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WAVELETS ON THE INTERVAL AND RELATED TOPICS

Abstract. We use an abstract framework to obtain a multilevel decomposition of a variety of functional spaces, using biorthogonal wavelet bases satisfying homogeneous boundary conditions on the unit interval.

1. Introduction

Wavelet bases and, more generally, multilevel decompositions of functional spaces are often presented as a powerful tool in many different areas of theoretical and numerical analysis. The construction of biorthogonal wavelet bases and decompositions of $L^p(\mathbb{R})$, $1 < p < +\infty$, is now well understood (see [17, 11, 15] and also [5]).

We aim at considering a similar problem on the interval $(0, 1)$. Many authors have studied this subject, but, to our knowledge, none of them gives a unitary and general approach. For example, the papers [3, 12, 13, 10, 22] deal only with orthogonal multiresolution analyses; in [2] the authors study biorthogonal wavelet systems, but they do not obtain polynomial exactness for the dual spaces and fundamental inequalities such as the generalized Jackson and Bernstein ones. Dahmen, Kunoth and Urban ([16]) get all such properties, but they address specifically the construction arising from (biorthogonal) B-splines on the real line. While their construction of the scaling functions is similar to ours, the wavelets are defined in a completely different way using the concept of stable completions (see [8]). Finally, we recall the work of Chiavassa and Liandrat ([9]), which is concerned with the characterization of spaces of functions satisfying homogeneous boundary value conditions (such as the Sobolev spaces $H_0^s(0, 1) = B_{22,0}^s(0, 1)$, see (2) for a definition) with orthogonal wavelet systems.

In this paper, we shall build multilevel decompositions of functional spaces (such as scales of Besov and Sobolev spaces) of functions defined on the unit interval and, possibly, subject to homogeneous boundary value conditions. Moreover, we do not work necessarily in an Hilbertian setting (as all the papers cited above do), but more generally we deal with subspaces of $L^p(0, 1)$, $1 < p < +\infty$.

Working on subintervals I of the real line, we cannot find, in general, a unique function whose dilates and translates form a multilevel decomposition of $L^p(I)$. Indeed, the main difference is the lack of translation invariance. We shall divide our construction in two steps. Firstly we shall deal with the half-line $(0, +\infty)$; secondly, in a suitable way, we shall glue together two such constructions (on $(0, +\infty)$ and $(-\infty, 1)$) to get the final result on the unit interval $(0, 1)$.

The outline of the paper is as follows. In Section 2, we recall some basic facts about abstract multilevel decompositions and characterization of subspaces. In Section 3 and 5, we construct scaling function and wavelet spaces for the half-line. In Section 4, we prove all the needed properties, such as the Jackson- and Bernstein-type inequalities, to get the characterization of Besov spaces. To get a similar result for spaces of functions satisfying homogeneous boundary value conditions (result contained in Section 7), in Section 6 we study the boundary values of the scaling functions and wavelets. In Section 8, we collect all our results to obtain a multilevel decomposition of spaces of functions defined on the unit interval $(0, 1)$. Finally, in Section 9, we describe in detail how our construction applies to the B-spline case.

We set here some basic notation that will be useful in the sequel.

Throughout the paper, C will denote a strictly positive constant, which may take different values in different places.

Given two functions $N_i : V \to \mathbb{R}_+$ $(i = 1, 2)$ defined on a set V, we shall use the notation $N_1(v) \lesssim N_2(v)$, if there exists a constant $C > 0$ such that $N_1(v) \le C N_2(v)$, for all $v \in V$. We say that $N_1(v) \asymp N_2(v)$, if $N_1(v) \lesssim N_2(v)$ and $N_2(v) \lesssim N_1(v)$.

Moreover, for any $x \in \mathbb{R}$ we will indicate by $\lceil x \rceil$ (or $\lfloor x \rfloor$) the smallest (largest) integer greater (less) than or equal to x.

Let Ω denote either $\mathbb{R}$ or the half-line $(0, +\infty)$ or the unit interval $(0, 1)$. In this paper, we will use the Besov spaces $B^s_{pq}(\Omega)$ as defined in [23] (see also [21]) and the Sobolev spaces $W^{s,p}(\Omega)$ $(= B^s_{pp}(\Omega)$, unless $p \neq 2$ and $s \in \mathbb{N})$, and $H^s(\Omega) = B^s_{22}(\Omega)$. For the sake of completeness, we recall the definition of Besov space $B^s_{pq}(\Omega)$. For $v \in L^p(\Omega)$, $1 < p < +\infty$, let us denote by Δ^r_h the difference of order $r \in \mathbb{N} \setminus \{0\}$ and step $h \in \mathbb{R}$, defined as

$$\Delta^r_h v(x) = \sum_{j=0}^{r} (-1)^{r+j} \binom{r}{j} v(x + jh), \qquad \forall x \in \Omega_{rh} = \{x \in \Omega : x + rh \in \Omega\}\,.$$

For $t > 0$, let $\omega^{(r)}_p$ be the modulus of smoothness of order r defined as

$$\omega^{(r)}_p(v, t) = \sup_{|h| \le t} \left\| \Delta^r_h v \right\|_{L^p(\Omega_{rh})}\,.$$

For $s > 0$ and $1 < p,\ q < +\infty$, we say $v \in B^s_{pq}(\Omega)$ whenever the semi-norm

$$|v|_{B^s_{pq}(\Omega)} = \left(\int_0^{+\infty} \left[t^{-s} \omega^{(r)}_p(v, t) \right]^q \frac{dt}{t} \right)^{1/q}$$

is finite, provided r is any integer $> s$ (different values of r give equivalent semi-norm). $B^s_{pq}(\Omega)$ is a Banach space endowed with the norm

$$\|v\|_{B^s_{pq}(\Omega)} := \|v\|_{L^p(\Omega)} + |v|_{B^s_{pq}(\Omega)}\,.$$

Moreover, we recall the following real interpolation result:

$$\left(L^p(\Omega), B^{s_1}_{pq}(\Omega)\right)_{\frac{s_2}{s_1}, q} = B^{s_2}_{pq}(\Omega) \tag{1}$$

where $1 < p, q < +\infty$ and $0 < s_2 < s_1$.

Finally, we will be interested in spaces of functions satisfying homogeneous boundary value conditions. To this end, let $C^\infty_0(\Omega)$ denote the space of $C^\infty(\Omega)$-functions whose support is a compact subset of Ω; here Ω is either $(0, +\infty)$ or $(0, 1)$, i.e., a proper subset of $\mathbb{R}$. We consider the Besov spaces $B^s_{pq,0}(\Omega)$ $(s \ge 0,\ 1 < p, q < +\infty)$ defined as the completion of $C^\infty_0(\Omega)$ in $B^s_{pq}(\Omega)$. One can show that, if $s \le \frac{1}{p}$, $B^s_{pq,0}(\Omega) = B^s_{pq}(\Omega)$, while, if $s > \frac{1}{p}$, $B^s_{pq,0}(\Omega)$ is strictly contained in $B^s_{pq}(\Omega)$. In this second case one also has

$$B^s_{pq,0}(\Omega) = \left\{ v \in B^s_{pq}(\Omega) : \frac{d^j v}{dx^j} = 0 \text{ on } \partial\Omega \text{ if } 0 \le j < s - \frac{1}{p} \right\}. \tag{2}$$

Moreover, for $s \geq 0$, $1 < p, q < +\infty$, let

$$B^s_{pq,00}(\Omega) = \left\{ v \in B^s_{pq}(\mathbb{R}) : \operatorname{supp} v \subseteq \overline{\Omega} \right\} ; \tag{3}$$

then, if $s - \frac{1}{p} \notin \mathbb{N}$,

$$B^s_{pq,00}(\Omega) = B^s_{pq,0}(\Omega) . \tag{4}$$

For these spaces, we have the real interpolation result:

$$\left(L^p(\Omega), B^{s_1}_{pq,0}(\Omega) \right)_{s_2/s_1, p} = B^{s_2}_{pq,00}(\Omega) , \tag{5}$$

where $1 < p, q < +\infty$ and $0 < s_2 < s_1$. In particular, if $1 < p < +\infty$, $0 < s_2 < s_1$, we have

$$\left(L^p(\Omega), W^{s_1,p}_0(\Omega) \right)_{s_2/s_1, p} = \begin{cases} W^{s_2,p}_{00}(\Omega) & \text{if } s_2 - \frac{1}{p} \in \mathbb{N} , \\ B^{s_2}_{pq,0}(\Omega) & \text{if } s_2 \in \mathbb{N} , \\ W^{s_2,p}_0(\Omega) & \text{if } s_2, s_2 - \frac{1}{p} \notin \mathbb{N} , \end{cases} \tag{6}$$

where $W^{s,p}_0(\Omega)$, $W^{s,p}_{00}(\Omega)$ are defined similarly to the Besov spaces (2) and (3). We will also consider negative values of s. For $s < 0$, $1 < p, q < +\infty$, let us denote by p' and q' the conjugate index of p and q respectively (i.e. $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$). Then, we set

$$B^s_{pq}(\Omega) = \left(B^{-s}_{p'q',0}(\Omega) \right)' .$$

2. Multilevel decompositions

We will recall how to define an abstract multilevel decomposition of a separable Banach space V, with norm denoted by $\| \cdot \|$, and how to obtain, using Jackson- and Bernstein-type inequalities, characterization of subspaces of V. For more details and proofs we refer, among the others, to [5, 6, 15].

2.1. Abstract setting

Let $\{V_j\}_{j \in \mathcal{J}}$ ($\mathcal{J} = \mathbb{Z}$ or $\mathcal{J} = \{j \in \mathbb{Z} : j \geq j_0 \in \mathbb{Z}\}$) be a family of closed subspaces of V such that $V_j \subset V_{j+1}$, $\forall j \in \mathcal{J}$. For all $j \in \mathcal{J}$, let $P_j : V \to V_j$ be a continuous linear operator satisfying the following properties:

$$\|P_j\|_{\mathcal{L}(V,V_j)} \leq C \qquad \text{(independent of } j) , \tag{7}$$

$$P_j v = v , \qquad \forall v \in V_j , \tag{8}$$

$$P_j \circ P_{j+1} = P_j . \tag{9}$$

Observe that (7) and (8) imply

$$\|v - P_j v\| \leq C \inf_{u \in V_j} \|v - u\| , \qquad \forall v \in V ,$$

where C is a constant independent of j. Through P_j, we define another set of operators $Q_j : V \to V_{j+1}$ by

$$Q_j v = P_{j+1} v - P_j v , \qquad \forall v \in V , \ \forall j \in \mathcal{J} ,$$

and the detail spaces

$$W_j := \operatorname{Im} Q_j\,, \qquad \forall j \in \mathcal{J}\,.$$

If $\mathcal{J}$ is bounded from below, it will be convenient to set $P_{j_0-1} = 0$ and $V_{j_0-1} = \{0\}$; thus, $Q_{j_0-1} = P_{j_0}$ and $W_{j_0-1} = V_{j_0}$. In this case, let us also set $\mathcal{I} = \mathcal{J} \cup \{j_0 - 1\}$, otherwise $\mathcal{I} = \mathcal{J}$. Thanks to the assumptions (7)-(8)-(9), every Q_j is a continuous linear projection on W_j, the sequence $\{Q_j\}_{j\in\mathcal{J}}$ is uniformly bounded in $\mathcal{L}(V, V_{j+1})$ and each space W_j is a complement space of V_j in V_{j+1}, i.e.,

$$V_{j+1} = V_j \oplus W_j\,. \tag{10}$$

By iterating the decomposition (10), we get for any two integers $j_1, j_2 \in \mathcal{I}$ such that $j_1 < j_2$

$$V_{j_2} = V_{j_1} \oplus \left(\bigoplus_{j=j_1}^{j_2-1} W_j \right), \tag{11}$$

so that every element in V_{j_2} can be viewed as a rough approximation of itself on a coarse level plus a sum of refinement details. Making some more assumptions on the operators P_j, we obtain a similar result for any $v \in V$. In fact, if

$$P_j v \to v \quad \text{as} \quad j \to +\infty\,, \tag{12}$$

and

$$\begin{cases} P_{j_0-1} = 0 & \text{if } \mathcal{J} = \{j \in \mathbb{Z} : j \geq j_0 \in \mathbb{Z}\}\,, \\ P_j v \to 0 \text{ as } j \to -\infty & \text{if } \mathcal{J} = \mathbb{Z}\,, \end{cases} \tag{13}$$

then

$$V = \bigoplus_{j=\inf \mathcal{I}}^{+\infty} W_j\,, \tag{14}$$

and

$$v = \sum_{j \geq \inf \mathcal{I}} Q_j v\,, \qquad \forall v \in V\,. \tag{15}$$

The decomposition (15) is said to be *q-stable*, $1 < q < \infty$, if

$$\|v\| \asymp \left(\sum_{j \geq \inf \mathcal{I}} \|Q_j v\|^q \right)^{1/q}, \qquad \forall v \in V\,. \tag{16}$$

For each $j \in \mathcal{J}$, let us fix a basis for the subspaces V_j

$$\Phi_j = \left\{ \varphi_{jk} : k \in \check{\mathcal{K}}_j \right\}, \tag{17}$$

and for the subspaces W_j

$$\Psi_j = \left\{ \psi_{jk} : k \in \hat{\mathcal{K}}_j \right\}, \tag{18}$$

with $\check{\mathcal{K}}_j$ and $\hat{\mathcal{K}}_j$ suitable sets of indices.

We can represent the operators P_j and Q_j in the form

$$(19)\qquad P_j v = \sum_{k\in\check{\mathcal{K}}_j} \check{v}_{jk}\varphi_{jk}\,, \qquad Q_j v = \sum_{k\in\hat{\mathcal{K}}_j} \hat{v}_{jk}\psi_{jk}\,, \qquad \forall v \in V\,.$$

Thus, if (12) and (13) hold, (15) can be rewritten as

$$(20)\qquad v = \sum_{j\ge\inf\mathcal{I}} \sum_{k\in\hat{\mathcal{K}}_j} \hat{v}_{jk}\psi_{jk}\,, \qquad \forall v \in V\,.$$

The bases chosen for the spaces V_j (and W_j) are called *uniformly p-stable* if, for a certain $1 < p < \infty$,

$$V_j = \left\{ \sum_{k\in\check{\mathcal{K}}_j} \alpha_k\varphi_{jk} : \{\alpha_k\}_{k\in\check{\mathcal{K}}_j} \in \ell^p \right\}$$

and

$$\left\| \sum_{k\in\check{\mathcal{K}}_j} \alpha_k\varphi_{jk} \right\| \asymp \left\| \{\alpha_k\}_{k\in\check{\mathcal{K}}_j} \right\|_{\ell^p}\,, \qquad \forall\,\{\alpha_k\} \in \ell^p\,,$$

the constants involved in the definition of $\asymp$ being independent of j.

If the multilevel decomposition is q-stable, the bases (18) of each W_j are uniformly p-stable, (12) and (13) hold, we can further transform (16) as

$$\|v\| \asymp \left(\sum_{j\ge\inf\mathcal{I}} \left(\sum_{k\in\hat{\mathcal{K}}_j} |\hat{v}_{jk}|^p \right)^{q/p} \right)^{1/q}\,, \qquad \forall v \in V\,.$$

2.2. Characterization of intermediate spaces

Let us consider a Banach space $Z \subset V$, whose norm will be denoted by $\|\cdot\|_Z$. We assume that there exists a semi-norm $|\cdot|_Z$ in Z such that

$$(21)\qquad \|v\|_Z \asymp \|v\| + |v|_Z\,, \qquad \forall v \in Z\,.$$

In addition, we assume that

$$(22)\qquad V_j \subset Z\,, \qquad \forall j \in \mathcal{J}\,.$$

Thus, Z is included in V with continuous embedding.

We recall that the real interpolation method ([4]) allows us to define a family of intermediate spaces Z^α_q, with $0 < \alpha < 1$ and $1 < q < \infty$, such that

$$Z \subset Z^{\alpha_2}_{q_2} \subset Z^{\alpha_1}_{q_1} \subset V\,, \qquad 0 < \alpha_1 < \alpha_2 < 1\,, \qquad 1 < q_1, q_2 < \infty\,,$$

with continuous inclusion. The space Z^α_q is defined as

$$Z^\alpha_q = (V, Z)_{\alpha,q} = \left\{ v \in V : |v|^q_{\alpha,q} := \int_0^\infty [t^{-\alpha}K(v,t)]^q \frac{dt}{t} < \infty \right\},$$

where

$$K(v,t) = \inf_{z\in Z}\{\|v-z\| + t\,|z|_Z\}\,, \qquad v \in V,\ t>0\,.$$

Z^α_q is equipped with the norm $\|v\|_{\alpha,q} = \left(\|v\|^q + |v|^q_{\alpha,q}\right)^{1/q}$. We note that one can replace the semi-norm $|v|_{\alpha,q}$ by an equivalent, discrete version as follows:

$$|v|_{\alpha,q} \asymp \left(\sum_{j\in\mathcal{J}} b^{\alpha q j} K(v, b^{-j})^q\right)^{1/q}, \qquad \forall v \in V\,,$$

where b is any real number > 1.

We can characterize the space Z^α_q in terms of the multilevel decomposition introduced in the previous Subsection (see [5, 6] and also [20]). This general result is based on two inequalities classically known as Bernstein and Jackson inequalities. In our framework, these inequalities read as follows: there exists a constant $b > 1$ such that the *Bernstein inequality*

$$|v|_Z \lesssim b^j\,\|v\|\,, \qquad \forall v \in V_j\,, \qquad \forall j \in \mathcal{J} \tag{23}$$

and the *Jackson inequality*

$$\|v - P_j v\| \lesssim b^{-j}\,|v|_Z\,, \qquad \forall v \in Z\,, \qquad \forall j \in \mathcal{J} \tag{24}$$

hold. The Bernstein inequality is also known as an *inverse inequality*, since it allows the stronger norm $\|v\|_Z$ to be bounded by the weaker norm $\|v\|$, provided $v \in V_j$. The Jackson inequality is an approximation result, which yields the rate of decay of the approximation error by P_j for an element belonging to Z. Note that, if we assume Z to be dense in V, then the Jackson inequality implies the consistency condition (12). The following characterization theorem holds.

THEOREM 1. *Let $\{V_j, P_j\}_{j\in\mathcal{J}}$ be a family as described in Section 2.1. Let Z be a subspace of V satisfying the hypotheses (21) and (22). If the Bernstein and Jackson inequalities (23) and (24) hold, then for all $0 < \alpha < 1$, $1 < q < \infty$ one has*

$$Z^\alpha_q = \left\{ v \in V : \sum_{j\geq \inf\mathcal{I}} b^{\alpha q j}\,\|Q_j v\|^q < +\infty \right\}$$

with

$$|v|_{\alpha,q} \asymp \left(\sum_{j\geq \inf\mathcal{I}} b^{\alpha q j}\,\|Q_j v\|^q\right)^{1/q}, \qquad \forall v \in Z^\alpha_q\,.$$

If in addition both (12) and (13) are satisfied, the bases (18) of each W_j ($j \in \mathcal{J}$) are uniformly p-stable (for suitable $1 < p < \infty$) and the multilevel decomposition (15) is q-stable, then the following representation of the norm of Z^α_q holds:

$$\|v\|_{\alpha,q} \asymp \left(\sum_{j\in\mathcal{J}} (1 + b^{\alpha q j}) \left(\sum_{k\in\hat{\mathcal{K}}_j} |\hat{v}_{jk}|^p\right)^{q/p}\right)^{1/q}, \qquad \forall v \in Z^\alpha_q\,,\ \textit{if}\ \mathcal{J} = \mathbb{Z}\,,$$

or

$$\|v\|_{\alpha,q} \asymp \|P_{j_0}v\| + \left(\sum_{j\in\mathcal{J}} b^{\alpha q j}\left(\sum_{k\in\hat{\mathcal{K}}_j}|\hat{v}_{jk}|^p\right)^{q/p}\right)^{1/q}, \qquad \forall v \in Z^\alpha_q\,,$$

if $\mathcal{J} = \{j \in \mathbb{Z} : j \geq j_0 \in \mathbb{Z}\}$.

It is possible to prove a similar result for dual topological spaces (see e.g. [5]). Indeed, let V be reflexive and $\langle f, v\rangle$ denote the dual pairing between the topological dual space V' and V. For each $j \in \mathcal{J}$, let $\widetilde{P}_j : V' \to V'$ be the adjoint operator of P_j and $\widetilde{V}_j = \mathrm{Im}\,\widetilde{P}_j \subset V'$. It is possible to show that the family $\{\widetilde{V}_j, \widetilde{P}_j\}$ satisfies the same abstract assumptions of $\{V_j, P_j\}$. Moreover, let $1 < q < \infty$ and let q' be the conjugate index of q $\left(\frac{1}{q} + \frac{1}{q'} = 1\right)$. Set $Z^{-\alpha}_q = \left(Z^\alpha_{q'}\right)'$, for $0 < \alpha < 1$, then one has:

THEOREM 2. *Under the same assumptions of Theorem 1 and with the notations of this Section, we have*

$$Z^{-\alpha}_q = \left\{ f \in Z' : \sum_{j\geq j_0} b^{-\alpha q j}\,\|\widetilde{Q}_j f\|^q_{V'} < +\infty \right\},$$

with

$$\|f\|_{Z^{-\alpha}_q} \asymp \left\|\widetilde{P}_{j_0} f\right\|_{V'} + \left(\sum_{j\geq j_0} b^{-\alpha q j}\,\|\widetilde{Q}_j v\|^q_{V'}\right)^{1/q}, \qquad \forall f \in Z^{-\alpha}_q\,.$$

2.3. Biorthogonal decomposition in $\mathbb{R}$

The abstract setting can be applied to construct a biorthogonal system of compactly supported wavelets on the real line and a biorthogonal decomposition of $L^p(\mathbb{R})$ $(1 < p < \infty)$. We will be very brief and we refer to [5, 6, 11, 17] for proofs and more details.

We suppose to have a couple of dual compactly supported scaling functions $\varphi \in L^p(\mathbb{R})$ and $\widetilde{\varphi} \in L^{p'}(\mathbb{R})$ $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ satisfying the following conditions. There exists a couple of finite real filters $h = \{h_n\}^{n_1}_{n=n_0}$, $\tilde{h} = \left\{\tilde{h}_n\right\}^{\tilde{n}_1}_{n=\tilde{n}_0}$ with $n_0, \tilde{n}_0 \leq 0$ and $n_1, \tilde{n}_1 \geq 0$, so that φ and $\widetilde{\varphi}$ satisfy the refinement equations:

$$\varphi(x) = \sqrt{2}\sum_{n=n_0}^{n_1} h_n\varphi(2x-n)\,, \qquad \widetilde{\varphi}(x) = \sqrt{2}\sum_{n=\tilde{n}_0}^{\tilde{n}_1} \tilde{h}_n\widetilde{\varphi}(2x-n)\,, \tag{25}$$

and

$$\mathrm{supp}\,\varphi = [n_0, n_1]\,, \qquad \mathrm{supp}\,\widetilde{\varphi} = \left[\tilde{n}_0, \tilde{n}_1\right]\,.$$

From now on, we will only describe the primal setting, the parallel ˜ construction following by analogy; it will be understood that there p has to be replaced by the conjugate index p'.

Setting, as usual, for $j, k \in \mathbb{Z}$, $\varphi_{jk}(x) = 2^{j/p}\varphi(2^j x - k)$, we have the biorthogonality relations

$$\langle\varphi_{jk}, \widetilde{\varphi}_{jk'}\rangle = \int_{\mathbb{R}} \varphi_{jk}(x)\widetilde{\varphi}_{jk'}(x)\,dx = \delta_{kk'}\,, \qquad \forall j, k, k' \in \mathbb{Z}\,.$$

Thus $\Phi_j = \{\varphi_{jk} : k \in \mathbb{Z}\}$ are uniformly p-stable bases for the spaces

$$V_j = V_j(\mathbb{R}) = \operatorname{span}_{L^p}\{\varphi_{jk} : k \in \mathbb{Z}\} = \left\{ \sum_{k\in\mathbb{Z}} \alpha_k \varphi_{jk} : \{\alpha_k\}_{k\in\mathbb{Z}} \in \ell^p \right\},$$

and, for any $v \in V_j$, we can write

$$v = \sum_{k\in\mathbb{Z}} \alpha_k \varphi_{jk} = \sum_{k\in\mathbb{Z}} \check{v}_{jk} \varphi_{jk} \qquad \text{with } \check{v}_{jk} = \langle v, \widetilde{\varphi}_{jk} \rangle$$

and

$$\|v\|_{L^p(\mathbb{R})} \asymp \left(\sum_{k\in\mathbb{Z}} |\check{v}_{jk}|^p \right)^{1/p}. \tag{26}$$

We have

$$V_j \subset V_{j+1}, \qquad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \qquad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^p(\mathbb{R}).$$

We suppose there exists an integer $L \geq 1$ so that, locally, the polynomials of degree up to $L-1$ (we will indicate this set $\mathbb{P}_{L-1}$) are contained in V_j. It is not difficult to show that L must satisfy the relation

$$L \leq n_1 - n_0 \tag{27}$$

(see [6], equation (3.2)).
We set

$$\psi(x) = \sqrt{2} \sum_{n=1-\widetilde{n}_1}^{1-\widetilde{n}_0} g_n \varphi(2x-n), \qquad \text{with } g_n = (-1)^n \widetilde{h}_{1-n},$$

and $\psi_{jk}(x) = 2^{j/p}\psi(2^j x - k)$, $\forall j, k \in \mathbb{Z}$. Then $\langle \psi_{jk}, \widetilde{\psi}_{j'k'} \rangle = \delta_{jj'}\delta_{kk'}$, $\forall j, j', k, k' \in \mathbb{Z}$. The wavelet spaces

$$W_j = \left\{ \sum_{k\in\mathbb{Z}} \alpha_k \psi_{jk} : \{\alpha_k\}_{k\in\mathbb{Z}} \in \ell^p \right\}, \qquad \forall j \in \mathbb{Z},$$

satisfy $L^p(\mathbb{R}) = \oplus_{j\in\mathbb{Z}} W_j$. For any $v \in L^p(\mathbb{R})$, this implies the expansion

$$v = \sum_{j,k\in\mathbb{Z}} \hat{v}_{jk} \psi_{jk}, \qquad \text{with } \hat{v}_{jk} = \langle v, \tilde{\psi}_{jk} \rangle; \tag{28}$$

in addition, if $p = 2$,

$$\|v\|_{L^2(\mathbb{R})} \asymp \left(\sum_{j,k\in\mathbb{Z}} |\hat{v}_{jk}|^2 \right)^{1/2}, \qquad \forall v \in L^2(\mathbb{R}). \tag{29}$$

Next, let us recall that if $\varphi \in B^{s_0}_{pq}(\mathbb{R})$ ($s_0 > 0$, $1 < p, q < +\infty$) then the Bernstein and Jackson inequalities hold for every space $Z = B^s_{pq}(\mathbb{R})$ with $0 \leq s < \min(s_0, L)$. Indeed, we have

$$|v|_{B^s_{pq}(\mathbb{R})} \lesssim 2^{js} \|v\|_{L^p(\mathbb{R})}, \qquad \forall v \in V_j, \ \forall j \in \mathbb{Z} \tag{30}$$

and

$$\|v - P_j v\|_{L^p(\mathbb{R})} \lesssim 2^{-js}|v|_{B^s_{pq}(\mathbb{R})}\,, \qquad \forall v \in B^s_{pq}(\mathbb{R})\,,\ \forall j \in \mathbb{Z}\,. \tag{31}$$

Thus, taking into account (1), we can apply the characterization Theorems 1 and 2 to the Besov space Z.

3. Scaling function spaces for the half-line

Starting from a biorthogonal decomposition on $\mathbb{R}$ as described in Subsection 2.3, we aim to the construction of dual scaling function spaces $V_j\left(\mathbb{R}^+\right)$ and $\widetilde{V}_j\left(\mathbb{R}^+\right)$ which will form a multilevel decomposition of $L^p\left(\mathbb{R}^+\right)$ and of $L^{p'}\left(\mathbb{R}^+\right)$, respectively. For simplicity, we will work on the scale $j = 0$ and again we will not explicitly describe the dual $\tilde{}$ construction. Without loss of generality, we shall suppose $L \le \widetilde{L}$ and $\tilde{n}_0 \le n_0 \le 0 \le n_1 \le \tilde{n}_1$ so that $\operatorname{supp}\varphi \subseteq \operatorname{supp}\widetilde{\varphi}$; if this is not the case, it is enough to exchange the role of the primal and the dual spaces.

From now on, we will append a suffix $^{\mathbb{R}}$ to all the functions defined on the real line. Note that if $k \ge -n_0$, $\varphi^{\mathbb{R}}_{0k}$ have support contained in $[0, +\infty)$. More precisely

$$\operatorname{supp}\varphi^{\mathbb{R}}_{0k} = [n_0 + k, n_1 + k]\,.$$

Let us fix a nonnegative integer δ and set $k^*_0 = -n_0 + \delta$; observe that

$$k^*_0 = \min\left\{k \in \mathbb{Z} : \operatorname{supp}\varphi^{\mathbb{R}}_{0k} \subseteq [\delta, +\infty)\right\}\,.$$

Let us define

$$V^{(+)} = \operatorname{span}\left\{\varphi^{\mathbb{R}}_{0k}|_{[0,+\infty)} : k \ge k^*_0\right\}\,; \tag{32}$$

this space will be identified in a natural way with a subspace of $V_0(\mathbb{R})$ and will not be modified by the subsequent construction. To obtain the right scaling space $V_0\left(\mathbb{R}^+\right)$ for the half-line, we will add to the basis $\left\{\varphi^{\mathbb{R}}_{0k}|_{[0,+\infty)} : k \ge k^*_0\right\}$ of $V^{(+)}$ a finite number of new functions. These functions will be constructed so that the property of reproduction of polynomials is maintained. In fact we know that for any polynomial $p \in \mathbb{P}_{L-1}$ and every fixed $x \in \mathbb{R}$,

$$p(x) = \sum_{k\in\mathbb{Z}} \check{p}_{0k}\,\varphi^{\mathbb{R}}_{0k}(x)\,.$$

So, if $\{p_\alpha : \alpha = 0, \ldots, L-1\}$ is a basis for $\mathbb{P}_{L-1}$, for every $x \ge 0$, we have

$$\begin{aligned} p_\alpha(x) &= \sum_{k\ge -n_1+1} c_{\alpha k}\varphi^{\mathbb{R}}_{0k}(x) \\ &= \sum_{k=-n_1+1}^{k^*_0-1} c_{\alpha k}\varphi^{\mathbb{R}}_{0k}(x) + \sum_{k\ge k^*_0} c_{\alpha k}\varphi^{\mathbb{R}}_{0k}(x)\,, \end{aligned} \tag{33}$$

where

$$c_{\alpha k} := (\check{p}_\alpha)_{0k} = \int_{\mathbb{R}} p_\alpha(y)\widetilde{\varphi}(y-k)\,dy\,, \qquad \alpha = 0, \ldots, L-1\,. \tag{34}$$

Since the second sum in (33) is a linear combination of elements of $V^{(+)}$, in order to locally generate all polynomials of degree $\leq L-1$ on the half-line, we will add to this space all the linear combinations of the functions

$$\phi_\alpha(x) = \sum_{k=-n_1+1}^{k_0^*-1} c_{\alpha k} \varphi_{0k}^{\mathbb{R}}(x)\,, \qquad \alpha = 0, \ldots, L-1\,. \tag{35}$$

REMARK 1. Let $\{p_\alpha | \alpha = 0, \ldots, L-1\}$ and $\{p_\alpha^\star | \alpha = 0, \ldots, L-1\}$ be two bases of $\mathbb{P}_{L-1}$. Let us denote by ϕ_α and $\phi_\alpha^\star$ the functions defined by the previous argument and by M the matrix of the change of basis of $\mathbb{P}_{L-1}$, then one can prove that

$$\phi_\alpha^\star = \sum_{\beta=0}^{L-1} M_{\alpha\beta} \phi_\beta$$

for every α.

PROPOSITION 1. *The functions ϕ_α, $\alpha = 0, \ldots, L-1$ are linearly independent.*

Proof. If δ is strictly positive, the linear independence of the boundary functions is obvious. Indeed, on $[0, \delta]$, they coincide with linearly independent polynomials. If $\delta = 0$ let us observe that the functions $\varphi_{0k}^{\mathbb{R}}|_{[0,+\infty)}$ involved in (35) have staggered support, i.e., $\operatorname{supp} \varphi_{0k}^{\mathbb{R}}|_{[0,+\infty)} = [0, n_1+k]$, thus they are linearly independent. To obtain the linear independence of the functions ϕ_α it is sufficient to prove that the matrix

$$C = (c_{\alpha k})_{\alpha=0,\ldots,L-1}^{k=-n_1+1,\ldots,-n_0-1}$$

induces an injective transformation. Thanks to Remark 1, we can choose any polynomial bases to prove the maximality of rank (C); if $p_\alpha(x) = x^\alpha$ for any α, one has

$$c_{\alpha k} = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} k^\beta \widetilde{M}_{\alpha-\beta}\,,$$

where $\widetilde{M}_i = \int_{\mathbb{R}} x^i \widetilde{\varphi}(x) dx$ is the i-th moment of $\widetilde{\varphi}$ on $\mathbb{R}$. Let v be a vector in $\mathbb{R}^L$ such that $C^T v = 0$, then the polynomial of degree $L-1$

$$\sum_{\beta=0}^{L-1} \sum_{\alpha=\beta}^{L-1} \binom{\alpha}{\beta} v_\alpha \widetilde{M}_{\alpha-\beta} x^\beta$$

has $n_1 - n_0 - 1$ distinct zeros; thus, recalling the relation (27), it is identically zero. This means $Mv = 0$ where $M = (M_{ij})$ is an upper triangular matrix with $M_{ij} = \binom{j}{i} \widetilde{M}_{j-i}$ if $j \geq i$. M is non-singular, in fact $\det(M) = (\int_{\mathbb{R}} \widetilde{\varphi}(x) dx)^L \neq 0$ and the proof is complete. □

The building blocks of our multiresolution analysis on $(0, +\infty)$ will be the border functions (35) and the basis elements of $V^{(+)}$. Using Proposition 1 and the linear independence on $\mathbb{R}$ of the functions $\varphi_{0k}^{\mathbb{R}}$, one can easily check that

PROPOSITION 2. *The functions ϕ_α, $\alpha = 0, \ldots, L-1$ and $\varphi_{0k}^{\mathbb{R}}|_{[0,+\infty)}$, $k \geq k_0^*$, are linearly independent.*

Thus, it is natural to define

$$V_0\left(\mathbb{R}^+\right) = \operatorname{span}\{\phi_\alpha : \alpha = 0, \dots, L-1\} \oplus V^{(+)} . \tag{36}$$

We rename the functions in the following way

$$\varphi_{0k} = \begin{cases} \phi_k & \text{if } k = 0, \dots, L-1, \\ \varphi^{\mathbb{R}}_{0,k_0^*+k-L} & \text{if } k \geq L . \end{cases} \tag{37}$$

Observe that

$$V^{(+)} = \operatorname{span}\{\varphi_{0k} : k \geq L\} . \tag{38}$$

We study now the biorthogonality of the dual generators of $V_0\left(\mathbb{R}^+\right)$ and $\widetilde{V}_0\left(\mathbb{R}^+\right)$. Setting

$$k^* = \max\left\{k_0^*, \tilde{k}_0^*\right\} = \max\left\{-n_0 + \delta, -\tilde{n}_0 + \tilde{\delta}\right\}, \tag{39}$$

let us observe that $\{\varphi_{0k} : k \geq k^*\}$ and $\{\widetilde{\varphi}_{0k} : k \geq k^*\}$ are already biorthogonal. In order to get a pair of dual systems using our "blocks" we have therefore to match the dimensions of the spaces spanned by

$$\{\phi_\alpha : \alpha = 0, \dots, L-1\} \cup \left\{\varphi^{\mathbb{R}}_{0k} : k = k_0^*, \dots, k^*-1\right\}$$

and by

$$\left\{\widetilde{\phi}_\beta : \beta = 0, \dots, \widetilde{L}-1\right\} \cup \left\{\widetilde{\varphi}^{\mathbb{R}}_{0k} : k = \tilde{k}_0^*, \dots, k^*-1\right\} .$$

This requirement can be translated into an explicit relation between δ and $\tilde{\delta}$; indeed, we must have $L - k_0^* = \widetilde{L} - \tilde{k}_0^*$, i.e.,

$$\tilde{\delta} - \delta = \widetilde{L} - L + \tilde{n}_0 - n_0 . \tag{40}$$

Since $\widetilde{L} \geq L$, we get $k^* = -\tilde{n}_0 + \tilde{\delta}$.

REMARK 2. The two parameters δ and $\tilde{\delta}$ have been introduced exactly because we want the equality of the cardinality of the sets previously indicated. On the other hand, we want to choose them as small as possible in order to minimize the perturbation due to the boundary. Thus, it will be natural to fix one, between δ and $\tilde{\delta}$, equal to zero and determine the other one from the relation (40). In particular, if $\widetilde{L} - L + \tilde{n}_0 - n_0 \geq 0$, we set $\delta = 0$ and $\tilde{\delta} = \widetilde{L} - L + \tilde{n}_0 - n_0$; in this case $\tilde{\delta} < \widetilde{L}$; whereas if $\widetilde{L} - L + \tilde{n}_0 - n_0 < 0$, we choose $\tilde{\delta} = 0$ and $\delta = L - \widetilde{L} + n_0 - \tilde{n}_0$. In analogy with the previous case, we will suppose

$$0 \leq \delta < L . \tag{41}$$

Let us define the spaces V_0^B and $\widetilde{V}_0^B$ spanned by the so called *boundary scaling functions* as

$$V_0^B := \operatorname{span}\left\{\varphi_{0k} : k = 0, \dots, \widetilde{L}-1\right\}, \qquad \widetilde{V}_0^B = \operatorname{span}\left\{\widetilde{\varphi}_{0k} : k = 0, \dots, \widetilde{L}-1\right\}, \tag{42}$$

and the spaces V_0^I and $\widetilde{V}_0^I$ spanned by the *interior scaling functions* as

$$V_0^I := \operatorname{span}\left\{\varphi_{0k} : k \geq \widetilde{L}\right\}, \qquad \widetilde{V}_0^I := \operatorname{span}\left\{\widetilde{\varphi}_{0k} : k \geq \widetilde{L}\right\} . \tag{43}$$

Thus we have

$$V_0\left(\mathbb{R}^+\right) = V_0^B \oplus V_0^I\,, \qquad \widetilde{V}_0\left(\mathbb{R}^+\right) = \widetilde{V}_0^B \oplus \widetilde{V}_0^I\,. \tag{44}$$

Note that, if $L < \widetilde{L}$, the subspace V_0^I is strictly contained in $V^{(+)}$ (see (38)). In other words, some of the functions in $V^{(+)}$ (which are scaling functions on $\mathbb{R}$ supported in $[0, +\infty)$) are thought as boundary scaling functions, i.e., are included in V_0^B. As we already observed, the basis functions of V_0^I and $\widetilde{V}_0^I$ are biorthogonal. Recalling (35) and (37), if $0 \le k \le \widetilde{L} - 1$ there exist coefficients α_{km} such that $\varphi_{0k} = \sum_{m<k^*} \alpha_{km} \varphi_{0m}^{\mathbb{R}}$; thus, if $l \ge \widetilde{L}$, due to the position of the supports, we have

$$\langle \varphi_{0k}, \widetilde{\varphi}_{0l} \rangle = \sum_{m<k^*} \alpha_{km} \int_{\mathbb{R}} \varphi_{0m}^{\mathbb{R}}\, \widetilde{\varphi}_{0,k^*+l-\widetilde{L}}^{\mathbb{R}}\, dx = 0\,,$$

i.e., $V_0^B \subset \left(\widetilde{V}_0^I\right)^{\perp}$. The only functions that we have to modify in order to obtain biorthogonal systems are the border ones. The problem is to find a basis of V_0^B, say $\left\{\eta_{0k} : k = 0, \dots, \widetilde{L} - 1\right\}$, and one of $\widetilde{V}_0^B$, say $\left\{\tilde{\eta}_{0l} : l = 0, \dots, \widetilde{L} - 1\right\}$, such that

$$\langle \eta_{0k}, \tilde{\eta}_{0l} \rangle = \delta_{kl}\,, \qquad k, l = 0, \dots, \widetilde{L} - 1\,.$$

Setting $\eta_{0k} = \sum_{m=0}^{\widetilde{L}-1} d_{km} \varphi_{0m}$ and $\tilde{\eta}_{0k} = \sum_{m=0}^{\widetilde{L}-1} \tilde{d}_{km} \widetilde{\varphi}_{0m}$, and calling X the Gramian matrix of components

$$X_{kl} = \langle \varphi_{0k}, \widetilde{\varphi}_{0l} \rangle\,, \qquad k, l = 0, \dots, \widetilde{L} - 1\,, \tag{45}$$

this is equivalent to the problem of finding two $\widetilde{L} \times \widetilde{L}$ real matrices, say $D = (d_{km})$ and $\widetilde{D} = \left(\tilde{d}_{km}\right)$, satisfying

$$D X \widetilde{D}^T = I\,. \tag{46}$$

A necessary and sufficient condition for (46) to have solutions is clearly the non-singularity of X, or equivalently $V_0^B \cap \left(\widetilde{V}_0^B\right)^{\perp} = \{0\}$. If this is the case, there exist infinitely many couples which satisfy equation (46); indeed if we choose $\widetilde{D}$ non-singular then it is sufficient to set $D = \left(X \widetilde{D}^T\right)^{-1}$. We know at present of no general result establishing the invertibility of X, although it can be proved, e.g., for orthogonal systems and for systems arising from B-spline functions (see Section 9 and also [16]). From now on we will assume this condition is verified and we suppose (renaming if necessary) that $\{\varphi_{0k}\}_{k\ge 0}$ and $\{\widetilde{\varphi}_{0l}\}_{l\ge 0}$ are dual biorthogonal bases.

Let us prove that the functions φ_{0k}, $k \ge 0$, form a p-stable basis of $V_0\left(\mathbb{R}^+\right)$.

PROPOSITION 3. *We have*

$$V_0\left(\mathbb{R}^+\right) = \left\{ v = \sum_{k\ge 0} \alpha_k \varphi_{0k} : \{\alpha_k\}_{k\in\mathbb{N}} \in \ell^p \right\}$$

with

$$\|v\|_{L^p(\mathbb{R}^+)} \asymp \|\{\alpha_k\}_{k\in\mathbb{N}}\|_{\ell^p}\,, \qquad \forall v \in V_0\left(\mathbb{R}^+\right)\,. \tag{47}$$

Proof. Let v be any function in $V_0\left(\mathbb{R}^+\right)$; by (44) it can be written as $v = v_B + v_I$ with $v_B = \sum_{k=0}^{\widetilde{L}-1} \alpha_k \varphi_{0k}$ and $v_I = \sum_{k\geq\widetilde{L}} \alpha_k\varphi_{0k}$. The sequence $\{\alpha_k\}_{k\geq\widetilde{L}}$ is p-summable thanks to the "inclusion" of V_0^I in $V_0(\mathbb{R})$ and by (26), so the first part of the Proposition is proved.

To show (47), let us set $K = \max\left\{|\operatorname{supp}\varphi_{0k}| : k = 0, \dots, \widetilde{L}-1\right\}$ and note that

$$\|v_B\|^p_{L^p(0,K)} \asymp \sum_{k=0}^{\widetilde{L}-1} |\alpha_k|^p \lesssim \|v\|^p_{L^p(\mathbb{R}^+)} . \tag{48}$$

Indeed, since for any $N \in \mathbb{N}\setminus\{0\}$, the application

$$x = (x_0, \dots, x_N) \in \mathbb{R}^{N+1} \longmapsto \left\| \sum_{n=0}^{N} x_n\varphi_{0n} \right\|_{L^p(0,K)} ,$$

defines a norm on $\mathbb{R}^{N+1}$ (for a proof see, e.g., [5], Proposition 6.1), and every two norms on a finite dimensional space are equivalent, the first equivalence is proven. The second inequality follows from

$$\begin{aligned} |\alpha_k|^p &= \left| \int_{\operatorname{supp}\tilde{\varphi}_{0k}} v(x)\tilde{\varphi}_{0k}(x)\,dx \right|^p \\ &\leq \int_{\operatorname{supp}\tilde{\varphi}_{0k}} |v(x)|^p\,dx \cdot \left(\int_{\operatorname{supp}\tilde{\varphi}_{0k}} |\tilde{\varphi}_{0k}(x)|^{p'}dx \right)^{p/p'} \lesssim \|v\|^p_{L^p(\mathbb{R}^+)} . \end{aligned}$$

Thus, by (48) and the p-stability of φ_{0k} on the line (26), we have

$$\|v\|^p_{L^p(\mathbb{R}^+)} = \|v_B + v_I\|^p_{L^p(\mathbb{R}^+)} \lesssim \|v_B\|^p_{L^p(0,K)} + \|v_I\|^p_{L^p(\mathbb{R}^+)} \asymp \sum_{k\geq 0} |\alpha_k|^p .$$

On the other hand, we have

$$\|v_I\|_{L^p(0,K)} = \|v - v_B\|_{L^p(0,K)} \leq \|v\|_{L^p(0,K)} + \|v_B\|_{L^p(0,K)} \lesssim \|v\|_{L^p(\mathbb{R}^+)}$$

and so

$$\begin{aligned} \|v_I\|^p_{L^p(\mathbb{R}^+)} &\lesssim \|v_I\|^p_{L^p(0,K)} + \|v_I\|^p_{L^p([K,+\infty))} \\ &= \|v_I\|^p_{L^p(0,K)} + \|v\|^p_{L^p([K,+\infty))} \lesssim \|v\|^p_{L^p(\mathbb{R}^+)} . \end{aligned}$$

Then

$$\sum_{k\geq 0} |\alpha_k|^p = \sum_{k=0}^{\widetilde{L}-1} |\alpha_k|^p + \sum_{k\geq\widetilde{L}} |\alpha_k|^p \asymp \|v_B\|^p_{L^p(0,K)} + \|v_I\|^p_{L^p(\mathbb{R}^+)} \lesssim \|v\|^p_{L^p(\mathbb{R}^+)} ,$$

and the result is completely proven. □

Similarly, it is possible to show that the dual basis $\{\widetilde{\varphi}_{0k}\}_{k\geq 0}$ of $\widetilde{V}_0\left(\mathbb{R}^+\right)$ is a p'-stable basis. Let us introduce the isometries $T_j : L^p\left(\mathbb{R}^+\right) \to L^p\left(\mathbb{R}^+\right)$ and $\widetilde{T}_j : L^{p'}\left(\mathbb{R}^+\right) \to L^{p'}\left(\mathbb{R}^+\right)$ defined as

$$(T_j f)(x) = 2^{j/p} f(2^j x) , \qquad \left(\widetilde{T}_j f\right)(x) = 2^{j/p'} f(2^j x) . \tag{49}$$

and set $\varphi_{jk} = T_j\varphi_{0k}$, $\widetilde{\varphi}_{jk} = \widetilde{T}_j\widetilde{\varphi}_{0k}$, $\forall j, k \geq 0$. We define the j-th level scaling function spaces as

$$V_j\left(\mathbb{R}^+\right) := T_j\left(V_0\left(\mathbb{R}^+\right)\right), \qquad \widetilde{V}_j\left(\mathbb{R}^+\right) := \widetilde{T}_j\left(\widetilde{V}_0\left(\mathbb{R}^+\right)\right). \tag{50}$$

Let us now show that these are families of refinable spaces.

PROPOSITION 4. *For any $j \in \mathbb{N}$, one has the inclusions $V_j\left(\mathbb{R}^+\right) \subset V_{j+1}\left(\mathbb{R}^+\right)$ and $\widetilde{V}_j\left(\mathbb{R}^+\right) \subset \widetilde{V}_{j+1}\left(\mathbb{R}^+\right)$.*

Proof. As before, we will only prove the result for the primal setting.

By (50), we can restrict ourselves to $j = 0$ and show that $V_0\left(\mathbb{R}^+\right) \subset V_1\left(\mathbb{R}^+\right)$. For $k \geq L$, rewriting (25), one has

$$\varphi_{0k}(x) = \varphi^{\mathbb{R}}_{0,k_0^*+k-L}(x) = 2^{\left(\frac{1}{2}-\frac{1}{p}\right)}\sum_m h_{m-2(k_0^*+k-L)}\left(T_1\varphi^{\mathbb{R}}_{0m}\right)(x).$$

As the filter $\{h_n\}$ is finite, we see that the first non vanishing term in the sum corresponds to $\varphi^{\mathbb{R}}_{0,k_0^*+\delta+2(k-L)}$, which belongs to $V^{(+)}$ for any $k \geq L$, so that the function on the left hand side belongs to $V_1\left(\mathbb{R}^+\right)$.

Suppose now $k < L$; without loss of generality we can choose $p_\alpha(x) = x^\alpha$ for any α, so that, by (33), (34) and (35), one has, for $x \geq 0$,

$$\begin{aligned} 2^{1/p}(2x)^k &= 2^{1/p}\left[\phi_k(2x) + \sum_{m\geq k_0^*} c_{km}\varphi^{\mathbb{R}}_{0m}(2x)\right] \\ &= \varphi_{1k}(x) + \sum_{m\geq k_0^*} c_{km}\left(T_1\varphi^{\mathbb{R}}_{0m}\right)(x). \end{aligned}$$

Again, using (35),

$$\begin{aligned} \varphi_{0k} &= 2^{-(k+1/p)}\left[\varphi_{1k} + \sum_{m\geq k_0^*} c_{km}\varphi^{\mathbb{R}}_{1m}\right] - \sum_{m\geq k_0^*} c_{km}\varphi^{\mathbb{R}}_{0m} \\ &= 2^{-(k+1/p)}\left[\varphi_{1k} + \sum_{m\geq L} c_{k,k_0^*+m-L}\varphi_{1m}\right] - \sum_{m\geq L} c_{k,k_0^*+m-L}\varphi_{0m}, \end{aligned}$$

which completes the proof. □

REMARK 3. Note that, choosing the basis of monomials for $\mathbb{P}_{L-1}$, the *refinement equation* for the modified border functions in V_0^B takes the form

$$\phi_\alpha = 2^{-(\alpha+1/p)}T_1\phi_\alpha + \sum_{k\geq k_0^*} H_{\alpha k}\varphi^{\mathbb{R}}_{1k} \tag{51}$$

where

$$H_{\alpha k} = c_{\alpha k}2^{-(\alpha+1/p)} - 2^{\left(\frac{1}{2}-\frac{1}{p}\right)}\sum_{l\geq k_0^*} c_{\alpha l}h_{k-2l}, \tag{52}$$

involving only the respective modified border function in V_1^B.

4. Projection operators

Following the guiding lines of the abstract setting, we will define a sequence of continuous linear operators $P_j : L^p(\mathbb{R}^+) \to V_j$ for $j \in \mathbb{N}$, satisfying (7), (8) and (9).

By (50), it is obvious that the definition of P_0 gives naturally the complete sequence, by posing $P_j = T_j \circ P_0 \circ T_j^{-1}$, where T_j is the isometry defined in (49). For $v \in L^p(\mathbb{R}^+)$, let us set

$$P_0 v = \sum_{k\geq 0} \check{v}_{0k}\varphi_{0k}\,, \qquad \text{with} \quad \check{v}_{0k} = \int_{\mathbb{R}} v(x)\widetilde{\varphi}_{0k}(x)\,dx\,.$$

We will first prove that P_0 is a well-defined and continuous operator.

PROPOSITION 5. *We have*

$$\|P_0 v\|^p_{L^p(\mathbb{R}^+)} \asymp \sum_{k\geq 0} |\check{v}_{0k}|^p \lesssim \|v\|^p_{L^p(\mathbb{R}^+)}\,, \qquad \forall v \in L^p(\mathbb{R}^+)\,.$$

Proof. Let us write

$$P_0 v = \sum_{k=0}^{\widetilde{L}-1} \check{v}_{0k}\varphi_{0k} + \sum_{k\geq \widetilde{L}} \check{v}_{0k}\varphi_{0k}\,.$$

Observe that, by the Hölder inequality,

$$\sum_{k=0}^{\widetilde{L}-1} |\check{v}_{0k}|^p \leq \|v\|^p_{L^p(\mathbb{R}^+)} \left(\sum_{k=0}^{\widetilde{L}-1} \|\widetilde{\varphi}_{0k}\|^p_{L^{p'}(\mathbb{R}^+)}\right) = C\,\|v\|^p_{L^p(\mathbb{R}^+)}\,.$$

Thus, by the p-stability property on the line (26),

$$\sum_{k\geq 0} |\check{v}_{0k}|^p \lesssim \|v\|^p_{L^p(\mathbb{R}^+)}\,.$$

This implies $P_0 v \in V_0(\mathbb{R}^+)$ and the result follows by (47). □

Since T_j is an isometry for any j, we immediately get (7). Equality (8), follows by the biorthogonality of the systems. Equality (9) is a consequence of the inclusion $V_j(\mathbb{R}^+) \subset V_{j+1}(\mathbb{R}^+)$, proven in Proposition 4. Similarly, one can define a sequence of dual operators, $\widetilde{P}_j : L^{p'}(\mathbb{R}^+) \to \widetilde{V}_j(\mathbb{R}^+)$, satisfying the same properties of the primal sequence.

4.1. Jackson and Bernstein inequalities

The main property of the original decomposition on the real line we have inherited, is the way polynomials are reconstructed through basis functions. This is what we call the approximation property, and it is fundamental for the characterization of functional spaces. In this section we will exploit it to prove Bernstein- and Jackson-type inequalities on the half-line and then apply the general characterization results of the abstract setting (Theorems 1 and 2).

As in Section 2.2, we consider a Banach subspace Z of $L^p(\mathbb{R}^+)$ and suppose that the scaling function φ belongs to Z. In the following, Z will be the Besov space $B^{s_0}_{pq}(\mathbb{R}^+)$, with $s_0 > 0$ and $1 < p, q < +\infty$.

PROPOSITION 6. *For any* $0 \le s \le s_0$*, the Bernstein-type inequality*

$$|v|_{B^s_{pq}(\mathbb{R}^+)} \lesssim 2^{js}\|v\|_{L^p(\mathbb{R}^+)}\,, \qquad \forall v \in V_j\left(\mathbb{R}^+\right)\,,\ \forall j \in \mathbb{N}\,, \tag{53}$$

holds.

Proof. Applying the definition of the operator T_j, it is easy to see that

$$|T_j v|_{B^s_{pq}(\mathbb{R}^+)} = 2^{js}|v|_{B^s_{pq}(\mathbb{R}^+)}\,, \qquad \forall v \in B^s_{pq}\left(\mathbb{R}^+\right)\,,\ \forall j \in \mathbb{N}\,; \tag{54}$$

so, by (50), it is enough to prove the inequality for $j = 0$. Proceeding as in Proposition 3, we choose $v \in V_0\left(\mathbb{R}^+\right)$ and write $v = v_B + v_I$. Let us estimate separately the semi-norms of the two terms. We have

$$|v_B|^p_{B^s_{pq}(\mathbb{R}^+)} = \left|\sum_{k=0}^{\widetilde{L}-1} \check{v}_{0k}\varphi_{0k}\right|^p_{B^s_{pq}(\mathbb{R}^+)} \le \left(\sum_{k=0}^{\widetilde{L}-1} |\check{v}_{0k}|\,|\varphi_{0k}|_{B^s_{pq}(\mathbb{R}^+)}\right)^p \lesssim \sum_{k=0}^{\widetilde{L}-1} |\check{v}_{0k}|^p\,,$$

the constants depending on the semi-norms of the basis functions and the equivalence of norms in $\mathbb{R}^{\widetilde{L}}$. Observe that v_I is an element of $V_0 = V_0(\mathbb{R})$; using the Bernstein inequality (30) and the p-stability on the line (26), we get

$$|v_I|_{B^s_{pq}(\mathbb{R}^+)} \lesssim \|v_I\|_{L^p(\mathbb{R})} \asymp \left(\sum_{k \ge \widetilde{L}} |\check{v}_{0k}|^p\right)^{1/p}\,.$$

Thus, by (47),

$$|v|^p_{B^s_{pq}(\mathbb{R}^+)} \lesssim \left(|v_B|^p_{B^s_{pq}(\mathbb{R}^+)} + |v_I|^p_{B^s_{pq}(\mathbb{R}^+)}\right) \lesssim \|\{\check{v}_{0k}\}_{k\in\mathbb{N}}\|^p_{\ell^p} \lesssim \|v\|_{L^p(\mathbb{R}^+)}\,.$$

□

Next, we prove the generalized Jackson inequality, following the same ideas used in showing the analogous property (31) on the real line (see [5]).

PROPOSITION 7. *For each* $0 \le s < \min(s_0, L)$*, we have*

$$\|v - P_j v\|_{L^p(\mathbb{R}^+)} \lesssim 2^{-js}|v|_{B^s_{pq}(\mathbb{R}^+)}\,, \qquad \forall v \in B^s_{pq}\left(\mathbb{R}^+\right),\ \forall j \in \mathbb{N}\,. \tag{55}$$

Proof. As before (see (54)), it is enough to prove that

$$\|v - P_0 v\|_{L^p(\mathbb{R}^+)} \lesssim |v|_{B^s_{pq}(\mathbb{R}^+)}\,, \qquad \forall v \in B^s_{pq}\left(\mathbb{R}^+\right).$$

Let us divide the half-line into unitary intervals, $\mathbb{R}^+ = \cup_{l \ge 0} I_l$ where $I_l = [l, l+1]$, and estimate $\|v - P_0 v\|_{L^p(I_l)}$. Recalling that polynomials up to degree $L-1$ are locally reconstructed through the basis of $V_0\left(\mathbb{R}^+\right)$, it is easy to see that for any $q \in \mathbb{P}_{L-1}$ there exists v_q in $V_0\left(\mathbb{R}^+\right)$ such that $v_q = q$ on I_l. Then, using (8), we have

$$\begin{aligned}\|v - P_0 v\|_{L^p(I_l)} &= \|v - q + P_0 q - P_0 v\|_{L^p(I_l)}\\ &\le \|v - q\|_{L^p(I_l)} + \|P_0(v - q)\|_{L^p(I_l)}\,.\end{aligned}$$

Moreover, from the compactness of the supports of the basis functions, setting $R_l = \{k \in \mathbb{N} : \operatorname{supp} \varphi_{0k} \cap I_l \neq \emptyset\}$ and $J_l = \cup_{k \in R_l} \operatorname{supp} \widetilde{\varphi}_{0k}$, for any $f \in L^p\left(\mathbb{R}^+\right)$, one gets

$$\begin{aligned}
\|P_0 f\|^p_{L^p(I_l)} &= \int_{I_l} \left| \sum_{k \in R_l} \check{f}_{0k} \varphi_{0k}(x) \right|^p dx \\
&\leq \left(\max_{k \in R_l} \|f\|_{L^p(\operatorname{supp} \widetilde{\varphi}_{0k})} \|\widetilde{\varphi}_{0k}\|_{L^{p'}(\mathbb{R}^+)} \right)^p \int_{I_l} \left(\sum_{k \in R_l} |\varphi_{0k}(x)| \right)^p dx \\
&\lesssim \|f\|^p_{L^p(J_l)} .
\end{aligned}$$

Thus

$$\|v - P_0 v\|_{L^p(I_l)} \lesssim \|v - q\|_{L^p(J_l)} , \qquad \forall q \in \mathbb{P}_{L-1} .$$

Taking the infimum over all $q \in \mathbb{P}_{L-1}$, we end up with

$$\|v - P_0 v\|_{L^p(\mathbb{R}^+)} \lesssim \sum_{l \geq 0} \inf_{q \in \mathbb{P}_{L-1}} \|v - q\|_{L^p(J_l)} . \tag{56}$$

A local version of Whitney's Theorem (see [21]) for a given interval of the real line J, states that, for any $v \in L^p(J)$,

$$\inf_{q \in \mathbb{P}_{L-1}} \|v - q\|_{L^p(J)} \lesssim w^{(L)}(v, J) , \tag{57}$$

where

$$w^{(L)}(v, J) = \left[\frac{1}{2|J|} \int_{-|J|}^{|J|} dh \left(\int_{J(Lh)} |\Delta_h^L v|^p dx \right) \right]^{1/p}$$

and $J(s) = \{x \in J : x + s \in J\}$. Observing that $h^* = |J_l|$ (independent of l) and using (57), we have

$$\begin{aligned}
\|v - P_0 v\|^p_{L^p(\mathbb{R}^+)} &\lesssim \sum_{l \geq 0} \left(w^{(L)}(v, J_l) \right)^p \\
&= \frac{1}{2h^*} \int_{-h^*}^{h^*} dh \sum_{l \geq 0} \int_{J_l(Lh)} |\Delta_h^L v|^p dx \\
&\lesssim \sup_{|h| \leq h^*} \int_{\mathbb{R}^+} |\Delta_h^L v|^p dx \\
&= \left(\omega_p^{(L)}(v, h^*) \right)^p ,
\end{aligned}$$

where $\omega_p^{(L)}$ is the modulus of smoothness (see [21]). To conclude the proof, it is enough to observe that, for any $1 < q < \infty$,

$$\omega_p^{(L)}(v, h^*) \lesssim \left(\sum_{j \in \mathbb{N}} 2^{jsq} \left(\omega_p^{(L)}(v, 2^{-j}) \right)^q \right)^{1/q} \asymp |v|_{B^s_{pq}(\mathbb{R}^+)} .$$

□

Since $B^s_{pq}\left(\mathbb{R}^+\right)$ is dense in $L^p\left(\mathbb{R}^+\right)$, the following property immediately follows.

COROLLARY 1. *The union* $\cup_{j \in \mathbb{N}} V_j \left(\mathbb{R}^+\right)$ *is dense in* $L^p\left(\mathbb{R}^+\right)$.

5. Wavelet function spaces for the half-line

We now have all the tools to build the detail spaces and the wavelets on the half-line. Recalling the abstract construction, we start from level $j = 0$ and look for a complement space $W_0\left(\mathbb{R}^+\right)$ such that $V_1\left(\mathbb{R}^+\right) = V_0\left(\mathbb{R}^+\right) \oplus W_0\left(\mathbb{R}^+\right)$ (note that the sum is not, in general, orthogonal) and $W_0\left(\mathbb{R}^+\right) \perp \widetilde{V}_0$. To this end, let us consider the basis functions of $V_1\left(\mathbb{R}^+\right)$ and let us write them as a sum of a function of $V_0\left(\mathbb{R}^+\right)$ and a function which will be an element of $W_0\left(\mathbb{R}^+\right)$. Since we have based our construction on the existence of a multilevel decomposition on the real line, we report two equations that will be largely used in the sequel (see, e.g., [6]):

$$(58)\quad \varphi_{1k}^{\mathbb{R}} = 2^{\frac{1}{p}-\frac{1}{2}}\left(\sum_{\tilde{n}_0\le k-2m\le \tilde{n}_1} \tilde{h}_{k-2m}\varphi_{0m}^{\mathbb{R}} + \sum_{1-n_1\le k-2m\le 1-n_0} \tilde{g}_{k-2m}\psi_{0m}^{\mathbb{R}}\right),$$

$$(59)\quad \psi_{0m}^{\mathbb{R}} = 2^{\frac{1}{2}-\frac{1}{p}} \sum_{l=1+2m-\tilde{n}_1}^{1+2m-\tilde{n}_0} g_{l-2m}\varphi_{1l}^{\mathbb{R}}.$$

Interior wavelets. Since $V_0\left(\mathbb{R}^+\right)$ contains the subspace $V^{(+)}$ defined in (32), $V_1\left(\mathbb{R}^+\right)$ contains the subspace $T_1 V^{(+)} = \left\{\varphi_{1k}^{\mathbb{R}}|_{[0,+\infty)} : k \ge k_0^*\right\}$. Considering equation (59), let us determine the integer m such that all the indices l in the sum are greater or equal to k_0^*. This is equivalent to $2m \ge k_0^* + \tilde{n}_1 - 1$, so we set

$$(60)\qquad m \ge \left\lceil \frac{k_0^* + \tilde{n}_1 - 1}{2} \right\rceil =: m_0^*.$$

Since, (see (2.9) in [11]),

$$(61)\qquad \sum_{n\in\mathbb{Z}} \tilde{h}_n h_{n-2k} = \delta_{0k}, \qquad \forall k \in \mathbb{Z},$$

it is easy to see that $\tilde{n}_1 - n_0$ is always odd; thus, by (39),

$$(62)\qquad m_0^* = \frac{\tilde{n}_1 - n_0 - 1}{2} + \left\lceil \frac{\delta}{2} \right\rceil.$$

Let us set

$$(63)\qquad W_0^I := \operatorname{span}\left\{\psi_{0m}^{\mathbb{R}}|_{[0,+\infty)} : m \ge m_0^*\right\};$$

we observe that W_0^I can be identified with a subspace of $W_0(\mathbb{R})$, thus it is orthogonal to $\widetilde{V}_0$ and $W_0^I \subseteq W_0\left(\mathbb{R}^+\right)$. The functions

$$\psi_{0m} := \psi_{0m}^{\mathbb{R}}|_{[0,+\infty)}$$

are called *interior wavelets*.

Border wavelets. Let us now call W_0^B a generic supplementary space of W_0^I in $W_0\left(\mathbb{R}^+\right)$ and set $V_1^B = T_1 V_0^B$.

PROPOSITION 8. *The dimension of the space W_0^B is m_0^*.*

Proof. Let $K > 0$ be an integer such that on $[K, +\infty)$ all non-vanishing wavelets and scaling functions are interior ones. Then

$$V_1^B \oplus \text{span}\left\{\varphi_{1k}^{\mathbb{R}} : k_0^* \le k < -n_0 + 2K\right\} = \\ \left[V_0^B \oplus \text{span}\left\{\varphi_{0k}^{\mathbb{R}} : k_0^* \le k < -n_0 + K\right\}\right] \oplus \left[W_0^B \oplus \text{span}\left\{\psi_{0m}^{\mathbb{R}} : m_0^* \le m \le K - 1\right\}\right],$$

since, by (58), the first interior wavelet used to generate $\varphi_{1,-n_0+2K}^{\mathbb{R}}$ is $\psi_{0K}^{\mathbb{R}}$. Then the result easily follows. □

To build W_0^B we need some functions that, added to W_0^I, will generate both V_1^B and the interior scaling functions that cannot be obtained in (58) using $V^{(+)}$ and W_0^I. Thanks to (51), we only have to consider the problem of generating interior scaling functions. Let us now look for the functions $\varphi_{1k}^{\mathbb{R}}$ generated by $\varphi_{0m}^{\mathbb{R}}$, for $m \ge k_0^*$, and by $\psi_{0m}^{\mathbb{R}}$, with $m \ge m_0^*$. Let us work separately on the two sums of (58):

(a) we must have $m \ge (k - \tilde{n}_1)/2$. Imposing $m \ge k_0^*$ and seeking for integer solutions, we get

$$\left\lceil \frac{k - \tilde{n}_1}{2} \right\rceil \ge k_0^* = -n_0 + \delta\,; \tag{64}$$

(b) similarly, we obtain $m \ge (n_0 - 1 + k)/2$. Again, we want $m \ge m_0^*$, so we must have

$$\left\lceil \frac{n_0 - 1 + k}{2} \right\rceil \ge m_0^*\,.$$

Using (62), this means

$$\left\lceil \frac{n_0 - 1 + k}{2} \right\rceil \ge \frac{-n_0 + \tilde{n}_1 - 1}{2} + \left\lceil \frac{\delta}{2} \right\rceil. \tag{65}$$

Since (64) and (65) have to be both satisfied, we obtain the following condition

$$k \ge -2n_0 + \tilde{n}_1 + 2\delta - 1 = 2k_0^* + \tilde{n}_1 - 1\,. \tag{66}$$

Indeed, this can be seen considering all possible situations. For instance, if n_0 and δ are even, then $\tilde{n}_1$ is odd and $\left\lceil \frac{\delta}{2} \right\rceil = \frac{\delta}{2}$. If k satisfies both (64) and (65), so does $k-1$; thus we can look for the least k as an even integer. In this case $\left\lceil \frac{k-\tilde{n}_1}{2} \right\rceil = \frac{k-\tilde{n}_1}{2} + \frac{1}{2}$ and $\left\lceil \frac{n_0-1+k}{2} \right\rceil = \frac{n_0-1+k}{2} + \frac{1}{2}$, and (66) easily follows. The other cases are dealt with similarly. Let us set

$$\overline{k} = 2k_0^* + \tilde{n}_1 - 1\,, \tag{67}$$

so that

$$\text{span}\left\{\varphi_{1k}^{\mathbb{R}}\big|_{[0,+\infty)} : k \ge \overline{k}\right\} \subseteq V^{(+)} \oplus W_0^I\,.$$

We are left with the problem of generating some functions of $V_1\left(\mathbb{R}^+\right)$, precisely $\varphi_{1k}^{\mathbb{R}}$ with $k_0^* \le k < \overline{k}$. Observe that we have to generate $\overline{k} - k_0^*$ functions using a space of dimension $m_0^* = \left\lceil \frac{\overline{k}-k_0^*}{2} \right\rceil$. In fact one can show that one out of two $\varphi_{1k}^{\mathbb{R}}$, for $k = k_0^*, \dots, \overline{k} - 1$, depends on the previous ones through elements of level zero.

PROPOSITION 9. *Let $e = 0$ if δ is even, $e = 1$ if δ is odd. For all $1 \le m \le m_0^* - e$, one has*

$$\varphi^{\mathbb{R}}_{1,\overline{k}-2m}\big|_{[0,+\infty)} \in S_m \oplus V_0^I \oplus W_0^I\,,$$

with

$$S_m := \operatorname{span}\left\{\varphi^{\mathbb{R}}_{1,\overline{k}-2l+1}\big|_{[0,+\infty)} : 1 \le l \le m\right\}.$$

Proof. Let us set $\tilde{n}_1 - n_0 = 2r + 1$ with $r > 0$ (recall that $\tilde{n}_1 - n_0$ is odd). Indeed, it is not difficult to see that for $r = 0$ there is nothing to prove. Observe that, by (58), for any $l \in \mathbb{Z}$, we have

$$\begin{aligned}\varphi^{\mathbb{R}}_{1,\overline{k}-2l} =& 2^{\left(\frac{1}{p}-\frac{1}{2}\right)}\left(\sum_{n\ge k_0^*-l}\tilde{h}_{\overline{k}-2l-2n}\varphi^{\mathbb{R}}_{0n} + \sum_{n\ge m_0^*+\left\lfloor\frac{\delta}{2}\right\rfloor-l}\tilde{g}_{\overline{k}-2l-2n}\psi^{\mathbb{R}}_{0n}\right)\\ =& 2^{\left(\frac{1}{p}-\frac{1}{2}\right)}\Big(\tilde{h}_{\tilde{n}_1-1}\varphi^{\mathbb{R}}_{0,k_0^*-l} + \tilde{h}_{\tilde{n}_1-3}\varphi^{\mathbb{R}}_{0,k_0^*-l+1} + \dots\\ &+ \tilde{g}_{-n_0}\psi^{\mathbb{R}}_{0,m_0^*+\left\lfloor\frac{\delta}{2}\right\rfloor-l} + \tilde{g}_{-n_0-2}\psi^{\mathbb{R}}_{0,m_0^*+\left\lfloor\frac{\delta}{2}\right\rfloor-l+1} + \dots\Big)\end{aligned}\tag{68}$$

and

$$\begin{aligned}\varphi^{\mathbb{R}}_{1,\overline{k}-2l+1} =& 2^{\left(\frac{1}{p}-\frac{1}{2}\right)}\left(\sum_{n\ge k_0^*-l}\tilde{h}_{\overline{k}-2l+1-2n}\varphi^{\mathbb{R}}_{0n} + \sum_{n\ge m_0^*+\left\lfloor\frac{\delta}{2}\right\rfloor-l}\tilde{g}_{\overline{k}-2l+1-2n}\psi^{\mathbb{R}}_{0n}\right)\\ =& 2^{\left(\frac{1}{p}-\frac{1}{2}\right)}\Big(\tilde{h}_{\tilde{n}_1}\varphi^{\mathbb{R}}_{0,k_0^*-l} + \tilde{h}_{\tilde{n}_1-2}\varphi^{\mathbb{R}}_{0,k_0^*-l+1} + \dots\\ &+ \tilde{g}_{-n_0+1}\psi^{\mathbb{R}}_{0,m_0^*+\left\lfloor\frac{\delta}{2}\right\rfloor-l} + \tilde{g}_{-n_0-1}\psi^{\mathbb{R}}_{0,m_0^*+\left\lfloor\frac{\delta}{2}\right\rfloor-l+1} + \dots\Big).\end{aligned}\tag{69}$$

Let us prove the stated result by induction on m. For $m = 1$ we consider the linear combination

$$\begin{aligned}&h_{n_0}\varphi^{\mathbb{R}}_{1,\overline{k}-2} + h_{n_0+1}\varphi^{\mathbb{R}}_{1,\overline{k}-1}\\ &\quad= 2^{\left(\frac{1}{p}-\frac{1}{2}\right)}\left[\left(h_{n_0}\tilde{h}_{\tilde{n}_1-1} + h_{n_0+1}\tilde{h}_{\tilde{n}_1}\right)\varphi^{\mathbb{R}}_{0,k_0^*-1} + \sum_{n\ge k_0^*}c_{1n}\varphi^{\mathbb{R}}_{0n}\right.\\ &\qquad\left. + \left(h_{n_0}\tilde{g}_{-n_0} + h_{n_0+1}\tilde{g}_{-n_0+1}\right)\psi^{\mathbb{R}}_{0,m_0^*+\left\lfloor\frac{\delta}{2}\right\rfloor-1} + \sum_{n\ge m_0^*+\left\lfloor\frac{\delta}{2}\right\rfloor}d_{1n}\psi^{\mathbb{R}}_{0n}\right],\end{aligned}$$

for some coefficients c_{1n} and d_{1n}. Writing (61) with $k = \frac{n_0-\tilde{n}_1+1}{2} \neq 0$ and

$$\sum_{n\in\mathbb{Z}}\tilde{g}_n h_{n-2k} = 0$$

(see (3.29) in [11]) with $k = -n_0$, we have

$$h_{n_0}\tilde{h}_{\tilde{n}_1-1} + h_{n_0+1}\tilde{h}_{\tilde{n}_1} = 0\,, \qquad h_{n_0}\tilde{g}_{-n_0} + h_{n_0+1}\tilde{g}_{-n_0+1} = 0\,.$$

Thus we have

$$h_{n_0}\varphi^{\mathbb{R}}_{1,\bar{k}-2}\big|_{[0,+\infty)} + h_{n_0+1}\varphi^{\mathbb{R}}_{1,\bar{k}-1}\big|_{[0,+\infty)} \in V_0^I \oplus W_0^I\,,$$

and the result follows because $h_{n_0} \neq 0$.

Set now $1 < m \le m_0^* - e$. As before, we choose a certain linear combination of the scaling functions $\varphi^{\mathbb{R}}_{1,\bar{k}-2l}$ and $\varphi^{\mathbb{R}}_{1,\bar{k}-2l+1}$ with $1 \le l \le m$. Then we use (68), (69) to represent them through functions of level 0. More precisely

$$\begin{aligned}
&h_{n_0}\varphi^{\mathbb{R}}_{1,\bar{k}-2m} + h_{n_0+1}\varphi^{\mathbb{R}}_{1,\bar{k}-2m+1} + \ldots + h_{n_0+2m-2}\varphi^{\mathbb{R}}_{1,\bar{k}-2} + h_{n_0+2m-1}\varphi^{\mathbb{R}}_{1,\bar{k}-1}\\
&= \left(h_{n_0}\tilde{h}_{\tilde{n}_1-1} + h_{n_0+1}\tilde{h}_{\tilde{n}_1}\right)\varphi^{\mathbb{R}}_{0,k_0^*-m}\\
&\quad + \left(h_{n_0}\tilde{h}_{\tilde{n}_1-3} + h_{n_0+1}\tilde{h}_{\tilde{n}_1-2} + h_{n_0+2}\tilde{h}_{\tilde{n}_1-1} + h_{n_0+3}\tilde{h}_{\tilde{n}_1}\right)\varphi^{\mathbb{R}}_{0,k_0^*-m+1}\\
&\quad + \ldots\\
&\quad + \left(h_{n_0}\tilde{h}_{\tilde{n}_1-2m+1} + h_{n_0+1}\tilde{h}_{\tilde{n}_1-2m+2} + \ldots + h_{n_0+2m-1}\tilde{h}_{\tilde{n}_1}\right)\varphi^{\mathbb{R}}_{0,k_0^*-1}\\
&\quad + \sum_{n\ge k_0^*} c_{mn}\varphi^{\mathbb{R}}_{0n}\\
&\quad + \left(h_{n_0}\tilde{g}_{-n_0} + h_{n_0+1}\tilde{g}_{-n_0+1}\right)\psi^{\mathbb{R}}_{0,m_0^*+\lfloor\frac{\delta}{2}\rfloor-m}\\
&\quad + \left(h_{n_0}\tilde{g}_{-n_0-2} + h_{n_0+1}\tilde{g}_{-n_0-1} + h_{n_0+2}\tilde{g}_{-n_0} + h_{n_0+3}\tilde{g}_{-n_0+1}\right)\psi^{\mathbb{R}}_{0,m_0^*+\lfloor\frac{\delta}{2}\rfloor-m+1}\\
&\quad + \ldots\\
&\quad + \left(h_{n_0}\tilde{g}_{-n_0-2m+2} + h_{n_0+1}\tilde{g}_{-n_0-2m+1} + \ldots + h_{n_0+2m-2}\tilde{g}_{-n_0+1}\right)\psi^{\mathbb{R}}_{0,m_0^*+\lfloor\frac{\delta}{2}\rfloor-1}\\
&\quad + \sum_{n\ge m_0^*+\lfloor\frac{\delta}{2}\rfloor} d_{mn}\psi^{\mathbb{R}}_{0n}\,,
\end{aligned} \tag{70}$$

for some c_{mn} and d_{mn}. The coefficients of the functions $\varphi^{\mathbb{R}}_{0,k_0^*-m}, \ldots, \varphi^{\mathbb{R}}_{0,k_0^*-1}$ can be written as

$$\sum_{n\in\mathbb{Z}} h_n\tilde{h}_{n-2k} = \delta_{0k}\,, \tag{71}$$

with $k = \frac{n_0-\tilde{n}_1+1}{2}, \ldots, \frac{n_0-\tilde{n}_1+1}{2} + m - 1 = -r, \cdots, -r+m-1$, respectively. Similarly, the coefficients of the functions $\psi^{\mathbb{R}}_{0,m_0^*+\lfloor\frac{\delta}{2}\rfloor-m}, \ldots, \psi^{\mathbb{R}}_{0,m_0^*+\lfloor\frac{\delta}{2}\rfloor-1}$ can be written as

$$\sum_{n\in\mathbb{Z}} h_{n-2k}\tilde{g}_n = 0\,,$$

with $k = -n_0, \cdots, -n_0-m+1$, respectively. Observe now that $m \le m_0^* - e = r + \lfloor\frac{\delta}{2}\rfloor$. If $m \le r$, all indices k in (71) are negative, so

$$\begin{aligned}
&\left(h_{n_0}\varphi^{\mathbb{R}}_{1,\bar{k}-2m} + h_{n_0+1}\varphi^{\mathbb{R}}_{1,\bar{k}-2m+1}\right)\Big|_{[0,+\infty)} \in\\
&\quad -\Big(h_{n_0+2}\varphi^{\mathbb{R}}_{1,\bar{k}-2m+2} + h_{n_0+3}\varphi^{\mathbb{R}}_{1,\bar{k}-2m+3} + \ldots\\
&\quad + h_{n_0+2m-2}\varphi^{\mathbb{R}}_{1,\bar{k}-2} + h_{n_0+2m-1}\varphi^{\mathbb{R}}_{1,\bar{k}-1}\Big)\Big|_{[0,+\infty)} + V_0^I \oplus W_0^I
\end{aligned}$$

and the result is proven by induction since $h_{n_0} \neq 0$. If $m > r$ (i.e., $\delta \geq 2$), we get

$$\begin{aligned}\left(h_{n_0}\varphi^{\mathbb{R}}_{1,\overline{k}-2m} + h_{n_0+1}\varphi^{\mathbb{R}}_{1,\overline{k}-2m+1}\right)\Big|_{[0,+\infty)} \in \\ -\Big(h_{n_0+2}\varphi^{\mathbb{R}}_{1,\overline{k}-2m+2} + h_{n_0+3}\varphi^{\mathbb{R}}_{1,\overline{k}-2m+3} + \dots \\ + h_{n_0+2m-2}\varphi^{\mathbb{R}}_{1,\overline{k}-2} + h_{n_0+2m-1}\varphi^{\mathbb{R}}_{1,\overline{k}-1}\Big)\Big|_{[0,+\infty)} \\ +2^{\left(\frac{1}{p}-\frac{1}{2}\right)}\varphi^{\mathbb{R}}_{0,k_0^*-m+r}\big|_{[0,+\infty)} + V_0^I \oplus W_0^I\,.\end{aligned}$$

Therefore, by the induction hypothesis, we only have to prove that

$$\varphi^{\mathbb{R}}_{0,k_0^*-m+r}\big|_{[0,+\infty)} \in S_m \oplus V_0^I \oplus W_0^I\,.$$

We immediately get $m - r \leq m_0^* - e - r = \left\lfloor \frac{\delta}{2} \right\rfloor$, so we show that

$$\varphi^{\mathbb{R}}_{0,k_0^*-l}\big|_{[0,+\infty)} \in S_l\,, \qquad 1 \leq l \leq \left\lfloor \frac{\delta}{2} \right\rfloor, \tag{72}$$

by induction on l. If $l = 1$, from (69) we have

$$\begin{aligned}\varphi^{\mathbb{R}}_{0,k_0^*-1}\big|_{[0,+\infty)} &= 2^{\left(\frac{1}{2}-\frac{1}{p}\right)}\tilde{h}_{\tilde{n}_1}\varphi^{\mathbb{R}}_{1,\overline{k}-1}\big|_{[0,+\infty)} + \sum_{n\geq k_0^*} c_{1n}\varphi^{\mathbb{R}}_{0n}\big|_{[0,+\infty)} \\ &\quad + \sum_{n\geq m_0^*+\left\lfloor\frac{\delta}{2}\right\rfloor-1} d_{1n}\psi^{\mathbb{R}}_{0n}\big|_{[0,+\infty)} \in S_1 \oplus V_0^I \oplus W_0^I\end{aligned}$$

(for some c_{1n} and d_{1n}). If $l > 1$, using induction, we similarly get

$$\begin{aligned}\varphi^{\mathbb{R}}_{0,k_0^*-l}\big|_{[0,+\infty)} &= 2^{\left(\frac{1}{2}-\frac{1}{p}\right)}\tilde{h}_{\tilde{n}_1}\varphi^{\mathbb{R}}_{1,\overline{k}-2l+1}\big|_{[0,+\infty)} + \sum_{n\geq k_0^*-l+1} c_{l,n}\varphi^{\mathbb{R}}_{0n}\big|_{[0,+\infty)} \\ &\quad + \sum_{n\geq m_0^*+\left\lfloor\frac{\delta}{2}\right\rfloor-l} d_{l,n}\psi^{\mathbb{R}}_{0n}\big|_{[0,+\infty)} \in S_l \oplus V_0^I \oplus W_0^I\end{aligned}$$

(again c_{ln} and d_{ln} are fixed coefficients). Thus we have proven (72), and this completes the proof. □

Using this result, setting

$$\psi_{0,m_0^*-l} := \varphi^{\mathbb{R}}_{1,\overline{k}-2l+1}\big|_{[0,+\infty)} - P_0\left(\varphi^{\mathbb{R}}_{1,\overline{k}-2l+1}\big|_{[0,+\infty)}\right), \qquad l = 1, \dots, m_0^*\,, \tag{73}$$

and

$$W_0^B = \{\psi_{0m} \mid m = 0, \dots, m_0^* - 1\}\,, \tag{74}$$

we get

$$W_0\left(\mathbb{R}^+\right) = W_0^B \oplus W_0^I\,, \tag{75}$$

with W_0^I as defined in (63). With the same process we can build

$$\widetilde{W}_0\left(\mathbb{R}^+\right) = \widetilde{W}_0^B \oplus \widetilde{W}_0^I \, .$$

As for the scaling functions, we must find a couple of biorthogonal bases for the spaces $W_0\left(\mathbb{R}^+\right)$ and $\widetilde{W}_0\left(\mathbb{R}^+\right)$. To fix notations, we suppose that $\widetilde{m}_0^* \leq m_0^*$ (otherwise we only have to exchange $\widetilde{m}_0^*$ and m_0^* in what follows). From the biorthogonality properties on the real line we have

$$\langle \psi_{0m}, \widetilde{\psi}_{0n} \rangle = \delta_{mn} \, , \qquad \forall m, n \geq m_0^* \, .$$

Note, however, that the modified wavelets we have defined are no longer orthogonal to the interior ones. In fact, from definition (73), it follows that

$$\langle \psi_{0m}, \widetilde{\psi}_{0n} \rangle = \left\langle \varphi^{\mathbb{R}}_{1,\overline{k}-2(m_0^*-m)+1}, \widetilde{\psi}_{0n} \right\rangle , \qquad m = 0, \dots, m_0^* - 1 \, , \; n \geq m_0^* \, .$$

Using the refinement equation for wavelets on the real line, we easily show that

(76)
$$\begin{aligned} &\langle \psi_{0m}, \widetilde{\psi}_{0n} \rangle = 0, \quad n = 0, \dots, \widetilde{m}_0^* - 1 \quad \text{if } m \geq \left\lceil \frac{\overline{\tilde{k}} + \tilde{n}_1 - 1}{2} \right\rceil =: m^* \\ &\langle \psi_{0m}, \widetilde{\psi}_{0n} \rangle = 0, \quad m = 0, \dots, m_0^* - 1 \quad \text{if} n \geq \left\lceil \frac{\overline{k} + n_1 - 1}{2} \right\rceil =: \widetilde{m}^* \, . \end{aligned}$$

Observing that $m^* \geq \widetilde{m}^*$, it is sufficient to find two $m^* \times m^*$ matrices $E = (e_{mr})$ and $\widetilde{E} = (\tilde{e}_{ns})$ such that

$$\left\langle \sum_{r=0}^{m^*-1} e_{mr} \psi_{0r}, \sum_{s=0}^{m^*-1} \tilde{e}_{ns} \widetilde{\psi}_{0s} \right\rangle = \delta_{mn} \, , \qquad \forall m, n = 0, \dots, m^* - 1 \, .$$

Calling Y the $m^* \times m^*$ matrix of components $Y_{mn} = \langle \psi_{0m}, \widetilde{\psi}_{0n} \rangle$, this condition is equivalent to

$$E Y \widetilde{E}^T = I \, .$$

Again, it is enough to prove that the matrix Y is non-singular. In fact this follows from the assumed invertibility of the matrix X (defined in (45)); since

$$\det Y \neq 0 \;\text{ iff }\; W_0\left(\mathbb{R}^+\right) \cap \widetilde{W}_0\left(\mathbb{R}^+\right)^{\perp} = \{0\} \, ,$$

we immediately get the result observing that

$$W_0\left(\mathbb{R}^+\right) \cap \widetilde{W}_0\left(\mathbb{R}^+\right)^{\perp} \subset V_1\left(\mathbb{R}^+\right) \cap \widetilde{V}_1^{\perp} = \{0\} \, .$$

Moreover

$$W_0\left(\mathbb{R}^+\right) \subset \widetilde{V}_0^{\perp} \, ;$$

indeed, for any $v \in L^p\left(\mathbb{R}^+\right)$, we have

$$\langle v - P_0 v, \widetilde{\varphi}_{0k} \rangle = \check{v}_{0k} - \sum_{l \geq 0} \check{v}_{0l} \langle \varphi_{0l}, \widetilde{\varphi}_{0k} \rangle = 0 \, .$$

Finally, for any $j \in \mathbb{N}$, we set

(77)
$$W_j\left(\mathbb{R}^+\right) = T_j W_0\left(\mathbb{R}^+\right) \qquad \text{and} \qquad \widetilde{W}_j\left(\mathbb{R}^+\right) = \widetilde{T}_j \widetilde{W}_0\left(\mathbb{R}^+\right) \; ;$$

setting $\psi_{jm} = T_j \psi_{0m}$, $\widetilde{\psi}_{jm} = \widetilde{T}_j \widetilde{\psi}_{0m}$ for every $j, m \in \mathbb{N}$, it is easy to check that the biorthogonality relations

$$\langle \psi_{jm}, \widetilde{\psi}_{j'n} \rangle = \delta_{jj'} \delta_{mn}\,, \qquad \forall j, j', m, n \geq 0\,,$$

hold. Moreover, with a proof similar to the one of Proposition 3, one has

PROPOSITION 10. *The bases $\Psi_j = \{\psi_{jm} : m \in \mathbb{N}\}$ of $W_j\left(\mathbb{R}^+\right)$, are uniformly p-stable bases and the bases $\widetilde{\Psi}_j = \left\{\widetilde{\psi}_{jm} : m \in \mathbb{N}\right\}$ of $\widetilde{W}_j\left(\mathbb{R}^+\right)$, are uniformly p'-stable bases for all $j \geq 0$.*

Moreover, let us state a characterization theorem for Besov spaces based on the biorthogonal multilevel decomposition $\left(V_j\left(\mathbb{R}^+\right), \widetilde{V}_j\right)_{j\geq 0}$ as described in Section 3. With the notation of the Introduction, for $1 < p, q < +\infty$, let us set

$$X^s_{pq} := \begin{cases} B^s_{pq}\left(\mathbb{R}^+\right) & \text{if} \quad s \geq 0\,, \\ \left(B^{-s}_{p'q'}\left(\mathbb{R}^+\right)\right)' & \text{if} \quad s < 0\,, \end{cases}$$

and denote by $\mathcal{D}\left(\mathbb{R}^+\right)$ the space of distributions.

THEOREM 3. *Let $\varphi \in B^{s_0}_{pq}\left(\mathbb{R}^+\right)$ for some $s_0 > 0$, $1 < p, q < +\infty$. For all $s \in S := (-\min(s_0, L), \min(s_0, L)) \setminus \{0\}$, the following characterization holds:*

$$X^s_{pq} = \begin{cases} \left\{ v \in L^p\left(\mathbb{R}^+\right) : \sum_{j\geq 0} 2^{sqj} \left(\sum_{k\geq 0} |v_{jk}|^p\right)^{q/p} < +\infty \right\} & \text{if} \quad s > 0\,, \\ \\ \Big\{ v \in \mathcal{D}\left(\mathbb{R}^+\right) : v \in X^{\overline{s}}_{pq},\ \text{for some } \overline{s} \in S \text{ and} \\ \qquad\qquad \sum_{j\geq 0} 2^{sqj} \left(\sum_{k\geq 0} |v_{jk}|^p\right)^{q/p} < +\infty \Big\} & \text{if} \quad s < 0\,, \end{cases}$$

where

$$v_{jk} = \begin{cases} \hat{v}_{jk} = \langle v, \widetilde{\psi}_{jk} \rangle & \text{if} \quad s > 0\,, \\ \tilde{\hat{v}}_{jk} = \langle v, \psi_{jk} \rangle & \text{if} \quad s < 0\,. \end{cases}$$

In addition, for all $s \in S$ and $v \in X^s_{pq}$, we have

$$|v|_{X^s_{pq}} \asymp \left(\sum_{j\geq 0} 2^{sqj} \left(\sum_{k\geq 0} |v_{jk}|^p \right)^{q/p} \right)^{1/q}. \tag{78}$$

Finally, if $p = q = 2$, the characterization and the norm equivalence hold for all index $s \in (-\min(s_0, L), \min(s_0, L))$.

Proof. It is sufficient to apply Theorems 1 and 2 (since the Bernstein- and Jackson- type inequalities have been proven in Proposition 6 and 7, respectively) and remember the interpolation result (1). $\square$

REMARK 4. It is possible to obtain a characterization result using the same representation of a function $v \in X^s_{pq}$ for both positive and negative s. Indeed, if $\varphi \in B^{s_0}_{pq}(\mathbb{R})$, $\widetilde{\varphi} \in B^{\tilde{s}_0}_{pq}(\mathbb{R})$, given any $s \in (-\min(\tilde{s}_0, \widetilde{L}), \min(s_0, L)) \setminus \{0\}$ and $v = \sum_{j,k\geq 0} \hat{v}_{jk} \psi_{jk} \in X^s_{pq}$, we have the same norm equivalence as in (78) with $\hat{v}_{jk}$ instead of v_{jk} (see, e.g., [14]). We also observe that, in general, $\tilde{s}_0 < s_0$ and thus Theorem 3 gives characterization for a larger interval.

6. Boundary values of scaling functions and wavelets

The aim of this section is to construct scaling functions and wavelets satisfying certain boundary value conditions, in view of the characterization of spaces arising from homogeneous boundary value problems. More precisely, we will see that it is possible to construct a basis of scaling functions in such a way that only one scaling function is non-zero at zero both for the primal and the dual systems. A similar property will be shown for the wavelet basis.

Let us start considering the scaling function case. Observe that all the interior scaling functions are zero at zero, while the value at zero of the boundary scaling functions depends upon the choice of the polynomial basis $\{p_\alpha\}$ of $\mathbb{P}_{L-1}$ and the biorthogonalization. If we start with a properly chosen polynomial basis (e.g., $p_\alpha(x) = x^\alpha$), only φ_{00} and $\widetilde{\varphi}_{00}$ (see (35)) do not vanish at zero. Thus, the idea is to biorthogonalize first the functions in the sets $\Phi^* = \{\varphi_{0k} : k = 1, \dots, \widetilde{L}-1\}$ and $\widetilde{\Phi}^* = \{\widetilde{\varphi}_{0k} : k = 1, \dots, \widetilde{L}-1\}$, so that the resulting systems contain functions which all vanish at zero. To do this, we need to check the non-singularity of the Gramiam matrix $\{\langle \varphi_{0k}, \widetilde{\varphi}_{0k'}\rangle\}_{k,k'=1,\dots,\widetilde{L}-1}$ obtained from the matrix X (see (45)) by deleting the first row and column. As for the non-singularity of the whole matrix we have to check this case by case. For example, for the B-splines case, this property is satisfied due to the total positivity of the associated matrix X (see Proposition 12 and also [16]). From now on we suppose this property is verified and biorthogonalize Φ^* and $\widetilde{\Phi}^*$. For simplicity, we will maintain the same notations for the new basis functions.

The second step consists in the biorthogonalization of the complete systems, keeping invariant the functions in Φ^* and $\widetilde{\Phi}^*$. Precisely, we have the following general result.

PROPERTY 1. *Let Φ^* and $\widetilde{\Phi}^*$ be the two biorthogonal systems described above. Consider*

$$\Phi = \{\varphi_{00}\} \cup \Phi^*, \qquad \widetilde{\Phi} = \{\widetilde{\varphi}_{00}\} \cup \widetilde{\Phi}^*$$

and suppose that the matrix $(\langle \varphi_{0k}, \widetilde{\varphi}_{0k'}\rangle)$, $k, k' = 0, \dots, \widetilde{L}-1$, is non-singular. Then, it is possible to construct new biorthogonal systems spanning the same sets as Φ and $\widetilde{\Phi}$, respectively, in which only the two functions $\varphi_{00}, \widetilde{\varphi}_{00}$ have been modified.

Proof. Let us set

$$\varphi_{00}^{\#} = \sum_{k=1}^{\widetilde{L}-1} \alpha_k \varphi_{0k} + \alpha_0 \varphi_{00}, \qquad \widetilde{\varphi}_{00}^{\#} = \sum_{l=1}^{\widetilde{L}-1} \beta_l \widetilde{\varphi}_{0l} + \beta_0 \widetilde{\varphi}_{00}.$$

We want to prove that we can find $\alpha_k, \beta_l, k, l = 0, \dots, \widetilde{L}-1$ so that $\alpha_0\beta_0 \neq 0$,

(79) $$\langle \varphi_{00}^{\#}, \widetilde{\varphi}_{0l}\rangle = \langle \varphi_{0k}, \widetilde{\varphi}_{00}^{\#}\rangle = 0, \qquad k, l = 1, \dots, \widetilde{L}-1,$$

and

(80) $$\langle \varphi_{00}^{\#}, \widetilde{\varphi}_{00}^{\#}\rangle = 1.$$

Imposing the conditions (79) and using the biorthogonality of the systems Φ^*, $\widetilde{\Phi}^*$, we get

$$\alpha_k = -\alpha_0 \langle \varphi_{00}, \widetilde{\varphi}_{0k}\rangle \qquad \text{and} \qquad \beta_l = -\beta_0 \langle \widetilde{\varphi}_{0l}, \widetilde{\varphi}_{00}\rangle, \qquad k, l = 1, \dots, \widetilde{L}-1.$$

Substituting these relations in (80), we end up with the identity

$$K\alpha_0\beta_0 = 1 \qquad \text{where} \qquad K = \left\langle \varphi_{00}, \widetilde{\varphi}_{00} - \sum_{k=1}^{\widetilde{L}-1} \langle \widetilde{\varphi}_{00}, \varphi_{0k}\rangle \widetilde{\varphi}_{0k} \right\rangle.$$

To conclude the proof it is enough to show that $K \neq 0$. Indeed, if $K = 0$, the function $\eta = \widetilde{\varphi}_{00} - \sum_{k=1}^{\widetilde{L}-1} \langle \widetilde{\varphi}_{00}, \varphi_{0k} \rangle \widetilde{\varphi}_{0k} \in (\text{span}\, \Phi^*)^{\perp}$, would also be orthogonal to φ_{00}; moreover $\eta \in \text{span}\, \widetilde{\Phi}$ and the systems Φ and $\widetilde{\Phi}$ are biorthogonalizable, i.e., $(\text{span}\, \Phi)^{\perp} \cap (\text{span}\, \widetilde{\Phi}) = \{0\}$; this would mean $\eta = 0$, contradicting the linear independence of the functions in $\widetilde{\Phi}$. □

Next we consider the wavelet case. Suppose we have constructed biorthogonal wavelets $\{\psi_{jk}\}_{k\geq 0}$, $\{\widetilde{\psi}_{jk}\}_{k\geq 0}$ starting from biorthogonal scaling systems $\{\varphi_{jk}\}_{k\geq 0}$, $\{\widetilde{\varphi}_{jk}\}_{k\geq 0}$ such that

$$\varphi_{j0}(0)\widetilde{\varphi}_{j0}(0) \neq 0 \quad \text{and} \quad \varphi_{jk}(0) = \widetilde{\varphi}_{jk}(0) = 0\,, \quad \forall k \geq 1\,,\ \forall j \geq 0\,. \tag{81}$$

Recalling that $\varphi_{j+1,k} = T_1 \varphi_{jk}$, one has

$$\varphi_{j+1,0}(0) = 2^{1/p} \varphi_{j0}(0)\,, \qquad \widetilde{\varphi}_{j+1,0}(0) = 2^{1/p'} \widetilde{\varphi}_{j0}(0)\,. \tag{82}$$

We want to prove that we can modify the wavelet systems and obtain a property similar to (81).

To this end, we report general observations about biorthogonal bases that can be found in the Appendix of [7]. Let S, $\widetilde{S}$ be two spaces of functions defined on some set Ω with biorthogonal bases $E = \{\eta_l\}_{l\in\mathcal{L}}$ and $\widetilde{E} = \{\tilde{\eta}_l\}_{l\in\mathcal{L}}$ (here $\mathcal{L}$ is some set of indices) with respect to some bilinear form ${}_S\langle \cdot, \cdot \rangle_{\widetilde{S}}$ on $S \times \widetilde{S}$. Let $F = \{\nu_l\}_{l\in\mathcal{L}}$ and $\widetilde{F} = \{\tilde{\nu}_l\}_{l\in\mathcal{L}}$ be two other bases such that $\nu_l = K_{lm}\eta_m$, $\tilde{\nu}_l = \widetilde{K}_{lm}\tilde{\eta}_m$, where K and $\widetilde{K}$ are suitable generalized matrices. It is easy to check that, to preserve the biorthogonality, we must have $\widetilde{K} = K^{-T}$.

LEMMA 1. *With the previous notation, suppose the elements of E and $\widetilde{E}$ are continuous functions, then the quantity*

$$\sum_{l\in\mathcal{L}} \eta_l(x)\tilde{\eta}_l(x), \quad \forall x \in \Omega$$

is invariant under any change of biorthogonal basis.

Proof. Let us denote by $e(x)$ the vector $(\eta_l(x))_{l\in\mathcal{L}}$, and similarly for $\widetilde{e}(x)$, $f(x)$, $\widetilde{f}(x)$. Note that $f(x) = Ke(x)$ and $\widetilde{f}(x) = K^{-T}\widetilde{e}(x)$; thus

$$\sum_{l\in\mathcal{L}} \nu_l(x)\tilde{\nu}_l(x) = f(x)^T \cdot \widetilde{f}(x) = e(x)^T K^T \cdot K^{-T}\widetilde{e}(x) = e(x)^T \cdot \widetilde{e}(x) = \sum_{l\in\mathcal{L}} \eta_l(x)\tilde{\eta}_l(x)\,.$$

□

We will apply this result to our biorthogonal wavelets.

COROLLARY 2. *Suppose the scaling functions φ, $\widetilde{\varphi}$ are continuous on $\mathbb{R}$, then*

$$\sum_{k\geq 0} \psi_{0k}(0)\widetilde{\psi}_{0k}(0) \neq 0\,. \tag{83}$$

Proof. Let $S = V_1(\mathbb{R}^+)$, $\widetilde{S} = \widetilde{V}_1(\mathbb{R}^+)$. Using the relations $V_1(\mathbb{R}^+) = V_0(\mathbb{R}^+) \oplus W_0(\mathbb{R}^+)$ and $\widetilde{V}_1(\mathbb{R}^+) = \widetilde{V}_0(\mathbb{R}^+) \oplus \widetilde{W}_0(\mathbb{R}^+)$, we have two couples of biorthogonal bases on S and $\widetilde{S}$: $E = \{\varphi_{1k}\}_{k\geq 0}$, $\widetilde{E} = \{\widetilde{\varphi}_{1k}\}_{k\geq 0}$ and $F = \{\varphi_{0k}\}_{k\geq 0} \cup \{\psi_{0k}\}_{k\geq 0}$, $\widetilde{F} = \{\widetilde{\varphi}_{0k}\}_{k\geq 0} \cup \{\widetilde{\psi}_{0k}\}_{k\geq 0}$. Using the previous Lemma, (81) and (82), we have

$$\varphi_{10}(0)\widetilde{\varphi}_{10}(0) = 2\varphi_{00}(0)\widetilde{\varphi}_{00}(0) = \varphi_{00}(0)\widetilde{\varphi}_{00}(0) + \sum_{k\geq 0} \psi_{0k}(0)\widetilde{\psi}_{0k}(0)\,.$$

Thus

$$\sum_{k\geq 0}\psi_{0k}(0)\widetilde{\psi}_{0k}(0)=\varphi_{00}(0)\widetilde{\varphi}_{00}(0)\neq 0\,.$$

□

This implies that we can always find $k \geq 0$ such that $\psi_{0k}(0)\widetilde{\psi}_{0k}(0) \neq 0$. Without loss of generality, we can suppose $k = 0$. Let us define, for $k \geq 1$,

$$\psi^*_{0k}(x) := \psi_{0k}(x) - \frac{\psi_{0k}(0)}{\psi_{00}(0)}\psi_{00}(x) =: \psi_{0k}(x) - c_k\psi_{00}(x) \tag{84}$$

and

$$\widetilde{\psi}^*_{0k}(x) := \widetilde{\psi}_{0k}(x) - \frac{\widetilde{\psi}_{0k}(0)}{\widetilde{\psi}_{00}(0)}\widetilde{\psi}_{00}(x) =: \widetilde{\psi}_{0k}(x) - \tilde{c}_k\widetilde{\psi}_{00}(x)\,. \tag{85}$$

Observe that $\psi^*_{0k}(0) = \widetilde{\psi}^*_{0k}(0) = 0$, $\forall k \geq 1$; moreover only a finite number of wavelets are modified, since all the interior ones vanish at the origin. Thus, to end up our construction, it is enough to show that it is possible to biorthogonalize the systems $\{\psi^*_{0k}\}_{k\geq 1}$ and $\{\widetilde{\psi}^*_{0k}\}_{k\geq 1}$. Indeed

LEMMA 2. *The matrix $Y^* = \{\langle\psi^*_{0k}, \widetilde{\psi}^*_{0l}\rangle\}_{k,l=1}^{m^*-1}$ is non-singular.*

Proof. Let us set $M = m^* - 1$; for $k, l = 1, \dots, M$ we have

$$\langle\psi^*_{0k}, \widetilde{\psi}^*_{0l}\rangle = \begin{cases} 1 + c_k\tilde{c}_k & \text{if } k = l \\ c_k\tilde{c}_l & \text{if } k \neq l\,. \end{cases}$$

It is easy to see that Y^* has only two different eigenvalues: $\lambda_1 = 1$, with multiplicity $M - 1$ and $\lambda_2 = 1 + \sum_{k=1}^{M} c_k\tilde{c}_k$, with multiplicity 1. Thus, by Corollary 2,

$$\det Y^* = 1 + \sum_{k=1}^{M} c_k\tilde{c}_k = \frac{\sum_{k\geq 0}\psi_{0k}(0)\widetilde{\psi}_{0k}(0)}{\psi_{00}(0)\widetilde{\psi}_{00}(0)} \neq 0\,.$$

□

Finally we construct our wavelet systems as described in Property 1.

7. Characterization of Besov spaces $B^s_{pq,00}(\mathbb{R}^+)$

We now prove that we can build multiresolution analyses on $L^p(\mathbb{R}^+)$ of functions satisfying homogeneous boundary value conditions in zero. This can be done by choosing sufficiently regular scaling functions φ and $\widetilde{\varphi}$ and by building boundary functions starting from particular bases of polynomials.

Suppose $\varphi \in B^{s_0}_{pq}(\mathbb{R})$ (or $W^{s_0,p}(\mathbb{R})$) and $S = [s_0] < L$. Let us consider the basis $p(x) = x^\alpha$ of the monomials, and let us build the functions on the boundary as in (35). We remove the first S boundary functions and define (see (42))

$${}_0V^B_0(\mathbb{R}^+) = \operatorname{span}\{\varphi_{0k} : S \leq k \leq \widetilde{L} - 1\} := \operatorname{span}{}_0\Phi_0 \tag{86}$$

and

$$
{}_0\widetilde{V}_0^B\left(\mathbb{R}^+\right) = \operatorname{span}\left\{\widetilde{\varphi}_{0k} : S \le k \le \widetilde{L} - 1\right\} := \operatorname{span}{}_0\widetilde{\Phi}_0 . \tag{87}
$$

The systems ${}_0\Phi_0$ and ${}_0\widetilde{\Phi}_0$ can be biorthogonalized if the matrix ${}_0X = (\langle\varphi_{0k}, \widetilde{\varphi}_{0l}\rangle)$, with $k, l = S, \dots, \widetilde{L} - 1$ is non-singular. For example, this condition is verified in the B-spline case (see Proposition 12 and also [16]).

As before, we set

$$
{}_0V_j\left(\mathbb{R}^+\right) := T_j\left({}_0V_0\left(\mathbb{R}^+\right)\right) \quad \text{and} \quad {}_0\widetilde{V}_j\left(\mathbb{R}^+\right) := \widetilde{T}_j\left({}_0\widetilde{V}_0\left(\mathbb{R}^+\right)\right).
$$

Let us define ${}_0P_0 : L^p\left(\mathbb{R}^+\right) \longrightarrow {}_0V_0\left(\mathbb{R}^+\right)$ as

$$
{}_0P_0 v = \sum_{k\ge S} \check{v}_{0k}\varphi_{0k}\,, \qquad \forall v \in L^p\left(\mathbb{R}^+\right). \tag{88}
$$

For any level $j > 0$, let us set

$$
{}_0P_j = T_j \circ {}_0P_0 \circ T_j^{-1}\,. \tag{89}
$$

Similar definitions hold for the dual operators ${}_0\widetilde{P}_j$. These sequences of operators satisfy the requirements (7), (8) and (9). Following the same construction as in Sections 3 and 5, we build a multiresolution analysis that will be used to characterize the Besov spaces $B^s_{pq,00}$ (see (3)). For the scaling spaces the situation is basically the same, in fact we have only dropped some functions on the boundary.

Many results hold in this context; for example, since ${}_0V_0\left(\mathbb{R}^+\right)$ is a subspace of $V_0\left(\mathbb{R}^+\right)$, one has

$$
\|v\|_{L^p(\mathbb{R}^+)} \asymp \left(\sum_{k\ge S} |\check{v}_{0k}|^p\right)^{1/p},
$$

for any $v \in {}_0V_0\left(\mathbb{R}^+\right)$ (see Proposition 5).

We note that while the number of boundary wavelets does not change, they are defined in a different way since their definition depends on the projector ${}_0P_0$. In fact, one has

$$
\begin{aligned}
{}_0\psi_{0,m_0^*-k} &:= \varphi^{\mathbb{R}}_{1,\overline{k}-2k+1}\big|_{[0,+\infty)} - {}_0P_0\varphi^{\mathbb{R}}_{1,\overline{k}-2k+1}\big|_{[0,+\infty)} \\
&= \psi_{0,m_0^*-k} + \sum_{l=0}^{S-1}\left\langle\varphi^{\mathbb{R}}_{1,\overline{k}-2k+1}, \widetilde{\varphi}_{0l}\right\rangle\varphi_{0l}
\end{aligned}
$$

for $k = 1, \dots, m_0^*$.

7.1. Bernstein and Jackson inequalities

In order to use the characterization Theorems 1 and 2, we will prove Jackson- and Bernstein-type inequalities for Besov spaces $B^s_{pq,0}\left(\mathbb{R}^+\right)$ or Sobolev spaces $W_0^{s,p}\left(\mathbb{R}^+\right)$. Let us observe that the Bernstein inequality follows from (53), because ${}_0V_0\left(\mathbb{R}^+\right) \subset V_0\left(\mathbb{R}^+\right)$ and $B^s_{pq,0}\left(\mathbb{R}^+\right) \subset B^s_{pq}\left(\mathbb{R}^+\right)$ with the same semi-norms. It only remains to prove a Jackson-type inequality, that cannot be deduced directly from (55) because it depends on the projectors (88).

PROPOSITION 11. *For each s such that $0 < S \le s \le s_0 < L$, we have*

$$\|v - {}_0P_j v\|_{L^p(\mathbb{R}^+)} \lesssim 2^{-js} |v|_{B^s_{pq}(\mathbb{R}^+)}\,, \quad \forall v \in B^s_{pq,0}\left(\mathbb{R}^+\right)\,,\ \forall j \in \mathbb{N}\,.$$

Proof. We can proceed as in Proposition 7, the only difference being on the first interval $I_0 = [0,1]$ and on the space of polynomials to be considered. Indeed, we consider the subspace ${}_0\mathbb{P}_{L-1}$ of $\mathbb{P}_{L-1}$, i.e., the space of polynomials which are zero at zero with all their derivatives of order less than S. Then, inequality (57) holds for every $v \in B^s_{pq,0}\left(\mathbb{R}^+\right)$ (straightforward modifications of the proof of Theorem 4.2, p. 183, in [19]). □

Finally, we state a characterization theorem for Besov spaces $B^s_{pq,00}\left(\mathbb{R}^+\right)$ based on the biorthogonal multilevel decomposition $\left({}_0V_j\left(\mathbb{R}^+\right), {}_0\widetilde{V}_j\left(\mathbb{R}^+\right)\right)$ as described before. This result immediately follows from Theorem 1.

THEOREM 4. *Let $\varphi \in B^{s_0}_{pq}\left(\mathbb{R}^+\right)$ for some $0 < s_0 < L$, $1 < p, q < \infty$. For all $0 < s < s_0$, we have*

$$B^s_{pq,00}\left(\mathbb{R}^+\right) = \left\{ v \in L^p\left(\mathbb{R}^+\right) : \sum_{j \ge 0} 2^{sqj} \left(\sum_{k \ge S} |\hat{v}_{jk}|^p \right)^{q/p} < \infty \right\} \tag{90}$$

and

$$|v|_{B^s_{pq}(\mathbb{R}^+)} \asymp \left(\sum_{j \ge 0} 2^{sqj} \left(\sum_{k \ge S} |\hat{v}_{jk}|^p \right)^{q/p} \right)^{1/p}, \quad \forall v \in B^s_{pq,00}\left(\mathbb{R}^+\right)\,. \tag{91}$$

REMARK 5. Since, for any $s \in \mathbb{R}$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$,

$$\left(B^s_{pq,00}\left(\mathbb{R}^+\right) \right)' = B^{-s}_{p'q'}\left(\mathbb{R}^+\right),$$

(see [23] p. 235) the extension of the previous theorem to the dual spaces (negative s) gives the same result of Theorem 3.

REMARK 6. As usual, if $p = q = 2$, s can assume the value 0. In this particular case, if $0 \le s_0 < L$, we get

$$\begin{aligned} H^s_{00}\left(\mathbb{R}^+\right) &= \left\{ v \in L^2\left(\mathbb{R}^+\right) : \sum_{j \ge 0} \sum_{k \ge S} 2^{2sj} |\hat{v}_{jk}|^2 < +\infty \right\} \\ &= \begin{cases} H^s_0\left(\mathbb{R}^+\right) & \text{if} \quad s - \frac{1}{2} \notin \mathbb{N}\,, \\ H^s_{00}\left(\mathbb{R}^+\right) & \text{if} \quad s - \frac{1}{2} \in \mathbb{N}\,, \end{cases} \end{aligned}$$

and

$$\|v\|_{H^s(\mathbb{R}^+)} \asymp \left(\sum_{j \ge 0} \sum_{k \ge S} (1 + 2^{2sj}) |\hat{v}_{jk}|^2 \right)^{1/2}, \quad \forall v \in H^s_{00}\left(\mathbb{R}^+\right)\,. \tag{92}$$

REMARK 7. In Theorem 4, we have described how to characterize the family of Besov spaces $B^s_{pq,00}\left(\mathbb{R}^+\right)$ using a scaling fuction $\varphi \in B^{s_0}_{pq}\left(\mathbb{R}^+\right)$ and removing $S = [s_0]$ boundary scaling functions. Of course, one can remove only $\overline{S} < S$ boundary scaling functions and proceed as before to construct a multiresolution analysis. In this case we characterize $B^s_{pq,00}\left(\mathbb{R}^+\right)$ with $0 < s \le \overline{S}$ and $B^{\overline{S}}_{pq,00}\left(\mathbb{R}^+\right) \cap B^s_{pq}\left(\mathbb{R}^+\right)$ for every $\overline{S} < s < s_0$.

8. Biorthogonal decomposition of the unit interval

In the previous sections, we have carried out the construction of a multiresolution analysis taking into account the presence of a boundary point. Now we want to exploit it to build a multilevel decomposition of the bounded interval $(0, 1)$. Intuitively one easily sees that, provided the scale is finer enough, the presence of the left boundary point does not influence the construction at the right one. More precisely, we will choose a level j_0 such that

$$V_j(0,1) = \operatorname{span}\left\{\varphi^{(0)}_{jl} : l \in \mathcal{I}_L\right\} \oplus \operatorname{span}\{\varphi_{jk} : k \in \mathcal{I}_I\} \oplus \operatorname{span}\left\{\varphi^{(1)}_{jr} : r \in \mathcal{I}_R\right\}\ , \ \forall j \ge j_0\,,$$

with $\mathcal{I}_I \neq \emptyset$ and where the boundary functions $\varphi^{(0)}_{jl}$ and $\varphi^{(1)}_{jr}$ are constructed independently. Here, and from now on, the suffix $^{(0)}$ or $^{(1)}$ refers to the boundary point 0 or 1, respectively.

The basic idea is to start from two multiresolution analyses on the half-lines $I_0 = (0, +\infty)$ and $I_1 = (-\infty, 1)$, and paste them together in a suitable way to get the spaces $V_j(0, 1)$. In turns, to obtain a decomposition on I_1, we first consider a decomposition on $\mathbb{R}^- = (-\infty, 0)$ and then we translate it of a unit.

Let us choose two bases for $\mathbb{P}_{L-1}$ and $\mathbb{P}_{\widetilde{L}-1}$, say $\left\{p^{(1)}_\alpha : \alpha = 0, \dots, L-1\right\}$ and $\left\{\tilde{q}^{(1)}_\beta : \beta = 0, \dots, \widetilde{L}-1\right\}$, possibly different from the ones used to build V^B_0 and $\widetilde{V}^B_0$ for $(0, +\infty)$. Fixing two nonnegative integers δ_1 and $\tilde{\delta}_1$, let us define the boundary functions as in (35)

$$\phi^{(0^-)}_\alpha(x) = \sum_{k=1-\delta_1-n_1}^{-n_0-1} c^{(1)}_{\alpha k}\varphi^{\mathbb{R}}_{0k}(x)\,, \qquad x \le 0\,, \qquad \forall \alpha = 0, \dots, L-1\,.$$

Matching the dimensions of $V_0\left(\mathbb{R}^-\right)$ and $\widetilde{V}_0\left(\mathbb{R}^-\right)$, we obtain a relation similar to (40):

$$\tilde{\delta}_1 - \delta_1 = \widetilde{L} - L - (\tilde{n}_1 - n_1)\,. \tag{93}$$

Recalling the definition of the isometries T_j (see (49)), we define

$$V_j\left(\mathbb{R}^-\right) = \operatorname{span}\left\{\phi^{(0^-)}_{j\alpha} = T_j\left(\phi^{(0^-)}_\alpha\right) : \alpha = 0, \dots, L-1\right\} \oplus \operatorname{span}\left\{\varphi^{\mathbb{R}}_{jk} : k \le -\delta_1 - n_1\right\};$$

using the operator $\tau : x \mapsto x - 1$, we translate the origin into the right edge of our interval. It is easy to see that, calling

$$\phi^{(1)}_{j\alpha}(x) = \sum_{k=2^j+1-\delta_1-n_1}^{2^j-n_0-1} c^{(1)}_{\alpha,k-2^j}\,\varphi^{\mathbb{R}}_{jk}(x)\,, \qquad x \le 1\,,$$

we have

$$V_j(-\infty,1) = \operatorname{span}\left\{\phi^{(1)}_{j\alpha} : \alpha = 0, \dots, L-1\right\} \oplus \operatorname{span}\left\{\varphi^{\mathbb{R}}_{jk} : k \le 2^j - \delta_1 - n_1\right\}.$$

As said before, we wish to maintain the situation at the two boundary points decoupled. This means we want to have at least one interior function in $V_j(0,1)$. This requirement yields the condition

$$-n_0 + \delta_0 \leq 2^j - \delta_1 - n_1 \,.$$

Keeping also into account the dual relation, we have to set a coarsest level j_0 such that for all $j \geq j_0$,

$$2^j \geq \max\{n_1 - n_0 + \delta_0 + \delta_1, \tilde{n}_1 - \tilde{n}_0 + \tilde{\delta}_0 + \tilde{\delta}_1\}. \tag{94}$$

By (40) and (93), we have

$$\left(\tilde{n}_1 - \tilde{n}_0 + \tilde{\delta}_0 + \tilde{\delta}_1\right) - (n_1 - n_0 + \delta_0 + \delta_1) = 2\left(\widetilde{L} - L\right) \geq 0\,,$$

so we must fix

$$j_0 \geq \left\lceil \log_2\left(\tilde{n}_1 - \tilde{n}_0 + \tilde{\delta}_0 + \tilde{\delta}_1\right)\right\rceil. \tag{95}$$

Thus, we have, for $j \geq j_0$,

$$\begin{aligned} V_j(0,1) \quad &= \quad \operatorname{span}\left\{\phi_{jk}^{(0)} : k = 0, \dots, L-1\right\} \oplus \\ &\oplus \quad \operatorname{span}\left\{\varphi_{jk}^{\mathbb{R}} : k = -n_0 + \delta_0, \dots, 2^j - \delta_1 - n_1\right\} \oplus \\ &\oplus \quad \operatorname{span}\left\{\phi_{jk}^{(1)} : k = 0, \dots, L-1\right\}, \end{aligned}$$

and similarly for the dual spaces $\widetilde{V}_j(0,1)$. By construction, and thanks to the choice of j_0, all the biorthogonality properties are maintained. Finally, we observe that

$$\dim V_j(0,1) = 2^j + 2L + 1 - \delta_0 - \delta_1 - \tilde{n}_1 + \tilde{n}_0\,, \qquad \forall j \geq j_0\,.$$

Since $V_{j+1}(0,1) = V_j(0,1) \oplus W_j(0,1)$, this implies

$$\dim W_j(0,1) = 2^{j+1} - 2^j = 2^j\,.$$

Going through the construction in Section 5, one easily proves that, setting

$$m_0^{\#} := \left\lceil \frac{\delta_1 + n_1 - \tilde{n}_0 + 1}{2} \right\rceil$$

one has

$$\left\{\varphi_{1k}^{\mathbb{R}} : k \leq 2(-\delta_1 - n_1) + \tilde{n}_0 + 1\right\} \subseteq V_0\left(\mathbb{R}^-\right) \oplus \operatorname{span}\left\{\psi_{0m}^{\mathbb{R}} : m \leq -m_0^{\#}\right\}.$$

As before, one out of two φ_{1k}, $k = -\delta_1 - n_1, \dots, 2(-\delta_1 - n_1) + \tilde{n}_0 + 2$ is linearly dependent modulus $V_0\left(\mathbb{R}^-\right)$ on the previous ones; observe then that $\dim W_0^B\left(\mathbb{R}^-\right) = m_0^{\#} - 1$. Therefore, defining the projection operator $P_0^{(0^-)}$ on $V_0\left(\mathbb{R}^-\right)$ and

$$\psi_{0,m_0^{\#}-k}^{(0^-)} := \varphi_{1,-2(\delta_1+n_1)+n_0+2k}^{\mathbb{R}}\big|_{(-\infty,0]} - P_0^{(0^-)}\varphi_{1,-2(\delta_1+n_1)+n_0+2k}^{\mathbb{R}}\big|_{(-\infty,0]}\,,$$

$$k = 1, \dots, m_0^{\#} - 1$$

one gets

$$W_0\left(\mathbb{R}^-\right) = \operatorname{span}\left\{\psi_{0m}^{\mathbb{R}} : m \le -m_0^{\#}\right\} \oplus \operatorname{span}\left\{\psi_{0m}^{(0^-)} : m = 1, \dots, m_0^{\#} - 1\right\}.$$

As in Section 5, we observe that

$$\left\langle \psi_{0m}^{\mathbb{R}}, \widetilde{\psi}_{0n} \right\rangle = 0, \qquad \forall n = 1, \dots, \widetilde{m}_0^{\#} - 1 \quad \text{if} \quad m \le -\left\lceil \frac{2(\tilde{n}_1 + \tilde{\delta}_1) - n_0 - \tilde{n}_0}{2} \right\rceil =: -m^{\#}$$

$$\left\langle \psi_{0m}, \widetilde{\psi}_{0n}^{\mathbb{R}} \right\rangle = 0, \qquad \forall m = 1, \dots, m_0^{\#} - 1 \quad \text{if} \quad n \le -\left\lceil \frac{2(n_1 + \delta_1) - n_0 - \tilde{n}_0}{2} \right\rceil =: -\widetilde{m}^{\#}.$$

Through the operators T_j we can define $W_j\left(\mathbb{R}^-\right) = T_j W_0\left(\mathbb{R}^-\right)$ and operating with the translation $\tau : x \mapsto x - 1$ we finally have

$$\begin{aligned} W_j(-\infty, 1) = \operatorname{span}\Big\{\psi_{jm}^{\mathbb{R}} \quad &: \quad m \le -m_0^{\#} + 2^j\Big\} \oplus \\ &\oplus \quad \operatorname{span}\left\{\psi_{jm}^{(1)} := \tau\left(T_j \psi_{0m}^{(0^-)}\right) : m = 1, \dots, m_0^{\#} - 1\right\}. \end{aligned}$$

As for the scaling spaces, we can paste the wavelet spaces at each level and we obtain

$$\begin{aligned} W_j(0,1) \quad &= \quad \operatorname{span}\left\{\psi_{jm}^{(0)} : m = 0, \dots, m_0^{*} - 1\right\} \oplus \\ &\oplus \quad \operatorname{span}\left\{\psi_{jm} : m = m_0^{*}, \dots, 2^j - m_0^{\#}\right\} \oplus \\ &\oplus \quad \operatorname{span}\left\{\psi_{jm}^{(1)} : m = 1, \dots, m_0^{\#} - 1\right\}. \end{aligned}$$

The minimum level j_0 must be taken in order to avoid intersection between the supports of the modified border wavelets corresponding to two different edges. It is not possible to have an easy and general formulation for it, but it will be computed later on for B-spline scaling functions.

REMARK 8. Obviously, we can state characterization theorems for the scales of Besov spaces $B_{pq}^{s}(0,1)$ and $B_{pq,00}^{s}(0,1)$ (see Theorems 3, 4 and subsequent Remarks). Moreover, we can also characterize spaces of functions possibly satisfying different boundary conditions at the two edges 0 and 1.

9. The B-spline multiresolution

We will detail here the construction of biorthogonal systems on the unit interval starting from B-splines multiresolution analyses on $\mathbb{R}$. Using particular properties of the basis functions, we will show that in this case we can prove, for example, the non-singularity of the matrix X (see (45)).

For any positive integer $l > 0$, let φ be a particular B-spline function of order l with integer nodes (see, e.g., [18] for definition and basic properties). More precisely, denoting by $[x_1, \dots, x_n]f$ the n-th order divided difference of f with respect to the nodes $x_1, \dots, x_n$ and by $f_+(x) = \max\{f(x), 0\}$, we have

$$\varphi^{(l)}(x) = l[0, 1, \dots, l]\left(\cdot - x - \left\lfloor \frac{l}{2} \right\rfloor\right)_+^{l-1}.$$

It is easy to see that

$$\operatorname{supp}\varphi^{(l)} = \left[-\left\lfloor\frac{l}{2}\right\rfloor, \left\lceil\frac{l}{2}\right\rceil\right] =: [n_0, n_1]\,,$$

and that $\varphi^{(l)}$ satisfies the refinement equation

$$\varphi^{(l)}(x) = \sum_{k=n_0}^{n_1} 2^{1-l}\binom{l}{k+\lfloor\frac{l}{2}\rfloor}\varphi^{(l)}(2x-k)\,.$$

It has been shown in [11] that for any integer $\tilde{l} \geq l$ such that $l+\tilde{l}$ is even, there exists a function $\widetilde{\varphi}^{(l,\tilde{l})} \in L^2(\mathbb{R})$ that satisfies the conditions in Section 2.3 with $L = l$ and $\widetilde{L} = \tilde{l}$. This is a compactly supported function such that

$$\operatorname{supp}\widetilde{\varphi}^{(l,\tilde{l})} = \left[-\left\lfloor\frac{l}{2}\right\rfloor - \tilde{l} + 1, \left\lceil\frac{l}{2}\right\rceil + \tilde{l} - 1\right] =: [\tilde{n}_0, \tilde{n}_1]\,.$$

Let us consider a fixed couple $(l, \tilde{l})$ and drop, for a moment, the superscripts. Substituting in (40) we must have

$$\tilde{\delta}_0 - \delta_0 = 1 - l\,. \tag{96}$$

For $l = \tilde{l} = 1$, we obtain the orthogonal Haar basis and we can obviously choose $\tilde{\delta}_0 = \delta_0 = 0$. In all the other cases, observing that $\tilde{n}_1 - \tilde{n}_0 - 1 = 2\tilde{l} + l - 3 \geq \tilde{l}$, we can choose $\tilde{\delta}_0 = 0$ and $\delta_0 = l - 1$, so $k_0^* = \lfloor\frac{l}{2}\rfloor + l - 1$ and $\tilde{k}_0^* = \lfloor\frac{l}{2}\rfloor + \tilde{l} - 1$. Considering now (93), we must have

$$\tilde{\delta}_1 - \delta_1 = 1 - l\,,$$

and again we can set $\tilde{\delta}_1 = 0$ and $\delta_1 = l - 1$. Recalling the definition (94), we must fix a coarsest level j_0 such that

$$j_0 \geq \left\lceil \log_2\left(l + 2\tilde{l} - 2\right)\right\rceil.$$

For $j \geq j_0$, we have (see Section 8)

$$\begin{aligned} V_j(0,1) \quad &= \quad \operatorname{span}\left\{\phi_{jk}^{(0)} : k = 0, \ldots, l-1\right\} \oplus \\ &\oplus \quad \operatorname{span}\left\{\varphi_{jk}^{\mathbb{R}} : k = l + \left\lfloor\frac{l}{2}\right\rfloor - 1, \ldots, 2^j - l - \left\lceil\frac{l}{2}\right\rceil + 1\right\} \oplus \\ &\oplus \quad \operatorname{span}\left\{\phi_{jk}^{(1)} : k = 0, \ldots, l-1\right\}. \end{aligned}$$

From the definition, it is easy to see that, for any integer p,

$$\varphi^{(2p)}(-x) = \varphi^{(2p)}(x)\,, \qquad \varphi^{(2p+1)}(1-x) = \varphi^{(2p+1)}(x)\,, \qquad \forall x \in \mathbb{R}\,,$$

and these properties are in fact maintained by the dual functions, i.e.,

$$\widetilde{\varphi}^{(2p,\tilde{l})}(-x) = \widetilde{\varphi}^{(2p,\tilde{l})}(x)\,, \qquad \widetilde{\varphi}^{(2p+1,\tilde{l})}(1-x) = \widetilde{\varphi}^{(2p+1,\tilde{l})}(x)\,, \qquad \forall x \in \mathbb{R}\,,$$

for any suitable integer $\tilde{l}$. These symmetry features can be used to generate boundary functions. In fact, if we choose $p_\alpha^{(1)}(y) = p_\alpha^{(0)}(-y)$, it is straightforward to see that

$$\phi_{j\alpha}^{(0)}(1-x) = \phi_{j\alpha}^{(1)}(x)\,, \qquad \forall x \in [0,1]\,.$$

As we said before, in this setting we can prove that the matrix we get imposing the biorthogonality conditions is non-singular. To simplify matters, we carry out the proof for the left boundary point only, the other one being treated analogously. To this end, let us set

$$\begin{aligned}
\varphi_{0k}^{(l,\tilde{l})} &= \begin{cases} \phi_{0k}^{(0)} & k = 0, \dots, l-1\,, \\ \varphi_{0,\lfloor \frac{l}{2} \rfloor + k - 1}^{(l)} & k = l, \dots, \tilde{l} - 1\,, \end{cases} \\
\tilde{\varphi}_{0k}^{(l,\tilde{l})} &= \tilde{\phi}_{0k}^{(0)} \qquad k = 0, \dots, \tilde{l} - 1\,,
\end{aligned}$$

and

$$X\big(l, \tilde{l}\big) = \left(\left\langle \varphi_{0\alpha}^{(l,\tilde{l})}, \tilde{\varphi}_{0\beta}^{(l,\tilde{l})} \right\rangle\right)_{\alpha,\beta=0,\dots,\tilde{l}-1}.$$

We prove now a stronger proposition about the non-singularity of $X\big(l, \tilde{l}\big)$. As the proof is done using induction, when necessary we will keep track of the dependence of the variables on the parameters l and $\tilde{l}$. Let $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_k\}$ be two selections of row and column indices, respectively; suppose $k \geq 2$, $i_n \neq i_m$ for every $n \neq m$ and $j_{n+1} = j_n + 1$. Let us denote by X_I^J the corresponding $k \times k$ submatrix of $X\big(l, \tilde{l}\big)$.

PROPOSITION 12. *For all $l, \tilde{l} \geq 1$ such that $l + \tilde{l}$ is even, every submatrix $X_{i_1,\dots,i_k}^{j_1,\dots,j_k}$ of $X(l, \tilde{l})$ is non-singular.*

Proof. Using biorthogonality on the line and the definition of the modified functions on the boundary, it is easy to see that, if $\tilde{l} \geq l$,

$$\left\langle \varphi_{0\alpha}^{(l,\tilde{l})}, \tilde{\varphi}_{0\beta}^{(l,\tilde{l})} \right\rangle = \left\langle \varphi_{0\alpha}^{(l,\tilde{l})}, (\cdot)^{\beta} \right\rangle$$

for all $\alpha, \beta = 0, \dots, \tilde{l} - 1$ and any couple $\big(l, \tilde{l}\big)$.

We will prove the result by induction on l. First, let us compute the matrix arising from the Haar function $\varphi = \chi_{[0,1)}$, that is the B-spline of order $l = 1$ (observe that in this case we do not have boundary scaling functions of the form (35)). Choosing $\delta_0, \tilde{\delta}_0 = 0$ one has

$$\left\langle \varphi_{0\alpha}^{(l,\tilde{l})}, \tilde{\varphi}_{0\beta}^{(l,\tilde{l})} \right\rangle = \int_{\alpha}^{\alpha+1} x^{\beta} dx = \frac{1}{\beta+1}\left[(\alpha+1)^{\beta+1} - \alpha^{\beta+1}\right].$$

Setting $P_\beta(x) = \frac{x^{\beta+1}}{\beta+1}$ we have, for any odd $\tilde{l}$,

$$X\big(1, \tilde{l}\big) = \big(P_\beta(\alpha+1) - P_\beta(\alpha)\big)_{\alpha,\beta=0,\dots\tilde{l}-1}.$$

If, starting from the second, we add to any row the preceding one and multiply the j-th column by j, we obtain the Vandermonde matrix $V = (V_{ij}) = (i^j)$, for $i, j = 1, \dots, \tilde{l}$. It is easy to prove that V has the property stated in the proposition, which in turn means that $X(1, \tilde{l})$ has it as well.

Secondly, let us suppose that the proposition is true for all matrices $X(n, m)$ with $n + m$ even and $n < l$ and observe that

$$\left\langle \varphi_{0\alpha}^{(l,\tilde{l})}, (\cdot)^{\beta} \right\rangle = -\left\langle \frac{d}{dx}\varphi_{0\alpha}^{(l,\tilde{l})}, P_\beta \right\rangle.$$

Using the relations (see [18])

$$\frac{d}{dx}\varphi^{(l)}(x) = \varphi^{(l-1)}(x + r_{l-1}) - \varphi^{(l-1)}(x + r_{l-1} - 1)$$

(where $r_s = s \bmod 2$),

$$c^{(l,\tilde{l})}_{\alpha,k} - c^{(l,\tilde{l})}_{\alpha,k-1} = \alpha\, c^{(l-1,\tilde{l}+1)}_{\alpha-1,k-r_{l-1}}$$

(see (34)) and the definition of boundary functions, we get

$$\begin{cases} \frac{d}{dx}\varphi^{(l,\tilde{l})}_{00} = \varphi^{(l-1,\tilde{l}+1)}_{0l} & \\ \frac{d}{dx}\varphi^{(l,\tilde{l})}_{0\alpha} = \alpha\varphi^{(l-1,\tilde{l}+1)}_{0,\alpha-1} - c^{(l,\tilde{l})}_{\alpha,k^*-1}\varphi^{(l-1,\tilde{l}+1)}_{0\alpha} & \alpha = 1, \dots, l-1, \\ \frac{d}{dx}\varphi^{(l,\tilde{l})}_{0\alpha} = \varphi^{(l-1,\tilde{l}+1)}_{0,\alpha-1} - \varphi^{(l-1,\tilde{l}+1)}_{0\alpha} & \alpha = l, \dots, \tilde{l}-1. \end{cases}$$

We can now express the elements of $B = X(l, \tilde{l})$ in terms of the components of $A = X(l - 1, \tilde{l} + 1)$ as follows:

$$B_{\alpha\beta} = \frac{1}{\beta+1}\begin{cases} -A_{l,\beta+1} & \alpha = 0, \\ -\alpha A_{\alpha-1,\beta+1} + c^{(l,\tilde{l})}_{\alpha,k^*-1}A_{l,\beta+1} & \alpha = 1, \dots, l-1, \\ -A_{\alpha-1,\beta+1} + A_{\alpha,\beta+1} & \alpha = l, \dots, \tilde{l}-1. \end{cases}$$

With elementary operations on the rows and columns of B, we can transform it in the matrix obtained deleting the last two rows and the first and last column of A. These operations never switch columns and do not affect the singularity of the minors of B, so by induction the proof is complete. □

Finally, with the notation of Section 8, we have

$$m_0^* = m_0^\# - 1 = \frac{l+\tilde{l}}{2} + \frac{l+r}{2} - 1, \qquad \widetilde{m}_0^* = \widetilde{m}_0^\# - 1 = \frac{l+\tilde{l}}{2} + r - 1,$$

and

$$m^* = m^\# - 1 = l + r + \tilde{l} - 2 + \left\lceil \frac{\tilde{l}-1}{2} \right\rceil, \qquad \widetilde{m}^* = \widetilde{m}^\# - 1 = 2l + r - 2 + \left\lceil \frac{\tilde{l}-1}{2} \right\rceil,$$

where r is 1 if l is odd, zero otherwise. Then it is easy to see that $\widetilde{m}^* \le m^*$ for all couples $(l, \tilde{l})$. Therefore, one has

$$\begin{aligned} W_j(0,1) &= \operatorname{span}\left\{\psi^{(0)}_{jm} : m = 0, \dots, m^*-1\right\} \oplus \\ &\oplus \operatorname{span}\left\{\psi_{jm} : m = m^*, \dots, 2^j - m^*\right\} \oplus \\ &\oplus \operatorname{span}\left\{\psi^{(1)}_{jm} : m = 1, \dots, m^*-1\right\}. \end{aligned}$$

Since the border wavelets have the same supports at the two edges, we only have to compute their lengths once. From the definition (74), substituting the values of $n_0, \tilde{n}_0$ and $n_1, \tilde{n}_1$ it is easy to see that

$$\max_{m=0,\dots,m^*-1} \left|\operatorname{supp} \psi^{(0)}_{jm}\right| = 2^{-j}\left(2\tilde{l} + \frac{3l}{2} + r - 3\right).$$

To obtain a lower limit for j_0 it is enough to require $2^{-j_0}\left(2\tilde{l}+\frac{3l}{2}+r-3\right)\leq\frac{1}{2}$. In this way we obtain

$$j_0 \geq \left\lceil \log_2\left(2\tilde{l}+\frac{3l}{2}+r-3\right)+1\right\rceil . \tag{97}$$

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