Rend. Sem. Mat. Univ. Pol. Torino
Vol. 57, 2 (1999)

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WEAK FORMULATION FOR NONLINEAR HYPERBOLIC STEFAN PROBLEMS

Abstract.
The aim of this paper is to analyze a suitable weak formulation of an abstract hyperbolic doubly nonlinear problem. The results apply to a general version of the hyperbolic Stefan problem with memory.

1. Introduction

Let V, H be a pair of real Hilbert spaces such that $V \subset H \subset V'$ with continuous and densely defined embeddings. Here V' denotes the antidual of V. Denote by $\langle v', v\rangle$ the duality pairing between v' in V' and v in V; if $v' \in H$ this is the ordinary inner product (v', v) in H. Let $((\cdot,\cdot))$ define the scalar product in V. The norms in V, H, V' will be denoted by $\|\cdot\|$, $|\cdot|$, $\|\cdot\|_*$, respectively.

Let $F : v \mapsto F[v]$ and $G : v \mapsto G[v]$ define two functional mappings such that

$$F : W^{1,1}(0,t;H) \to W^{1,1}(0,t;H), \quad G : W^{1,1}(0,t;V) \to W^{2,1}(0,t;V') \quad \forall t \in]0,T],$$

$$G[v](0) = g_0 \in V' \qquad \forall v \in W^{1,1}(0,T;V),$$

for some prescribed value g_0. Let B be a maximal monotone set in $V \times V'$ and $L : V \to V'$ a linear, continuous, selfadjoint, and weakly coercive operator.

Then, under suitable boundedness and continuity hypotheses on F and G, Colli and Grasselli [4] showed that there exists a unique function $u \in W^{2,\infty}(0,T;H) \cap W^{1,\infty}(0,T;V)$ satisfying

$$(1) \qquad u_{tt}(t) + Bu_t(t) + Lu(t) \ni F[u_t](t) + G[u](t), \qquad t \in]0,T[,$$

$$(2) \qquad u(0) = u_0 \in V, \qquad u_t(0) = v_0 \in D(B),$$

for some initial data u_0 and v_0, where u_t stands for the usual derivative of the vector-valued function u.

This paper is devoted to investigate the well-posedness of a weak formulation of (1-2), looking for solutions $u \in C^0([0,T];V) \cap C^1([0,T];H)$ in the special case when B is a maximal monotone and sublinear operator from H in H. Let us point out at once the interest of this case, since the main application of [4] fits into our framework. In fact, Section 6 is concerned with a nonlinear extension of the hyperbolic Stefan problem with memory discussed in [4].

In our setting, the right hand side $F+G$ of (1) is replaced by one map $S : v \mapsto S_1[v]+S_2[v]$ with the properties

$$S : L^1(0,t;V) \cap W^{1,1}(0,t;H) \to L^1(0,t;H) + W^{1,1}(0,t;V') \qquad \forall t \in]0,T],$$

$$S_2[v](0) = s_{20} \in V' \qquad \forall v \in L^1(0,T;V) \cap W^{1,1}(0,T;H)\,.$$

By assuming $u_0 \in V$ and $v_0 \in H$, we can establish existence, uniqueness and continuous dependence theorems. Then, these results will be applied to the general version of the hyperbolic Stefan problem with memory.

The study of the abstract problem is carried out as follows. First we consider the explicit problem, with a fixed right hand side, and analyze it by an approximation and passage to the limit procedure where we can recover strong convergences as in [2]. To show that the actual implicit version is well posed, we use the Contraction Mapping Principle locally in time and extend the solution by a global estimate, as in [4]. Then, we also deduce the Lipschitz continuous dependence of the solution on the initial data.

2. Main results

Keeping the same notation as above, we can specify our hypotheses.

a) L is linear continuous symmetric operator from V to V' such that

$$\langle Lv, v\rangle + \ell|v|^2 \geq c\|v\|^2 \qquad \forall v \in V\,, \tag{3}$$

for some $\ell, c > 0$. Hence, denoting by I the identity (or injection) operator in H, $\ell I + L$ is strongly monotone from V to V'. Besides, we fix a constant C such that

$$|v| \leq C\|v\|, \qquad \|Lv\|_* \leq C\|v\| \qquad \forall v \in V\,. \tag{4}$$

b) B is maximal monotone from $D(H) = H$ to H and satisfies

$$B(0) \ni 0\,, \tag{5}$$

$$\exists \Lambda > 0 \quad : \quad \forall u \in H, \quad \forall w \in Bu \qquad |w| \leq \Lambda(1 + |u|)\,. \tag{6}$$

c) $S = S_1 + S_2$ fulfills

$$S_1[0] \in L^1(0,T;H)\,, \tag{7}$$

$$S_2[0] \in W^{1,1}(0,T;V')\,, \tag{8}$$

$$S_2[v](0) = s_{20} \qquad \forall v \in L^1(0,T;V) \cap W^{1,1}(0,T;H) \tag{9}$$

and there is a positive constant C_S such that

$$\begin{aligned} &\|S_1[\hat{v}] - S_1[\tilde{v}]\|^2_{L^1(0,t;H)} + \|d_t(S_2[\hat{v}] - S_2[\tilde{v}])\|^2_{L^1(0,t;V')} \\ &\leq C_S\left(\|\hat{v} - \tilde{v}\|^2_{L^1(0,t;V)} + \|(\hat{v} - \tilde{v})_t\|^2_{L^1(0,t;H)}\right) \end{aligned} \tag{10}$$

for any $t \in]0,T]$, $\hat{v}, \tilde{v} \in L^1(0,T;V) \cap W^{1,1}(0,T;H)$, where $d_t := d/dt$.
From (7-10) it results that the inequality

$$\begin{aligned} &\|S_1[v]\|^2_{L^1(0,t;H)} + \|d_t S_2[v]\|^2_{L^1(0,t;V')} \\ &\leq C'_S\left(1 + \|v\|^2_{L^1(0,t;V)} + \|v_t\|^2_{L^1(0,t;H)}\right) \end{aligned} \tag{11}$$

holds for any $t \in]0,T]$, $v \in L^1(0,T;V) \cap W^{1,1}(0,T;H)$, with the constant C'_S depending only on $\|S_1[0]\|_{L^1(0,T;H)}$, $\|d_t S_2[0]\|_{L^1(0,T;V')}$, $\|s_{20}\|_{V'}$, and C_S.

PROBLEM 1. *Find* $u \in C^0([0,T];V) \cap C^1([0,T];H)$ *and* $w \in L^\infty(0,T;H)$ *satisfying*

$$u_{tt} \in L^1(0,T;V')\,, \tag{12}$$

$$\langle u_{tt}(t), v\rangle + \langle w(t), v\rangle + \langle Lu(t), v\rangle = \langle S[u](t), v\rangle \quad \text{for a.a. } t \in]0,T[, \qquad \forall v \in V\,, \tag{13}$$

$$w(t) \in B(u_t(t)) \qquad \text{for a.a. } t \in]0,T[\,, \tag{14}$$

$$u(0) = u_0, \qquad (du/dt)(0) = v_0\,. \tag{15}$$

THEOREM 1 (EXISTENCE AND UNIQUENESS). *Assume that Hypotheses a), b), c) hold and let* u_0, v_0, s_{20} *be given such that*

$$u_0 \in V, \qquad v_0 \in H, \qquad s_{20} \in V'\,. \tag{16}$$

Then there exists one and one solution of Problem 1.

THEOREM 2 (CONTINUOUS DEPENDENCE). *Assume that Hypotheses a), b), c) hold. Let* $\{\hat{u}_0, \hat{v}_0, \hat{s}_{20}\}$, $\{\tilde{u}_0, \tilde{v}_0, \tilde{s}_{20}\}$ *be two sets of data satisfying (16), and let* $(\hat{u}, \hat{w})$, $(\tilde{u}, \tilde{w})$ *be the corresponding solutions of Problem 1. Then there is a constant* N*, depending only on* ℓ, c, C, C_S *and* T*, such that*

$$\|\hat{u} - \tilde{u}\|_{C^0([0,T];V)\cap C^1([0,T];H)} \leq N(\|\hat{u}_0 - \tilde{u}_0)\| + |\hat{v}_0 - \tilde{v}_0| + \|\hat{s}_{20} - \tilde{s}_{20}\|_*)\,. \tag{17}$$

The above theorems are shown in Section 5, after proving an auxiliary lemma in Section 3 and looking at the explicit problem in Section 4. The last section is devoted to the mentioned application.

3. Auxiliary lemma

LEMMA 1. *Let* (V, H, V') *be a Hilbert triplet and* B *denote a maximal monotone operator from* $D(B) = H$ *to* H*. If the condition*

$$\exists \Lambda > 0 \quad : \quad \forall u \in H, \quad \forall \omega \in Bu, \qquad |\omega| \leq \Lambda(1 + |u|)\,, \tag{18}$$

is fulfilled, then the restriction A *of* B *to* V *is maximal monotone from* V *to* V'.

Proof. Without loss of generality we may assume that $0 \in B0$: this can be achieved by shifting the range of B. The monotonicity of A is obvious. We check its maximal monotonicity. Given f in V', we try to solve in V the equation $f \in \mathcal{T}u + Au$ by approximating it with the equation $f \in \mathcal{T}u^\varepsilon + B^\varepsilon u^\varepsilon$ ($\varepsilon > 0$), where $\mathcal{T}$ is the Riesz operator from V to V', that is,

$$\langle \mathcal{T}u, v\rangle = ((u, v)) \qquad \forall u, v \in V\,,$$

and B^ε the Yosida approximation of B in H. Being I the identity operator of H, we recall that

$$B^\varepsilon = \frac{I - J^\varepsilon}{\varepsilon}, \quad \text{where } J^\varepsilon = (I + \varepsilon B)^{-1} \text{ denotes the resolvent of } B\,,$$

$$B^\varepsilon u = \frac{u - u^\varepsilon}{\varepsilon}, \quad \text{where } u^\varepsilon \text{ is defined by } (I + \varepsilon B)u^\varepsilon = u\,.$$

It is important to distinguish between the single-valued operator B^ε of H and the multivalued operator BJ^ε. We have $B^\varepsilon u \in BJ^\varepsilon u$ for all $u \in H$. In fact, $B^\varepsilon u \in B(J^\varepsilon u)$ means $(I - J^\varepsilon)u \in$

$\varepsilon B(J^\varepsilon u)$, that is, $u \in (J^\varepsilon u) + \varepsilon B(J^\varepsilon u)$, and this is true owing to the definition of J^ε. As B^ε is maximal monotone and Lipschitz continuous of constant $1/\varepsilon$, $B^\varepsilon\big|_V$ is also monotone and Lipschitz continuous from V to V'. As $\mathcal{T} : V \to V'$ is obviously coercive, [1, Corollary 1.3, p. 48] ensures the existence of $\mathsf{u}^\varepsilon \in V$ satisfying $\mathcal{T}\mathsf{u}^\varepsilon + B^\varepsilon \mathsf{u}^\varepsilon = f$. Multiply this equation by u^ε. Note that $\langle \mathcal{T}\mathsf{u}^\varepsilon, \mathsf{u}^\varepsilon\rangle = \|\mathsf{u}^\varepsilon\|^2$, $(B^\varepsilon \mathsf{u}^\varepsilon, \mathsf{u}^\varepsilon) \geq 0$ (because $B^\varepsilon 0 = 0$), $\langle f, \mathsf{u}^\varepsilon\rangle \leq \|f\|_*\|\mathsf{u}^\varepsilon\|$. Then we easily get

$$\|\mathsf{u}^\varepsilon\|^2 \leq \|f\|_*\|\mathsf{u}^\varepsilon\|\,,$$

whence $\{\mathsf{u}^\varepsilon\}_{\varepsilon>0}$ is bounded in V. Setting $\mathsf{w}^\varepsilon := B^\varepsilon \mathsf{u}^\varepsilon$, from [1, Proposition 1.1. (iii), p. 42], (4), and (18) we recover the estimate

$$|\mathsf{w}^\varepsilon| = |B^\varepsilon \mathsf{u}^\varepsilon| \leq \inf_{\mathsf{w}\in B\mathsf{u}^\varepsilon} |\mathsf{w}| \leq \Lambda(1 + |\mathsf{u}^\varepsilon|) \leq \Lambda(1 + C\|\mathsf{u}^\varepsilon\|) \leq \mathsf{c}\,,$$

for some constant c independent of ε. Therefore, there are a subsequence $\{\mathsf{u}^{\varepsilon_n}\}$ weakly converging to u in V and a subsequence $\{\mathsf{w}^{\varepsilon_n}\}$ weakly converging to w in H. As n goes to ∞, $\varepsilon_n \downarrow 0$ and the equality

$$\langle \mathcal{T}\mathsf{u}^{\varepsilon_n}, v\rangle + (B^{\varepsilon_n}\mathsf{u}^{\varepsilon_n}, v) = \langle f, v\rangle, \qquad v \in V\,,$$

tends to

$$\langle \mathcal{T}\mathsf{u}, v\rangle + (\mathsf{w}, v) = \langle f, v\rangle, \qquad v \in V\,,$$

thanks to the continuity of $\mathcal{T}$. Now we show that

$$\limsup_{n\uparrow\infty}(\mathsf{w}^{\varepsilon_n}, \mathsf{u}^{\varepsilon_n}) \leq (\mathsf{w}, \mathsf{u})\,. \tag{19}$$

In order to simplify the notation, we replace ε_n with n. On account of the relation

$$\langle \mathcal{T}\mathsf{u}^n, \mathsf{u}^n\rangle + (\mathsf{w}^n, \mathsf{u}^n) = \langle f, \mathsf{u}^n\rangle\,,$$

we deduce that

$$\begin{aligned} \|\mathsf{u}\|^2 &\leq \liminf \|\mathsf{u}^n\|^2 \\ &= -\limsup(-\langle \mathcal{T}\mathsf{u}^n, \mathsf{u}^n\rangle) \\ &= -\limsup((\mathsf{w}^n, \mathsf{u}^n) - \langle f, \mathsf{u}^n\rangle) \\ &= \langle f, \mathsf{u}\rangle - \limsup(\mathsf{w}^n, \mathsf{u}^n) \end{aligned}$$

and consequently

$$\limsup(\mathsf{w}^n, \mathsf{u}^n) \leq \langle f, \mathsf{u}\rangle - \|\mathsf{u}\|^2 = \langle f, \mathsf{u}\rangle - \langle \mathcal{T}\mathsf{u}, \mathsf{u}\rangle = (\mathsf{w}, \mathsf{u})\,.$$

Hence, (19) is true and [1, Proposition 1.1 (iv), p. 42] allows us to conclude that $\mathsf{w} \in B\mathsf{u}$. Thus $\mathsf{u} \in V$ solves $\mathcal{T}\mathsf{u} + B\mathsf{u} \ni f$ and the lemma is completely proved.

□

REMARK 1. In fact, the assumption (18) has been used only to deduce the boundedness of $\{\mathsf{w}^\varepsilon\}$ from that of $\{\mathsf{u}^\varepsilon\}$. Therefore the same proof hold if (18) is replaced by the more general condition

$$B \text{ is bounded on bounded sets}\,.$$

4. The explicit problem

For $v \in C^0([0,T];V) \cap C^1([0,T];H)$ and $t \in]0,T]$, we introduce the auxiliary norm

$$\|v\|_t^2 := \|v\|^2_{C^0([0,t];V)} + \|v_t\|^2_{C^0([0,t];H)} . \tag{20}$$

and let

$$s = s_1 + s_2 \in L^1(0,T;H) + W^{1,1}(0,T;V') , \tag{21}$$

with

$$s_2(0) = s_{20} . \tag{22}$$

We recover useful properties for the following reduction of Problem 1, which is expressed in terms of the single variable function u, for the sake of convenience.

PROBLEM 2. *Find $u \in C^0([0,T];V) \cap C^1([0,T];H)$ fulfilling (12), (14-15), and*

$$\langle u_{tt}(t), v\rangle + \langle w(t), v\rangle + \langle Lu(t), v\rangle = \langle s_1(t) + s_2(t), v\rangle \tag{23}$$

for a.a. $t \in]0,T[$ and any $v \in V$, for some $w \in L^\infty(0,T;H)$.

Indeed, we can state

THEOREM 3. *Assume that Hypotheses a) and b), (21-22) and (16) are satisfied. Then Problem 2 has a unique solution. Moreover, letting $\{\hat u_0, \hat v_0, \hat s_{20}, \hat s_1, \hat s_2\}$, $\{\tilde u_0, \tilde v_0, \tilde s_{20}, \tilde s_1, \tilde s_2\}$ be two sets of data and letting $\hat u$, $\tilde u$ represent the related solutions, the estimate*

$$\begin{aligned} \|\hat u - \tilde u\|_t^2 \le C_1 \Big(& \|\hat u_0 - \tilde u_0\|^2 + |\hat v_0 - \tilde v_0|^2 + \|\hat s_{20} - \tilde s_{20}\|_*^2 \\ & + \|\hat s_1 - \tilde s_1\|^2_{L^1(0,t;H)} + \|(\hat s_2 - \tilde s_2)_t\|^2_{L^1(0,t;V')} \Big) \end{aligned} \tag{24}$$

holds for any $t \in]0,T]$, where the constant C_1 depends just on ℓ, c, C, and T.

Proof. We start from the existence result given in [4], where stronger assumptions are required on data. Therefore we regularize s, u_0, v_0 and choose three families $\{s^\varepsilon\}$, $\{u_0^\varepsilon\}$, $\{v_0^\varepsilon\}$ such that

$$\begin{aligned} &s^\varepsilon \in W^{1,1}(0,T;H), \quad s^\varepsilon \to s \quad \text{in} \quad L^1(0,T;H) \cap W^{1,1}(0,T;V') , \\ &u_0^\varepsilon \in V, \quad Lu_0^\varepsilon \in H, \quad u_0^\varepsilon \to u_0 \quad \text{in} \quad V , \\ &v_0^\varepsilon \in V, \quad v_0^\varepsilon \to v_0 \quad \text{in} \quad H \end{aligned}$$

as $\varepsilon \downarrow 0$. For instance, u_0^ε could be taken as the solution of the elliptic problem (see [3, Appendix])

$$u_0^\varepsilon + \varepsilon L u_0^\varepsilon = u_0 .$$

for ε sufficiently small (cf. (3)). Thanks to Lemma 1 and [4, Lemma 3.3, p. 88], Problem 2 with s, u_0, v_0 replaced by s^ε, v_0^ε, u_0^ε has a unique solution $(u^\varepsilon, w^\varepsilon)$ satisfying $u^\varepsilon \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H)$, $w^\varepsilon \in L^\infty(0,T;V')$. Actually, in our case w^ε belongs to $L^\infty(0,T;H)$ owing to (6). Now, we can use [4, estimate (3.5), p. 87] as contracting estimate. Indeed, since (14) and

(5) yield $(w, u_t) \geq 0$ a.e. in $]0, T[$, the procedure followed in [4, Lemma 3.2, p. 87] enables us to infer

$$
\begin{aligned}
\|\hat{u}^\varepsilon - \tilde{u}^\varepsilon\|_t^2 \leq & C_1 \Big(\|\hat{u}_0^\varepsilon - \tilde{u}_0^\varepsilon\|^2 + |\hat{v}_0^\varepsilon - \tilde{v}_0^\varepsilon|^2 + \|\hat{s}_{20}^\varepsilon - \tilde{s}_{20}^\varepsilon\|_*^2 \\
& + \|\hat{s}_1^\varepsilon - \tilde{s}_1^\varepsilon\|_{L^1(0,t;H)}^2 + \|(\hat{s}_2^\varepsilon - \tilde{s}_2^\varepsilon)_t\|_{L^1(0,t;V')}^2 \Big)
\end{aligned} \tag{25}
$$

with obvious notation, and (note that 0 solves Problem 2 with null data)

$$
\|u^\varepsilon\|_t^2 \leq C_1 \Big(\|u_0^\varepsilon\|^2 + |v_0^\varepsilon|^2 + \|s_{20}^\varepsilon\|_*^2 + \|s_1^\varepsilon\|_{L^1(0,t;H)}^2 + \|(s_2^\varepsilon)_t\|_{L^1(0,t;V')}^2 \Big) \tag{26}
$$

for all $t \in]0, T]$ and for some constant C_1 independent of ε. Consequently, also $\|w^\varepsilon\|_{L^\infty(0,T;H)}$ is uniformly bounded.

Due to well-known compactness results, we can find subsequences converging weakly star $\left(\overset{*}{\rightharpoonup}\right)$. Let u, w, and $\varepsilon_n \downarrow 0$ fulfill

$$
\begin{aligned}
u^n \overset{*}{\rightharpoonup} u \quad & \text{in} \quad L^\infty(0, T; V) \cap W^{1,\infty}(0, T; H), \\
w^n \overset{*}{\rightharpoonup} w \quad & \text{in} \quad L^\infty(0, T; H),
\end{aligned}
$$

where u^n and w^n stand for u^{ε_n} and w^{ε_n}, respectively. Now, one can show that the pair (u, w) solves the equations of Problem 2. In fact, it turns out that

$$
u^n \to u \qquad \text{strongly in } C^0([0, T]; V) \cap C^1([0, T]; H), \tag{27}
$$

$$
w \in B(u_t) \qquad \text{a.e. in }]0, T[\,. \tag{28}
$$

The proof of (27) consists in a direct check of the Cauchy condition in $C^0([0, T]; V) \cap C^1([0, T]; H)$ for u^n, by applying again [4, estimate (3.5), p. 87]. Further, accounting for the strong convergence of $\{u_t^n\}$ in $L^2(0, T; H)$ and arguing as in the proof of (19), thanks to [1, Lemma 1.3, p. 42] one verifies the second condition (28).

At this point, we can first take the limit in (25) on some subsequence $\varepsilon_n \downarrow 0$ and recover (24). Then, the uniqueness of u (and w) follows from (24) (and a comparison in (23)).

□

5. Existence, uniqueness and continuous dependence

. The next step for proving Theorem 1 consists in showing that Problem 1 has one and only one solution in some time interval $[0, \tau]$, $\tau \in]0, T]$. The main tool is the classical Contraction Mapping Principle.

Introduce the metric spaces

$$
\begin{aligned}
X \quad & := \quad C^0([0, \tau]; V) \cap C^1([0, \tau]; H), \\
X_0 \quad & := \quad \{\xi \in X : \xi(0) = u_0,\ \xi_t(0) = v_0,\ \|\xi\|_\tau^2 \leq C_2\},
\end{aligned}
$$

where the constants C_2 and τ are specified later. Fix an element z of X_0 and let $u \in X$ and $w \in L^\infty(0, \tau; V')$ satisfy (14-15) and

$$
\langle (d^2u/dt^2)(t), v\rangle + \langle Lu(t), v\rangle + \langle w(t), v\rangle = \langle S[z](t), v\rangle \tag{29}
$$

for a.a. $t \in]0, \tau[$ and any $v \in V$. Due to the assumptions on S and to Theorem 3, the operator D mapping z into the unique solution u of the above explicit problem is well defined. Since

$u(0) = u_0$ and $u_t(0) = v_0$, the claim is that $\mathsf{D}(X_0) \subseteq X_0$ provided C_2 is suitably chosen and τ is small enough. Indeed, (24) and (11) enable us to deduce that

$$\|u\|_\tau^2 \le C_1\Big(\|u_0\|^2 + |v_0|^2 + \|s_{20}\|_*^2\Big) + C_1 C_S'\Big(1 + \|z\|^2_{L^1(0,\tau;V)} + \|z_t\|^2_{L^1(0,\tau;H)}\Big),$$

whence, taking $C_2 = 2C_1(C_S' + \|u_0\|^2 + |v_0|^2 + \|s_{20}\|_*^2)$ and $C_3 = C_1 C_S'$, it results that

$$\|u\|_\tau^2 \le \frac{C_2}{2} + C_3\tau^2\|z\|_\tau^2\,, \tag{30}$$

on account of (20). Therefore, if $\tau^2 \le \frac{1}{2C_3}$ it is ensured that $\mathsf{D} : X_0 \to X_0$. On the other hand, X_0 is a complete metric space with respect to the distance $d(\hat z, \tilde z) := \|\hat z - \tilde z\|_\tau$, $\hat z, \tilde z \in X_0$. Thus, it becomes important to find conditions on τ in order that D yields a contraction mapping. Let $\hat z, \tilde z \in X_0$ and $\hat u = \mathsf{D}(\hat z)$, $\tilde u = \mathsf{D}(\tilde z)$. Owing to (24) and (10), one can infer that

$$\|\hat u - \tilde u\|_\tau^2 \le C_4\tau^2\|\hat z - \tilde z\|_\tau^2\,, \tag{31}$$

with $C_4 = C_1 C_S$. Then, choosing $\tau = \min\left\{T,\ \left(\frac{1}{2C_4}\right)^{\frac12}\right\}$, we can conclude that the operator D has a unique fixed point $u \in X_0$. Clearly, such u provides, along with the related w, the solution to Problem 1 in the preliminary time interval $[0, \tau]$.

To complete the proof of Theorem 1 it remains to verify the possibility of extending the solution to the whole interval $[0, T]$. Here we perform that by deriving a global estimate.

Owing to the assumptions (10-d2.9), we can argue as in the deduction of (30) (recalling (13) in place of (29)) and, in view of Theorem 3 and the Hölder inequality, conclude that

$$\|u\|_\tau^2 \le C_2 + C_3 T\int_0^\tau \|u\|_s^2 ds\,, \tag{32}$$

where τ is now an arbitrary value in $[0, T]$. As the function $t \mapsto \|u\|_t$ is continuous, the Gronwall lemma implies that $\|u\|_\tau^2 \le C_2\, e^{C_3 T\tau}$. Then, integrating from 0 to $t \in [0, T]$, by (4) we find a constant C_5, independent of t, such that

$$\|u\|_{C^0([0,t];V)\cap C^1([0,t];H)} \le C_5 \tag{33}$$

for any $t \in]0, T]$. Moreover, (14) and (6) provide us a bound for $\|w\|_{L^\infty(0,T;H)}$.

To prove Theorem 2, consider two sets of initial data $(\hat u_0,\, \hat v_0,\, \hat s_{20})$, $(\tilde u_0,\, \tilde v_0,\, \tilde s_{20})$ and denote by $(\hat u,\, \hat w)$, $(\tilde u,\, \tilde w)$ the corresponding solutions of Problem 1. Thanks to (24) and (10), we have that

$$\begin{aligned}\|\hat u - \tilde u\|_t^2 \quad &\le \quad C_1\Big(\|\hat u_0 - \tilde u_0\|^2 + |\hat v_0 - \tilde v_0|^2 + \|\hat s_{20} - \tilde s_{20}\|_*^2 \\ &\qquad + C_S\|\hat u - \tilde u\|^2_{L^1(0,t;V)} + C_S\|\hat u_t - \tilde u_t\|^2_{L^1(0,t;H)}\Big)\,,\end{aligned}$$

whence, estimating the norms in L^1 by the norms in L^2, we infer

$$\|\hat u - \tilde u\|_t^2 \le C_1\Big(\|\hat u_0 - \tilde u_0\|^2 + |\hat v_0 - \tilde v_0|^2 + \|\hat s_{20} - \tilde s_{20}\|_*^2\Big) + C_4 T\int_0^t \|\hat u - \tilde u\|_s^2 ds \tag{34}$$

for any $t \in]0, T]$. Finally, (17) comes out from applying the Gronwall lemma to (34).

□

6. The Stefan problem

Consider a two-phase material which occupies a bounded and smooth domain $\Omega \subset \mathbb{R}^N$, at any time $t \in [0, T]$, $T > 0$ being given. In addition, suppose that the material is homogeneous with unit density. Denote by $\theta : Q \to \mathbb{R}$ its relative temperature and by $\chi : Q \to [0, 1]$ the proportion of one of the two phases, where $Q := \Omega \times]0, T[$.

Assume that the evolution of (θ, χ) is governed by the system

(35) $\partial_t(\theta + \phi * \theta + \psi * \chi) - \Delta(k * \theta) = f + g(x, t, \theta(x, t), (\alpha * \theta)(x, t), (\gamma * \nabla\theta)(x, t))$,

(36) $\chi \in \mathcal{H}(\theta)$,

where these conditions must hold in the cylinder Q. Here the symbol ∂_t denotes the partial derivative $\partial/\partial t$, Δ is the standard Laplace operator in space variables and $\mathcal{H}$ is the Heaviside graph, i.e. $\mathcal{H}(\eta) = 0$ if $\eta < 0$, $\mathcal{H}(0) = [0, 1]$, $\mathcal{H}(\eta) = 1$ if $\eta > 0$. Moreover, the functions $\phi, \psi, k, \alpha, \gamma : [0, \infty[\to \mathbb{R}$ are memory kernels, and the symbol $*$ stands for the convolution product over $(0, t)$, that is,

$$(a * b)(t) := \int_0^t a(s)b(t - s)ds, \qquad t \in [0, T],$$

where a and b may depend also on the space variables. The source term $f : Q \to \mathbb{R}$ is a known function related to both the heat supply and the prescribed past histories of θ and χ up to $t = 0$, g is a Lipschitz continuous function with respect to the last three variables.

We recall that equation (35) is obtained, via energy balance, from a set of constitutive laws for the internal energy and the heat flux in the framework of the theory of materials with memory (cf. [4, 5] and references therein). On the other hand, relationship (36) represents the usual Stefan equilibrium condition when the phase change temperature is supposed to be zero.

Let us associate with equations (35) and (36) the initial condition

$$\theta(\cdot, 0) = \theta_0 \qquad \text{in } \Omega \tag{37}$$

and, for instance, the homogeneous Dirichlet boundary condition

$$\theta = 0 \qquad \text{on } \partial\Omega \times]0, T[\,. \tag{38}$$

The Stefan problem (35-38) may be termed *hyperbolic* as the *freezing index*

$$u(x, t) := (1 * \theta)(x, t) = \int_0^t \theta(x, s)ds, \qquad (x, t) \in Q \tag{39}$$

obeys a nonlinear Volterra integrodifferential equation of hyperbolic type, provided that $k(0)$ is positive and k is sufficiently smooth (see, e.g., [2]).

In order to state a variational formulation of the problem (35-38), we take $H = L^2(\Omega)$ and introduce a closed subspace V of $H^1(\Omega)$ containing $H^1_0(\Omega)$, whose choice depends on the type of boundary conditions one wants to deal with (see the later Remark 3). If we identify H with its dual space H' by the Riesz theorem, H turns out to be a subset of V' and

$$\langle v', v\rangle = (v', v) \qquad \forall v' \in H, \quad \forall v \in V \,,$$

where $\langle\cdot, \cdot\rangle$ is the duality pairing between V' and V and $(\cdot, \cdot)$ is the scalar product in H. The injections $V \hookrightarrow H \hookrightarrow V'$ are both dense and, thanks to the assumptions on Ω, compact. Owing to (35), and recalling (39) and the formula

$$\sigma * \nu_t = (\sigma * \nu)_t - \nu(0)\sigma = \sigma(0)\nu + \sigma_t * \nu - \nu(0)\sigma \,,$$

which holds whenever it makes sense, we derive immediately an equality in u,

$$\langle \partial_t (u_t + \phi * u_t + \psi * \chi)(t), v \rangle + \int_\Omega \nabla (k(0)u + k' * u)(t) \cdot \nabla v$$
$$= \langle f(t), v \rangle + \langle g(\cdot, t, u_t(t), (\alpha * u_t)(t), (\gamma(0)\nabla u + \gamma' * \nabla u)(t)), v \rangle$$

for all $v \in V$, for a.a. $t \in]0, T[$. Suppose that

$$\phi \in W^{1,1}(0, T) \tag{40}$$
$$\psi,\ k \in W^{2,1}(0, T), \qquad \psi(0),\ k(0) > 0 \tag{41}$$
$$f \in L^1(0, T; H) + W^{1,1}(0, T; V') \tag{42}$$
$$\theta_0 \in H\,. \tag{43}$$

Regarding the term

$$\mathcal{G}[u](x, t) := g(x, t, u_t(x, t), (\alpha * u_t)(x, t), (\gamma(0)\nabla u + \gamma' * \nabla u)(x, t))$$

with

$$g : \Omega \times]0, T[\times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}\,,$$

we require that

$$\alpha \in L^1(0, T), \qquad \gamma \in W^{1,1}(0, T)\,, \tag{44}$$

(45) $\forall (z, w, \vec{p}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ the function $(x, t) \to g(x, t, z, w, \vec{p})$ is measurable,

$$|g(x, t, z_1, w_1, \vec{p}_1) - g(x, t, z_2, w_2, \vec{p}_2)|$$
$$\leq M(|z_1 - z_2| + |w_1 - w_2| + |\vec{p}_1 - \vec{p}_2|) \qquad \text{for a.a. } (x, t) \in Q\,, \tag{46}$$

where M denotes a positive constant. Since we have a fixed datum f, possibly translating g, we can assume

$$g(x, t, 0, 0, \vec{0}) = 0 \qquad \text{for a.a. } (x, t) \in \Omega \times]0, T[\,. \tag{47}$$

Let us couple the variational problem (6) with

$$\chi \in \mathcal{H}(u_t) \qquad \text{a.e. in } Q\,, \tag{48}$$
$$u(0) = 0 \qquad \text{and} \qquad u_t(0) = \theta_0\,. \tag{49}$$

Of course, we aim to set the problem (6), (48-49) in some equivalent form which fits into the frame of Problem 1. The maximal monotone graph $\mathcal{H}$ is easily extended to an operator B fulfilling (6) with $\Lambda = 1$. Hence $\chi \in L^\infty(0, T; H)$ plays (not exactly, cf. (51)) the role of w and we look for solutions $u \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]); L^2(\Omega))$.

REMARK 2. Thanks to our positions for g,α,γ and to the regularizing properties of the convolution product (referring to [6], we just remind that $\|\sigma * \nu\|_{L^p(0,T)} \leq \|\sigma\|_{L^1(0,T)} \|\nu\|_{L^p(0,T)}$ whenever $\sigma \in L^1(0, T)$, $\nu \in L^p(0, T)$, $p \in [1, \infty]$), it turns out that $\mathcal{G}[u] \in L^\infty(0, T; H)$. Thus equation (6) makes sense.

REMARK 3. The homogeneous Dirichlet boundary condition (38) corresponds just to the choice $V = H^1_0(\Omega)$, whereas the option $V = H^1(\Omega)$ imply a Neumann boundary condition. For mixed boundary conditions one should choose V as the subspace of $H^1(\Omega)$ whose elements vanish at points of the boundary of Ω where the (homogeneous) Dirichlet condition is prescribed. Note moreover that the assumption on f allows to consider non homogeneous Neumann data, for instance in the case of the Neumann problem; indeed the formula

$$\langle f(t), v\rangle = \int_{\partial\Omega} h(\cdot, t)v\,, \qquad v \in V\,,$$

defines $f \in L^2(0, T; V')$ if, e.g., $h \in L^2(\partial\Omega\times]0, T[)$.

Now, we rewrite (6) in a suitable form. By a time differentiation of $\psi * \chi$ we deduce that

$$\psi(0)\chi + \psi' * \chi = \mathsf{R} \qquad \text{in } V'\,, \tag{50}$$

where

$$\langle \mathsf{R}, v\rangle := \langle f + \mathcal{G}[u] - \partial_t(u_t + \phi * u_t), v\rangle - \int_\Omega \nabla(k(0)u + k' * u)\cdot\nabla v \qquad \forall v \in V$$

in $]0, T[$. Equation (50) can be solved with respect to χ (see, e.g., [6, Ch. 2]) making use of the resolvent of $(\psi(0))^{-1}\psi'$. In fact, there is a unique function $\Psi \in W^{1,1}(0, T)$ such that $\psi(0)\Psi + \psi' * \Psi = \psi'$. This enables us to transform (50) into $\psi(0)\chi = \mathsf{R} - \Psi * \mathsf{R}$ (holding in $L^1(0, T; V')$). Now, since the convolution product is associative and commutative, with the help of integrations by parts and of (49), we conclude that

$$\langle u_{tt}, v\rangle + \psi(0)\langle\chi, v\rangle + k(0)\int_\Omega \nabla u\cdot\nabla v = \langle\mathcal{S}[u], v\rangle\,, \tag{51}$$

where

$$\begin{aligned}\langle\mathcal{S}[u], v\rangle \;:=\;& \langle f + \mathcal{G}[u] - \Psi * (f + \mathcal{G}[u]) - \Psi\theta_0 + (\Psi(0) - \phi(0))u_t, v\rangle\\ &+\langle(\Psi' - \phi' + \phi(0)\Psi + \Psi * \phi') * u_t, v\rangle\\ &+\int_\Omega \nabla((k(0)\Psi + \Psi * k' - k') * u)\cdot\nabla v\end{aligned}$$

for any $v \in V$, a.e. in $]0, T[$. Note that $\mathcal{S}[u]$ depends on u and u_t in a non local way and that the kernels $(\Psi' - \phi' + \phi(0)\Psi + \Psi * \phi')$ and $(k(0)\Psi + \Psi * k' - k')$ stay in $L^1(0, T)$ and $W^{1,1}(0, T)$, respectively. Moreover, it is not difficult to check (see [5] for details) that, due to (40-47), the mapping $\mathcal{S}$ satisfies (7-10).

Then the system (51), (48-49) fits into our abstract framework, and we can state

THEOREM 4. *Assume (40-47). Then there exists a unique pair*

$$u \in C^0([0, T]; V) \cap C^1([0, T]; H)\,, \tag{52}$$

$$\chi \in L^\infty(Q) \tag{53}$$

solving Problem (6), (48-49).

Note that Theorem 1 actually yields $\chi \in L^\infty(0, T; H)$; the additional regularity (53) plainly results from the boundedness of $\mathcal{H}$. At this point, let us stress that, due to the special dependence on ∇u of the nonlinearity g, the results of [4] do not apply to this case.

THEOREM 5. *Consider two sets of data* $\{\hat{f}, \hat{\theta}_0\}$, $\{\tilde{f}, \tilde{\theta}_0\}$ *satisfying (42-43) and denote by* $(\hat{u}, \hat{\chi})$, $(\tilde{u}, \tilde{\chi})$ *the corresponding solutions of Problem (6), (48-49). Then the following estimate holds*

$$\|\hat{u} - \tilde{u}\|_{C^0([0,T];V)\cap C^1([0,T];H)} \le \mathsf{N}\left(\|\hat{f} - \tilde{f}\|_{L^1(0,T;H)+W^{1,1}(0,T;V')} + |\hat{\theta}_0 - \tilde{\theta}_0|\right),$$

where the constant N *is independent of* $(\hat{u}, \hat{\chi})$ *and* $(\tilde{u}, \tilde{\chi})$.

We refer the reader to the dissertation [5] for further possible applications.

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AMS Subject Classification: 35R35.

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Lavoro pervenuto in redazione il 21.2.1997 e, in forma definitiva, il 31.3.1998.