma')\in\Xi_j$ the following estimate holds

$$\begin{aligned}&\left|{}_{\gamma'}\partial^y_{\alpha',\beta'}\left(\partial_{2y}^{N+1}\right)\ {}_\gamma\partial_{\alpha,\beta}M^{h-j-1}_{2k}F^h_{2k}\Big|_{\widetilde E_2\cup\widetilde E_4}\right|\\ &\quad\le\frac{C_{40}}{\tilde\sigma^{1+(h-j-1)}}\left|{}_{\gamma'+2k(h-j-1)}\partial^y_{\alpha',\beta'+h-j-1}\left(\partial_{2y}^{N+1-(h-j-1)}\right)\right|_{\widetilde E_2\cup\widetilde E_4},\end{aligned}$$

where ${}_{\gamma'+2k(h-j-1)}\partial^y_{\alpha',\beta'+h-j-1}\in\Xi_{h-1}$ and $\partial_{2y}^{N+1-(h-j-1)}\in\Gamma_{N+1-(h-j-1)}$. Therefore

$$\begin{aligned}&\left|\int_{\widetilde E_2\cup\widetilde E_4}\left(\sum_{j=0}^{h-2}(-1)^jM^j_{2k}\left(\partial_{1y}^{\alpha_1}\partial_{2y}^{\beta_1}u(y)\right)\ {}_\gamma\partial_{\alpha,\beta}M^{h-j-1}_{2k}F^h_{2k}\right)(n_1+iy_1^{2k}n_2)\,ds\right|\\ &\quad\le\frac{C_{41}H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{N-1-r_0}L_{N-r_0-1}.\end{aligned}$$

2) $j=h-1$. Integrating by parts gives:

$$\begin{aligned}\left|\int_{\widetilde E_2\cup\widetilde E_4}M^{h-1}_{2k}\left(\partial_{2y}^{N+1}u(y)\right)\ {}_\gamma\partial_{\alpha,\beta}F^h_{2k}\,dy_2\right| &\le C_{42}\frac{N\left|u,V^T_{\delta(1-6^k/N)}\right|_N}{\delta}\\ &\le\frac{C_{43}H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{N-r_0-1}L_{N-r_0-1}.\end{aligned}$$

B) Consider now integrals along the edges $\widetilde{E}_1$ and $\widetilde{E}_3$, where $\widetilde{E}_1$, $\widetilde{E}_3$ denote the edges of $S_{6^k\tilde{\sigma}}$, which are parallel to Ox_1.

1) $0 \leq j \leq h-2$. We have

$$\left| {}_{\gamma}\partial_{\alpha,\beta} M_{2k}^{h-j-1} F_{2k}^{h} \right| \Big|_{\widetilde{E}_1 \cup \widetilde{E}_3} \leq \frac{C_{44}}{\tilde{\sigma}^{\frac{h-j-1}{2k+1}+1}} \quad \text{and} \quad |y_1|\big|_{\widetilde{E}_1 \cup \widetilde{E}_3} \leq C_{45}\tilde{\sigma}^{\frac{1}{2k+1}}$$

Furthermore for $h-2-(2k+1)s-2k \leq j \leq h-2-(2k+1)s$ and every $(\alpha', \beta', \gamma') \in \Xi_j$ we have

$$\begin{aligned} y_1^{2k}\, {}_{\gamma'}\partial^{y}_{\alpha',\beta'} \left(\partial_{2y}^{N+1} u(y)\right) &= y_1^{h-2-j-(2k+1)s} \\ &\quad \times {}_{\gamma'+j+2k-(h-2-(2k+1)s)}\partial^{y}_{\alpha',\beta'+s+1} \left(\partial_{2y}^{N-s} u(y)\right), \end{aligned}$$

where $(\alpha', \beta'+s+1, \gamma'+j+2k-h+(2k+1)s+2) \in \Xi_{h-1}$ and $(0, N-s) \in \Gamma_{N-s}$. Hence we get

$$\begin{aligned} &\left| \int_{\widetilde{E}_1 \cup \widetilde{E}_3} \left(\sum_{j=0}^{h-2} (-1)^j M_{2k}^{j} \left(\partial_{2y}^{N+1} u(y)\right) {}_{\gamma}\partial_{\alpha,\beta} M_{2k}^{h-j-1} F_{2k}^{h} \right) y_1^{2k}\, dy_1 \right| \\ \leq\ & C_{46} \sum_{s=0}^{\left[\frac{h-2}{2k+1}\right]} \frac{\left|u, V^T_{\delta(1-6^k/N)}\right|_{N-s}}{(\delta-\delta')^{1+\frac{h-j-1}{2k+1}}} \int_{\widetilde{E}_1 \cup \widetilde{E}_3} |y_1|^{h-(2k+1)s-j-2}\, dy_1 \\ \leq\ & \frac{C_{47} H_0}{H_1} \left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}. \end{aligned}$$

2) $j = h-1$. We note that

$$\begin{aligned} y_1^{2k} M_{2k}^{h-1} \left(\partial_{2y}^{N+1} u\right) &= y_1^{2k} \partial_{2y} M_{2k}^{h-1} \left(\partial_{2y}^{N} u\right) = (M_{2k} - \partial_{1y}) M_{2k}^{h-1} \left(\partial_{2y}^{N} u\right) \\ &= M_{2k}^{h} \left(\partial_{2y}^{N} u\right) - \partial_{1y} M_{2k}^{h-1} \left(\partial_{2y}^{N} u\right) = \partial_{2y}^{N}\varphi - \partial_{1y} M_{2k}^{h-1} \left(\partial_{2y}^{N} u\right). \end{aligned}$$

It follows that

$$\begin{aligned} &\left| \int_{\widetilde{E}_1 \cup \widetilde{E}_3} M_{2k}^{h-1} \left(\partial_{2y}^{N+1} u(y)\right) {}_{\gamma}\partial_{\alpha,\beta} F_{2k}^{h} y_1^{2k}\, dy_1 \right| \\ &\leq \left| \int_{\widetilde{E}_1 \cup \widetilde{E}_3} \partial_{2y}^{N}\varphi\, {}_{\gamma}\partial_{\alpha,\beta} F_{2k}^{h}\, dy_1 \right| \\ &\quad + \left| \int_{\widetilde{E}_1 \cup \widetilde{E}_3} \partial_{1y} M_{2k}^{h-1} \left(\partial_{2y}^{N} u(y)\right) {}_{\gamma}\partial_{\alpha,\beta} F_{2k}^{h}\, dy_1 \right|. \end{aligned} \tag{16}$$

By Proposition 4.1 the first integral in the right hand side of (16) can be estimated by

$$\frac{C_{48} \left|u, V^T_{\delta(1-6^k/N)}\right|_{N}}{\delta-\delta'} \leq \frac{C_{49} H_0}{H_1} \left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}.$$

To estimate the second integral in the right hand side of (16) we now integrate by parts:

$$\begin{aligned}
&\left| \int_{\widetilde{E}_1 \cup \widetilde{E}_3} \partial_{1y} M_{2k}^{h-1} \left(\partial_{2y}^N u(y) \right) \ {}_\gamma \partial_{\alpha,\beta} F_{2k}^h \, dy_1 \right| \\
&\leq \quad C_{50} \left| u, V^T_{\delta(1-6^k/N)} \right|_N \left(\frac{N}{\sigma} + \frac{N^2}{\sigma^2} \left| \int_{\widetilde{E}_1 \cup \widetilde{E}_3} y_1^{2k} \, dy_1 \right| + \frac{N^{1+\frac{1}{2k+1}}}{\sigma^{1+\frac{1}{2k+1}}} \left| \int_{\widetilde{E}_1 \cup \widetilde{E}_3} dy_1 \right| \right) \\
&\leq \quad \frac{C_{51} H_0}{H_1} \left(\frac{H_1}{\delta} \right)^{N-r_0-1} L_{N-r_0-1}.
\end{aligned}$$

II) $|x_1| \geq 2\tilde{\sigma}^{\frac{1}{2k+1}}$. We now consider the representation formula for $\partial_2^{N+1} u(x)$ in $V_{\tilde{\sigma}}(x)$. As above the volume integral can be estimated by

$$C_{52} T^{\frac{1}{2k+1}} \left(\left| u, V^T_{\delta'} \right|_{N+1} + \left(\frac{H_1}{\delta} \right)^{N-r_0-1} L_{N-r_0-1} \right).$$

For the boundary integral we again split into 2 cases

A) $j = h - 2$. As in Lemma 4.5 we obtain the following estimate

$$\begin{aligned}
&\left| \int_{S_{\tilde{\sigma}}(x)} \left(\sum_{j=0}^{h-2} (-1)^j M_{2k}^j \left(\partial_{1y}^{\alpha_1} \partial_{2y}^{\beta_1} u(y) \right) \ {}_\gamma \partial_{\alpha,\beta} M_{2k}^{h-j-1} F_{2k}^h \right) (n_1 + i y_1^{2k} n_2) \, ds \right| \\
&\qquad \leq \frac{C_{53} H_0}{H_1} \left(\frac{H_1}{\delta} \right)^{N-r_0-1} L_{N-r_0-1}.
\end{aligned}$$

B) $j = h - 1$.

1) Along E_2 and E_4 we can estimate exactly as in Lemma 4.5.

2) Consider now the boundary integral along E_1 and E_3. As in this Lemma I) B) 2) we have

$$\begin{aligned}
&\left| \int_{E_1 \cup E_3} M_{2k}^{h-1} \left(\partial_{2y}^{N+1} u(y) \right) \ {}_\gamma \partial_{\alpha,\beta} F_{2k}^h y_1^{2k} \, dy_1 \right| \\
&\quad \leq \left| \int_{E_1 \cup E_3} \partial_{2y}^N \varphi \ {}_\gamma \partial_{\alpha,\beta} F_{2k}^h \, dy_1 \right| \\
&\quad\quad + \left| \int_{E_1 \cup E_3} \partial_{1y} M_{2k}^{h-1} \left(\partial_{2y}^N u(y) \right) \ {}_\gamma \partial_{\alpha,\beta} F_{2k}^h \, dy_1 \right|.
\end{aligned} \tag{17}$$

By Proposition 4.1 the first integral in the right hand side of (17) can be estimated by

$$\frac{C_{54} \left| u, V^T_{\delta'} \right|_N}{\delta - \delta'} \leq \frac{C_{55} H_0}{H_1} \left(\frac{H_1}{\delta} \right)^{N-r_0-1} L_{N-r_0-1}.$$

To estimate the second integral in the right hand side of (17) we now integrate by parts

$$\begin{aligned}
&\left|\int_{E_1\cup E_3} \partial_{1y} M_{2k}^{h-1}\left(\partial_{2y}^{N} u(y)\right)\ {}_{\gamma}\partial_{\alpha,\beta} F_{2k}^{h}\, dy_1\right| \\
\leq\ & \left| M_{2k}^{h-1}\left(\partial_{2y}^{N} u(y)\right)\ {}_{\gamma}\partial_{\alpha,\beta} F_{2k}^{h}\Big|_{\partial E_1\cup\partial E_3}\right| \\
&+\left|\int_{E_1\cup E_3} \partial_{1y} M_{2k}^{h-1}\left(\partial_{2y}^{N} u(y)\right)\frac{\partial}{\partial y_1}\left({}_{\gamma}\partial_{\alpha,\beta} F_{2k}^{h}\right) dy_1\right| \\
\leq\ & C_{56}\left|u, V_{\delta'}^{T}\right|_{N}\left(\frac{N}{\sigma}+\frac{N^2}{\sigma^2}\left|\int_{E_1\cup E_3} y_1^{2k}\, dy_1\right|\right) \\
\leq\ & \frac{C_{57}H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}.
\end{aligned}$$

□

LEMMA 4.7. *Assume that* $(\alpha,\beta,\gamma)\in\Xi_{h-1}$. *Then there exists a constant* C_{58} *such that*

$$\begin{aligned}
\max_{x\in V_{\delta}^{T}}\left|{}_{\gamma}\partial_{\alpha,\beta}\left(\partial_1^{N-r_0+1}u(x)\right)\right| \leq\ & C_{58}\Bigg(T^{\frac{1}{2k+1}}\left|u, V_{\delta'}^{T}\right|_{N+1} \\
&+H_0\left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}\left(T^{\frac{1}{2k+1}}+\frac{1}{H_1}\right)\Bigg).
\end{aligned}$$

Proof. We have the following representation formula for ${}_{\gamma}\partial_{\alpha,\beta}\left(\partial_1^{N-r_0+1}u(x)\right)$ in $V_{\tilde{\sigma}}(x)$

$$\begin{aligned}
{}_{\gamma}\partial_{\alpha,\beta}\left(\partial_1^{N-r_0+1}u(x)\right) =& \int_{V_{\tilde{\sigma}}(x)}(-1)^h\ {}_{\gamma}\partial_{\alpha,\beta}F_{2k}^{h}(y,x)\,(A(y)+B(y))\,dy_1dy_2 \\
&+\int_{S_{\tilde{\sigma}}(x)}\left(\sum_{j=0}^{h-1}(-1)^j M_{2k}^{j}\left(\partial_{1y}^{N-r_0+1}u(y)\right)\ {}_{\gamma}\partial_{\alpha,\beta}M_{2k}^{h-j-1}F_{2k}^{h}\right)(n_1+iy_1^{2k}n_2)\,ds\,,
\end{aligned} \tag{18}$$

where

$$A(y) = -\sum_{\substack{(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})\in\Xi_h^0\\ \tilde{\beta}\geq 1,\tilde{\gamma}\geq 1}}\ \sum_{m=1}^{\min\{2kh,N-r_0+1\}} C^{1}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}m}\binom{m}{N-r_0+1}_{(\tilde{\gamma}-m)}\partial^{y}_{N-r_0-m+1+\tilde{\alpha},\beta_1+\tilde{\beta}}u(y)\,,$$

and

$$B(y) = -\partial_{1y}^{N-r_0+1}\varphi(y_1,y_2,u,\ldots,{}_{\gamma}\partial^{(y)}_{\alpha,\beta}u)\,.$$

Therefore, as in Lemma 4.5, the first integral in (18) is estimated by

$$C_{59}T^{\frac{1}{2k+1}}\left(\left|u, V_{\delta'}^{T}\right|_{N+1}+H_0\left(\frac{H_1}{\delta}\right)^{N-r_0-1}L_{N-r_0-1}\right).$$

We now estimate the second integral in (18).

I) $0 \leq j \leq h-2$. As in Lemma 4.5 I) A) 2) we obtain

$$\left|\int_{S_{\tilde{\sigma}}(x)} \left(\sum_{j=0}^{h-2} (-1)^j M_{2k}^j \left(\partial_{1y}^{N-r_0+1} u(y)\right) \;{}_\gamma\partial_{\alpha,\beta} M_{2k}^{h-j-1} F_{2k}^h\right) \left(n_1 + i y_1^{2k} n_2\right) ds\right| \leq \frac{C_{60} H_0}{H_1} \left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}.$$

II) $j = h-1$.

A) We first estimate the integrals along E_2 and E_4. Note that

$$\begin{aligned}
&\left|M_{2k}^{h-1}\left(\partial_{1y}^{N-r_0+1} u(y)\right)\right| \\
&= \left|M_{2k}^h \left(\partial_{1y}^{N-r_0} u(y)\right) - i M_{2k}^{h-1} y_1^{2k} \partial_{2y} \left(\partial_{1y}^{N-r_0} u(y)\right)\right| \\
&\leq \left|\partial_{1y}^{N-r_0} \varphi\right| + \left|\sum_{\substack{(\tilde{\alpha},\tilde{\beta},\tilde{\gamma}) \in \Xi_h^0 \\ \tilde{\beta} \geq 1, \tilde{\gamma} \geq 1}} \sum_{m=1}^{\min\{2kh, N-r_0\}} C^1_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}m} \binom{m}{N-r_0} \;{}_{(\tilde{\gamma}-m)}\partial^y_{\alpha_1+\tilde{\alpha}-m, \beta_1+\tilde{\beta}} u\right| \\
&\quad + \left|\partial_{2y} M_{2k}^{h-1} y_1^{2k} \left(\partial_{1y}^{N-r_0} u(y)\right)\right| \\
&\leq \left|\partial_{2y} M_{2k}^{h-1} y_1^{2k} \left(\partial_{1y}^{N-r_0} u(y)\right)\right| + C_{61} \frac{H_0}{H_1} \left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left|\int_{E_2 \cup E_4} M_{2k}^{h-1} \left(\partial_{1y}^{N-r_0+1} u(y)\right) \;{}_\gamma\partial_{\alpha,\beta} F_{2k}^h \, dy_2\right| \\
&\leq C_{62} \frac{H_0}{H_1} \left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1} \\
&\quad + \left|\int_{E_2 \cup E_4} M_{2k}^{h-1} y_1^{2k} \left(\partial_{1y}^{N-r_0} u(y)\right) \partial_{2y} \left({}_\gamma\partial_{\alpha,\beta} F_{2k}^h\right) dy_2\right| \\
&\quad + \left|M_{2k}^{h-1} y_1^{2k} \left(\partial_{1y}^{N-r_0} u(y)\right) \left({}_\gamma\partial_{\alpha,\beta} F_{2k}^h\right)\Big|_{\partial E_2 \cup \partial E_4}\right| \\
&\leq C_{63} \frac{H_0}{H_1} \left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}.
\end{aligned}$$

B) The integrals along E_1 and E_3 now can be estimated in the same manner as in Lemma 4.5. □

LEMMA 4.8. *Assume that $(\alpha_1, \beta_1) \in \Gamma_{N+1} \setminus \Gamma_N$, $\alpha_1 \geq 1$, $\beta_1 \geq 1$. Then there exists a constant C_{64} such that*

$$\max_{x \in V_\delta^T} \left|\partial_1^h \left(\partial_1^{\alpha_1} \partial_2^{\beta_1} u(x)\right)\right| \leq C_{64} \Bigg(T^{\frac{1}{2k+1}} \left|u, V^T_{\delta(1-6^k/N)}\right|_{N+1} + H_0 \left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1} \left(T^{\frac{1}{2k+1}} + \frac{1}{H_1}\right)\Bigg).$$

Proof. This lemma can be proved as in [10]. We omit the details. □

(Continuing the proof of Theorem 4.3). Put $|u, V_\delta^T|_{N+1} = g(\delta)$. Combining Lemmas 4.5-4.8 gives

$$g(\delta) \le C_{65}\left(T^{\frac{1}{2k+1}} g\left(\delta(1-6^k/N)\right) + H_0\left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}\left(T^{\frac{1}{2k+1}} + \frac{1}{H_1}\right)\right).$$

Choosing $T \le \left(1/8^{12^k} C_{65}\right)^{2k+1}$ then by Lemma 4.3 we deduce that

$$g(\delta) \le C_{66} H_0 \left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}\left(T^{\frac{1}{2k+1}} + \frac{1}{H_1}\right).$$

If T is chosen to be small enough such that $T \le (1/2C_{66})^{2k+1}$ and choosing $H_1 \ge 2C_{66}$ (in addition to $H_1 \ge C_2 H_0^{2r_0+3}$) we arrive at

$$g(\delta) \le H_0\left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}.$$

That means

$$\left|u, V_\delta^T\right|_{N+1} \le H_0\left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}.$$

The proof of Theorem 4.3 is therefore completed. □

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